# APPENDIX III

**Mathematics for Option Valuation**

This Appendix contains derivations of the partial derivatives stated in Sec. 10.1. In addition, a number of useful results applicable to probability densities are also derived. These results, which are provided here for ease of reference, are available in a variety other sources, e.g., Cox and Rubinstein (1985, Chp.5), Stoll and Whalley (1993, Chp.11).

**Results for Probability Distributions and Densities**

**1.** Chapter 9 makes use of a result concerning the standard deviation of a sum of standard normal random variables. More precisely, the discrete random walk was given the form:

$$X(t+1) = X(t) + Z(t) \quad \text{where } X(0) = 0, \text{ and } t \,\varepsilon\, \{1,2,3.....\}$$

where $Z(1), Z(2), Z(3)$.... form a stochastic process of ***independent*** random variables with the standard normal probability distribution: $Z(t) \sim N[0,1]$. This requires the $Z(t)$ to be iid, identically, independent distributed random variables. Over any time interval 0 to $T$, the variance of $\Delta X(t)$ can be evaluated by determining the variance:

$$var\{ \sum_{t=1}^{T} Z(t) \} = E[ \sum_{t=1}^{T} Z(t)]^2 = \sum_{t=1}^{T} \sigma_Z^{\,2} = T\, \sigma_Z^{\,2} = T$$

Hence, when $\Delta t = T$, the $Z$ is $N[0,T]$. Now, consider what happens when the time interval $\Delta t$ shrinks. Because $Z(t)$ is $N[0,\Delta t]$ over any arbitrary time interval, the random walk now has the form:

$$X(t+\Delta t) = X(t) + Z(t) \sqrt{\Delta t}$$

Which is the result given in Chapter 9.

**2.** In the derivation of the Black-Scholes put option formula the following result is needed:

$$N[d] = 1 - N[-d] \qquad \Rightarrow \qquad N[-d] = 1 - N[d]$$

$$N[d] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-z^2/2}\, dz = \frac{1}{\sqrt{2\pi}} \int_{-d}^{\infty} e^{-z^2/2}\, dz \qquad (symmetry \;\; of \;\; N)$$

$$= \frac{1}{\sqrt{2\pi}} \{ \int_{-\infty}^{\infty} e^{-z^2/2}\, dz - \int_{-\infty}^{-d} e^{-z^2/2}\, dz \} = 1 - N[-d]$$

**3.** Another important property is the chain rule for partial differentiation of the cumulative distribution:

$$\frac{\partial N[d]}{\partial x} = \frac{\partial N[d]}{\partial d} \frac{\partial d}{\partial x} = N'[d] \frac{\partial d}{\partial x} \qquad (for \quad d = d[x])$$

where $N'[d]$ is the standard normal probability density evaluated at d (see n.1 of Chapter 10).

**4.a)** Application of the chain rule to the cumulative normal distribution function leads to results involving the normal probability density function evaluated at $d_2$ and $d_1$. Simplification of the derivatives requires the relationship between $N'[d_1]$ and $N'[d_2]$ to be identified. Observing that $d_2 = d_1 - \sigma\sqrt{t^*}$ produces:

$$d_2^2 = \{d_1 - \sigma \sqrt{t^*}\}^2 = d_1^2 - 2\, d_1\, \sigma \sqrt{t^*} + \sigma^2\, t^*$$

$$= d_1^2 - 2\{\ln\{S/X\} + (r + .5\sigma^2)\, t^*\} + \sigma^2\, t^* = d_1^2 - 2 \ln\{S/(X\, e^{-rt^*})\}$$

Direct substitution into the formula for the normal probability density function now produces:

$$N'[d_2] = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{d_2^2}{2}} = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{d_1^2}{2} + \ln\{S\,/\,\{Xe^{-rt^*}\}\}} = \frac{1}{2\pi}\, e^{-\frac{d_1^2}{2}}\, e^{\ln\{S\,/\,\{Xe^{-rt^*}\}}$$

$$= N'[d_1]\, \frac{S}{Xe^{-rt^*}} \qquad \Rightarrow \qquad N'[d_1] = N'[d_2]\, \frac{Xe^{-rt^*}}{S}$$

**4.b)** When the commodity involved has a carry return, as in the case of currency options, it is necessary to adjust $N'[d_1]$ to be:

$$N'[d_1] = N'[d_2]\, \frac{X\, e^{-rt^*}}{S\, e^{-r_f\, t^*}}$$

where $r_f$ is the return on the riskless foreign security.

