The Early History of Life Annuity Valuation

Geoffrey Poitras

Simon Fraser University

Vancouver BC CANADA

Overview

- Early History of Financial Markets
- De Witt's Theoretical Solution
- Halley's Life Table and Life Annuity Valuation
- De Moivre's Approximation: Simplified Pricing
- Bernoulli's Problem: Contingent claims versus annuities certain

Early History of Security Markets

- Reading: SAIS, sec. 2.1, Poitras (2000, ch.2)
- Census Contracts and the Prohibition on Usury
- Southern European practice of forced loans by governments, e.g., the prestiti in Venice
- Northern European practice of market based issuance of government debt.

Jump to first page

History of Life Contingencies

- The origins of life annuities can be traced to ancient times.
  - Socially determined rules of inheritance usually meant a sizable portion of the family estate would be left to a predetermined individual, often the first born son.
  - Bequests such as usufructs, maintenances and life incomes were common methods of providing security to family members and others not directly entitled to inheritances.
  - The Falcidian law of ancient Rome, effective from 40 BC, maintained that the rightful heir(s) to an estate was entitled to not less than one quarter of the property left by a testator, the ***"Falcidian fourth"***

The Judicial Quandary

- The judicial quandary is to determine a value for any other legacies: if the total legacy value exceeded three quarters of the value of the total estate, these bequests had to be reduced proportionately.
- The Falcidian fourth created a legitimate valuation problem for jurists because many types of bequests did not have observable market values.
- This was the case for bequests of life incomes. Some method was required to convert bequests of life incomes to a form that could be valued.

Ulpian's Conversion Table

- To address the problem of the Falcidian fourth, the Roman jurist Ulpian (Domitianus Ulpianus, ?-228) devised a table for the conversion of life annuities to annuities certain

Age of annuitant in years								
0-19	20-24	25-29	30-34	35-39	40 ... 49	50-54	55-59	60-
30	28	25	22	20	19 ... 10	9	7	5

Comparable Term to maturity of an annuity certain in years

Jump to first page

Problems of application

- In practice, values were often determined by taking the annual value of the legacy, and multiplying this value by the term to maturity of the annuity certain to get the associated legacy value, resulting in a significant overvaluation
  - due to discounting at r=0 (Nicholas Bernoulli in the De Usu Artis Conjectandi in Jure, 1709).
  - Social prohibitions on interest payments

Early Life Annuity Contracts

- Northern European towns favoured annuities or *rentes* secured by urban taxes. As early as 1260, such early issues of *rentes heritables* and *rentes vagieres* (life annuities) appeared in the French towns such as Calais.
- Municipal finance using life and fixed term annuities spread to the Low Countries and German towns
- Between 1275 and 1290 the city of Ghent in Flanders issued *lijfrenten* or life annuities followed by issues of *erfrenten* or redeemable *rentes*.
- Municipalities, particularly in Holland, Flanders and Brabant, continued to issue life and redeemable annuities leading to increasingly larger stocks of public debt and, ultimately, to repayment difficulties for some towns by the 16th century.

The Life Annuity Contract

- The life annuity usually was a contract involving three parties:
  - the subscriber who provided the initial capital
  - the shareholder who was entitled to receive the annuity payments
  - the nominee on whose life the payout was contingent
- Different variations are possible