**5.** The derivations of the Greeks for the Bachelier options require the following result for differentiating a normal probability density:

$$\frac{\partial n[g]}{\partial x} = \frac{\partial}{\partial x}\, \frac{1}{\sqrt{2\pi}}\, e^{-\frac{g^2}{2}} = n[g]\, \frac{\partial\{-g^2/2\}}{\partial g}\, \frac{\partial g}{\partial x} = n[g]\, (-g)\, \frac{\partial g}{\partial x}$$

**Greeks for Black-Scholes Call Options**

**Delta:**

$$\frac{\partial C}{\partial S} = N[d_1] + S\, \frac{\partial N[d_1]}{\partial S} - Xe^{-rt^*}\, \frac{\partial N[d_2]}{\partial S} = N[d_1] + SN'[d_1]\frac{\partial d_1}{\partial S} - Xe^{-rt^*} N'[d_2]\, \frac{\partial d_2}{\partial S}$$

$$= N[d_1] + S\, N'[d_1]\, \frac{\partial d_1}{\partial S} - Xe^{-rt^*}\, N'[d_1]\, \frac{S}{Xe^{-rt^*}}\, \frac{\partial d_1}{\partial S} = N[d_1]$$

The last line follows from Property 4 in Section A and observing that: $\partial d_1/\partial S = \partial d_2/\partial S$.

**Gamma:**

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial N[d_1]}{\partial S} = N'[d_1] \frac{\partial d_1}{\partial S} = \frac{1}{S \sigma \sqrt{t^*}} N'[d_1]$$

**Theta:**

$$-\frac{\partial C}{\partial t} = \frac{\partial C}{\partial t^*} = S \frac{\partial N[d_1]}{\partial t^*} + rXe^{-rt^*} N[d_2] - Xe^{-rt^*} \frac{\partial N[d_2]}{\partial t^*}$$

$$= S N'[d_1] \frac{\partial d_1}{\partial t^*} + rXe^{-rt^*} N[d_2] - Xe^{-rt^*} N'[d_2] \frac{\partial d_2}{\partial t^*}$$

$$= S N'[d_1] \frac{\partial d_1}{\partial t^*} - Xe^{-rt^*} N'[d_1] \frac{S}{Xe^{-rt^*}} \{\frac{\partial d_1}{\partial t^*} - \sigma \frac{\partial \sqrt{t^*}}{\partial t^*}\} + rXe^{-rt^*} N[d_2]$$

$$= \frac{S\sigma}{2\sqrt{t^*}} N'[d_1] + rXe^{-rt^*} N[d_2]$$

**Vega:**

$$\frac{\partial C}{\partial \sigma} = S \frac{\partial N[d_1]}{\partial \sigma} - Xe^{-rt^*} \frac{\partial N[d_2]}{\partial \sigma} = S N'[d_1] \frac{\partial d_1}{\partial \sigma} - Xe^{-rt^*} N'[d_1] \frac{S}{Xe^{-rt^*}} \frac{\partial d_2}{\partial \sigma}$$

$$= S N'[d_1] \{\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\} = S N'[d_1] \sqrt{t^*}$$

**Greeks for the Bachelier Call Option**

For the Bachelier call option for a dividend paying security given in Chapter 8, the delta, theta and gamma are:

$$\Delta_{BC} = \frac{\partial C}{\partial S} = e^{-\delta t^*} N[h] + (Se^{-\delta t^*} - Xe^{-rt^*}) \frac{\partial N}{\partial S} + V \frac{\partial n}{\partial S} = e^{-\delta t^*} N[h]$$

$$\Gamma_{BC} = \frac{\partial^2 C}{\partial S^2} = e^{-\delta t^*} \frac{\partial N}{\partial S} = e^{-\delta t^*} n[h] \frac{\partial h}{\partial S} = \frac{e^{-2\delta t^*}}{V} n[h]$$

$$-\theta_{BC} = \frac{\partial C}{\partial t^*} = (-\delta Se^{-\delta t^*} + rXe^{-rt^*}) N[h] + (Se^{-\delta t^*} - Xe^{-rt^*}) \frac{\partial N}{\partial t^*} + \frac{\partial V}{\partial t^*} n[h] + V \frac{\partial n}{\partial t^*}$$

$$= -\frac{\partial C}{\partial t} = (-\delta Se^{-\delta t^*} + rXe^{-rt^*}) N[h] + \frac{\partial V}{\partial t^*} n[h]$$

**Risk Neutral Valuation of Black-Scholes**

This part of Appendix III provides a description of the risk-neutral valuation approach together with applications to the valuation of both the Black-Scholes and Bachelier options. The solution to the risk-neutral Black-Scholes valuation problem for geometric Brownian motion is available in several sources, e.g., Stoll and Whalley (1995). The general risk neutral valuation problem starts by considering ***the valuation problem*** for a European call option on a non-dividend paying stock:

$$C_t = e^{-rt^*} E\{\max\ [0,\ S_T - X]\}$$

$$= e^{-rt^*}\{\ E[S_T - X \mid S_T \geq X] + E[0 \mid S_T < X]\}$$

$$= e^{-rt^*}\{E[S_T - X \mid S_T \geq X]\}$$

$$= e^{-rt^*}\ \{E[S_T \mid S_T \geq X] - X\ Prob[S_T \geq X]\}$$

where $t^* = T - t$, $E[\cdot]$ is the time t expectation taken with respect to the risk neutral density $Prob[\cdot]$ and $r$ is the riskless interest rate.