Quoting Prices: *Years' Purchase*

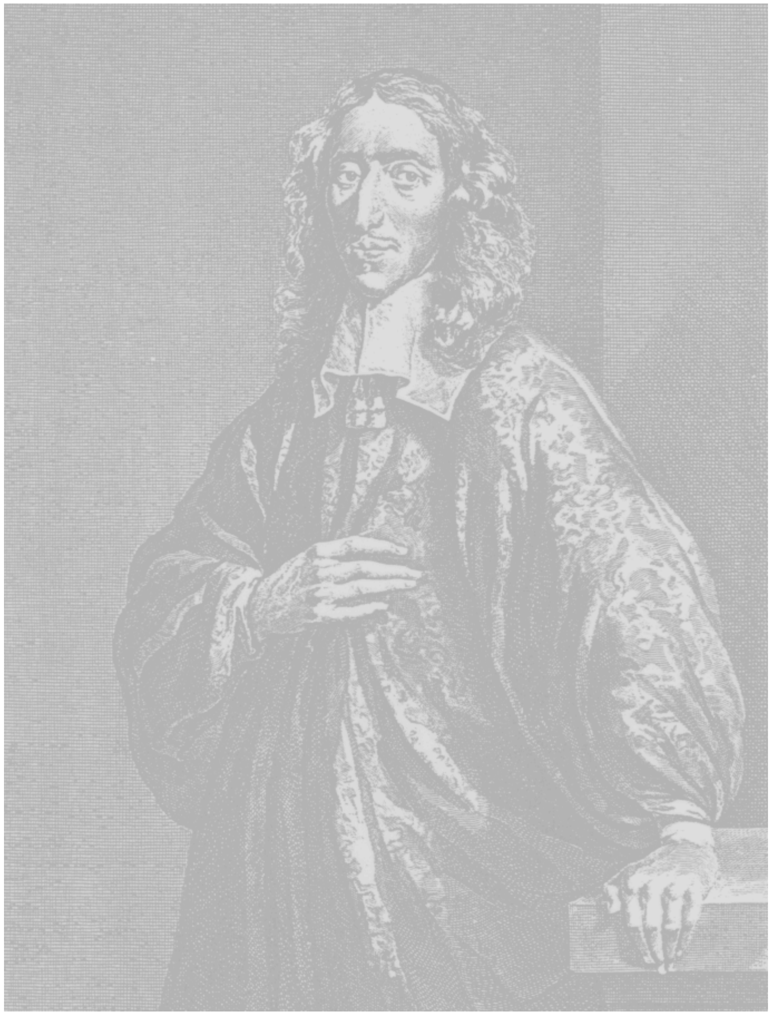
- Annuity prices were quoted in '***years' purchase***',
  - The price of the annuity divided by the annual annuity payment is the years' purchase
    - ☞ For a perpetual annuity, years purchase is the inverse of the annual yield to maturity.
    - ☞ Related to the current yield (see SAIS, sec. 4.1)

Preliminaries for Jan de Witt's Solution

- Who was Jan de Witt?

- Some notation is needed for the A_n
- For de Witt this was the present value of an annuity with a 4% annual rate to be paid at the end of the half year *n*

$$A_n = \sum_{t=1}^{n} \frac{1}{(1+r)^t} = \frac{1}{r} - \frac{1}{r(1+r)^n} \qquad where \quad (1+r) = \sqrt{1.04}$$



Jan de Witt (1625-72)

Jump to first page

Determining the Life Distribution

- Does **not** assume a uniform distribution where death at each age would be equally likely,
- De Witt divided the interval between 3 and 80 years into four subperiods: (3,53), (53,63), (63,73) and (73,80). Within each subperiod, an equal chance of mortality is assigned.
- The number of chances assigned to each subgroup is 1, 2/3, 1/2, 1/3.
- The chance of living beyond 80 is assumed to be zero.

De Witt's Solution for the Fair Value

To evaluate the expected present value of the life annuity, de Witt performs the calculation:

(Why is the divisor 128?)

$$E[A_n] = \frac{\sum_{n=1}^{99} A_n + \frac{2}{3}\sum_{n=100}^{119} A_n + \frac{1}{2}\sum_{n=120}^{139} A_n + \frac{1}{3}\sum_{n=140}^{153} A_n}{128}$$

Jump to first page

Bernoulli's problem

- "...I notice that the value of (life annuity) incomes is not correctly calculated by supposing that the return will last as many years as someone is supposed probably to live."
- The problem is to demonstrate that the fair value of a life annuity with differ from that for a term annuity with term to maturity equal to the expected duration of life

Edmond Halley: Demography and Life Annuity Valuation

- Edmond Halley (1656-1742)
  - published his influential paper “An Estimate of the Degrees of Mortality of Mankind, drawn from the curious Tables of the Births and Funerals at the City of Breslaw; with an Attempt to ascertain the Price of Annuities upon Lives” in 1694
  - At this time the English government was still selling life annuities at seven years' purchase, independent of age.

Edmond Halley (1656-1742)

Jump to first page

The Breslau Data

- The absence of empirical distribution for population demographics – a life table -- is a problem in de Witt's life annuity valuation.
- Earlier 17^{th} C, efforts at demographics in England (John Graunt) constructed a life table from the bills of mortality.
- From the end of the 16th century, Breslaw, a city in Silesia, had maintained a register of births and deaths, classified according to sex and age.
- The Breslau data is used in the preparation of Halley's "Estimate..."
  - For the purposes of constructing a precise life table, only the population size is missing.

Jump to first page

Preliminaries for Halley's Valuation

- The total number of annuities sold on a life starting at year x is ℓ_x
  - ℓ_x equals the sum of $d_x + d_{x+1} + \ldots\ldots + d_{w-1}$ where d_i is the number of annuities which terminate in period i due to the death of annuitant nominees in that half year
  - $d_i = 0$ for $x \geq w$.
  - Taking ℓ_{x+t} to be the number of nominees, starting in year x surviving in period $x + t$, it follows that: $d_{x+t} = \ell_{x+t} - \ell_{x+t+1}$ and that the probability of death in any given half year j is (d_{x+j} / ℓ_x).