In words, the valuation problem for a European call indicates that the value of the call option depends only on the paths where $S_T \geq X$. The density associated with $S_T < X$, which includes the potentially negative price paths, does not directly enter the valuation. While there is some difference in the shape of the upper portions of the normal and log-normal distributions, this difference provides a basis for contrasting the performance of the Bachelier and Black-Scholes options. Given that neither assumption generally provides a particularly close fit to observed price distributions, it does not follow that log-normality will necessarily provide better pricing accuracy than normality. This issue must be addressed empirically.

Evaluation of the valuation problem requires the integration of variables which follow the standard normal distribution. Two different specifications are needed to do this, one for the arithmetic Brownian motion and the other for geometric Brownian motion. For the case of arithmetic Brownian motion, over the time interval starting at t and ending at $T$, with $t^* = T - t$:

$$S_T = S_t + \alpha t^* + \sigma \sqrt{t^*}\ Z \qquad or \qquad Z = \frac{S_T - S_t - \alpha t^*}{\sigma \sqrt{t^*}}$$

where $Z$ is $N(0,1)$. The standard normal specification associated with geometric Brownian motion is stated in Appendix III.2, Sec. A.6.

Evaluating the valuation problem for the arithmetic Brownian motion case gives:

$$X\ Prob[S_T \geq X] = X\ N[\frac{S_t + \alpha t^* - X}{\sigma\sqrt{t^*}}]$$

$$E[S_T | S_T \geq X] = \int_{(X - S_t - \alpha t^*)/\sigma\sqrt{t^*}}^{+\infty} (S_t + \alpha t^* + \sigma\sqrt{t^*}\ Z)\ n[Z]\ dZ$$

$$= (S_t + \alpha t^*) \int_{-\infty}^{(S_t + \alpha t^* - X)/\sigma\sqrt{t^*}} n[Z]\ dZ + \sigma\sqrt{t^*} \int_{(X - S_t - \alpha t^*)/\sigma\sqrt{t^*}}^{+\infty} Z\ n[Z]\ dZ$$

$$= (S_t + \alpha t^*)\ N[\frac{S_t + \alpha t^* - X}{\sigma\sqrt{t^*}}] + \sigma\sqrt{t^*}\ n[\frac{S_t + \alpha t^* - X}{\sigma\sqrt{t^*}}]$$

Substituting these results into the valuation problem and observing the definition of g gives the Wilcox option result stated in Proposition 9.1. The absence-of-arbitrage consistent solution requires the derivatives of the fundamental PDE for the Bachelier call option formula stated in Proposition 9.1 to be evaluated.

The risk neutral solution for the geometric Brownian motion proceeds much as with the arithmetic Brownian solution:

$$X\ Prob[S_T \geq X] = X\ N[\frac{\ln\left[\frac{S_t}{X}\right] + \mu\ t^*}{\sigma\sqrt{t^*}}]$$

$$E[S_T | S_T \geq X] = \int_{(\ln[X/S_t] - \mu t^*)/\sigma\sqrt{t^*}}^{+\infty} (S_t\ e^{\mu\ t^* + \sigma\sqrt{t^*}\ Z})\ n[Z]\ dZ$$

$$= \left[S_t\ e^{\mu\ t^* + \frac{\sigma^2\ t^*}{2}}\right] \int_{-\infty}^{\frac{(\ln[S_t/X] + \mu t^*}{\sigma\sqrt{t^*}} + \sigma\sqrt{t^*}} n[Z]\ dZ$$

$$= (S_t\ e^{\mu t^* + \frac{\sigma^2\ t^*}{2}})\ N[\frac{\ln[S_t/X] + \mu t^*}{\sigma\sqrt{t^*}} + \sigma\sqrt{t^*}]$$

From the discussion in Appendix III.2, Sec. A.6, the ***risk neutral*** valuation approach now permits the result that the riskfree interest rate ($r$) can be used for continuously compounded rate of return on the stock ($\alpha$). This produces $r = \alpha = \mu + \sigma^2/2$ to be used to make a substitution for $\mu\ (= r - \sigma^2/2)$. Collecting all these results into the valuation problem gives:

$$C(t) = e^{-rt^*}\{E[S(T)\ |\ S(T) \geq X] - X\ Prob[S_T \geq X]\} = e^{-rt^*}\{S(t)\ e^{rt^*}\ N[d_1] - X\ N[d_2]\}$$

Taking the discounting operator *exp{-rt*}* through gives the Black-Scholes formula.