Halley's Fair Value Solution

- Halley used the Breslau data to provide empirical estimates for the probabilities used in the expectation.
- Halley observed the calculated solutions involved "a not ordinary number of Arithmetical operations"

$$E[A_n] = \frac{1}{\ell_x}\sum_{n=1}^{w-x-1} A_n \, d_{x+n} = \frac{1}{\ell_x}\sum_{n=1}^{w-x-1} d_{x+n} \sum_{t=1}^{n} \frac{1}{(1+r)^t}$$

$$= \frac{1}{\ell_x}\sum_{t=1}^{n} \sum_{n=1}^{w-x-1} d_{x+n} \frac{1}{(1+r)^t} = \frac{1}{\ell_x}\sum_{n=1}^{w-x-1} \ell_{x+n} \frac{1}{(1+r)^n}$$

Jump to first page

De Moivre's Approximations

- Abraham de Moivre (1667-1754), an expatriate Frenchman transplanted to London following the Repeal of the Edict of Nantes
- De Moivre laid the theoretical foundation for Richard Price, James Dodson and others to develop the actuarially sound principles required to implement modern life insurance
- Together with Laplace, one of two giants of probability theory in the 18th century

Abraham de Moivre (1667-1754)

Jump to first page

Solving the Valuation Formula

- De Moivre solved the valuation problem for both arithmetic (uniform) and geometrically declining death rates.
- Solving for the uniform case:

$$E[A_n] = \frac{n-1}{n}\frac{1}{1+r} + \frac{n-2}{n}\frac{1}{(1+r)^2} + \ldots + \frac{n-(n-1)}{n}\frac{1}{(1+r)^{n-1}} + \frac{n-n}{n}\frac{1}{(1+r)^n}$$

$$= \sum_{t=1}^{n}\frac{1}{(1+r)^t}\left(1-\frac{t}{n}\right) = \sum_{t=1}^{n}\frac{1}{(1+r)^t} - \sum_{t=1}^{n}\frac{t}{n\,(1+r)^t}$$

Jump to first page

The Final Steps in the Solution

- After some further manipulation, de Moivre is able to find the solution:

$$E[A_n] = [A_n + \frac{1+r}{n}\frac{dA_n}{d(1+r)}]$$

$$= (\frac{1}{r} - \frac{1}{r(1+r)^n}) + \frac{1+r}{n}[\frac{n}{r(1+r)^{n+1}} + \frac{1}{r^2(1+r)^n} - \frac{1}{r^2}]$$

$$= \frac{1}{r} - \frac{1+r}{n}[\frac{1}{r}\{\frac{1}{r} - \frac{1}{r(1+r)^n}\}] = \frac{1}{r}\{1 - \frac{1+r}{n}A_n\}$$

Jump to first page

Solving Bernoulli's Problem

- There are three values to consider: *D*, the expected duration of life; $E[A_n]$ the fair value of a life annuity; and A_d the value of a term annuity with term to maturity equal to *D.*

$$D = \sum_{t=1}^{w-x-1} t \frac{d_{x+t}}{\ell_x} \qquad E[A_n] = \sum_{n=1}^{w-x-1} A_n \frac{d_{x+n}}{\ell_x} \qquad A_d = \sum_{t=1}^{D} \frac{1}{(1+r)^t}$$

Jump to first page

Some Basic Results

- Comparing D with $E[A_n]$ and A_d it is apparent:
  - For $r > 0$, it follows that $D > E[A_n]$ and $D > A_d$, due to the impact of discounting on the terms in $E[A_n]$ and Ad.
  - Even if interest rates are zero and $D = A_d$, $E[A_n]$ and D are still not equal due to the $E[A_n]$ only crediting the cash flow if the end of period is reached. (This problem is avoided by making the simplifying assumption).

Using de Moivre's Approximation

- Though Bernoulli did not find the solution, using de Moivre's approximation, it can be shown that:

$$A_d - E[A_n] = \frac{1}{r(1+r)^D} - [\frac{1+r}{n}.\frac{1}{r}]\{\frac{1}{r} - \frac{1}{r(1+r)^n}\}$$

$$= \frac{1}{r^2}\{\frac{r}{(1+r)^D} - [\frac{1+r}{n}\{1 - \frac{1}{(1+r)^n}\}]\}$$

$$= \frac{1}{r^2}\{\frac{nr\ (1+r)^{n-D} + (1+r) - (1+r)^{n+1}}{n(1+r)^n}\} > 0$$

Jump to first page