4. Life Annuity Valuation: From de Witt and Halley to de Moivre and Simpson

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The roots of financial economics stretch back to antiquity. The valuation of financial transactions, such as determining payment on a loan or distributing profits from a partnership, is so ancient that any search for pioneers would be fruitless. A less ambitious beginning is required. A common thread connecting various elements of modern financial economics, such as asset pricing theory, financial risk management and contingent claims valuation, is the application of notions from probability theory to the valuation and management of financial securities and to the analysis of financial market equilibrium, for example, Rubinstein (2005). As discussed by Sylla (2005), the intellectual revolution associated with the Enlightenment marks the emergence of the modern frequentist interpretation of probability. Some of the earliest practical applications of this probability theory were concerned with a problem in financial economics: the valuation of life annuities. Not only were these securities important in state and municipal finance in France, England and Holland, in an era predating actuarially sound pension plans and life insurance the life annuity performed an essential social function. As such, those that first applied the frequentist concept of probability to the valuation of life annuities deserve special recognition as 'pioneers of financial economics'. Within this group of pioneers that includes Jan de Witt and Nicholas Bernoulli, the contributions of Edmond Halley (1656-1742) and Abraham de Moivre (1667-1754) are outstanding.

While detailed modern summaries of the early contributions to life annuity valuation are available in Pearson (1978), Alter and Riley (1986), Hald (1990) and Poitras (2000), there is little direct examination of the primary sources. The essence of the solutions in the contributions are converted to the modern context leaving in the background the original methods and arguments used to determine the solutions. As a consequence, Halley's novel geometric analysis of the joint life annuity for three lives or de Moivre's application of 'fluxions' to determine a life annuity valuation formula go unrecognized. With this in mind, the primary objective of this paper is to demonstrate how this work on life annuity valuation contributed to 'the genesis of ideas and the evolution of methods' (de Roover 1969, p.358) within financial economics. In the following, section 1 provides an historical overview of the development of life contingent contracts. Section 2 examines the contributions to life annuity valuation in the 17^{th} and early 18^{th} century, focusing on the insights of Jan de Witt (1625-1672) and Nicholas Bernoulli (1695-1726). Section 3 details the analysis of the single life annuity contained in Halley (1693). Section 4 discusses the de Moivre solution procedure for the single life annuity valuation formula. Section 5 gives specific consideration to the various solutions proposed to valuing the joint life annuity. In addition to Halley's geometric approach used to assess the joint life annuity, the evolution of de Moivre's solution to the valuation of joint life annuities is discussed, together with advances to the available solutions made by Thomas Simpson (1710-1761).

1. Development of Life Contingent Contracts[1]

An aleatory contract has a payoff that depends on a random outcome, for example, Daston (1988). Examples of such contracts arise in the early history of insurance where, say, a contract would be made to protect against the loss of a cargo at sea. In a sense, any security that is not riskless is aleatory but this stretches the notion in a direction that is not too helpful. In the early history of financial economics, it is aleatory contracts with outcomes dependent on life contingencies which are, by far, the most significant. Not only were such contracts socially important prior to the development of modern pension plans and life insurance schemes, life contingent contracts also provide the first instance of significant analytical solutions to aleatory security pricing problems. In providing contingent claims pricing formulas for the range of life annuity contracts that were traded in the late

17th and early 18th centuries, intellectual giants of that era, such as Edmond Halley and Abraham de Moivre, laid the foundations for modern financial economics.

The origins of life annuities can be traced to ancient times. Socially determined rules of inheritance usually meant a sizable portion of the family estate would be left to a predetermined individual, often the first born son. Bequests such as usufructs, maintenances and life incomes were common methods of making provisions for family members and others not directly entitled to inheritances.[2] One element of the Falcidian law (*Lex Falcidia)* of ancient Rome – law 68 of the Justinian Code effective from 40 BC – was that the rightful heir(s) to an estate was entitled to not less than one quarter of the property left by a testator, the so-called 'Falcidian fourth' (Bernoulli 1709, ch.5). This created a judicial quandary requiring any other legacies to be valued and, if the total value of those other legacies exceeded three quarters of the value of the total estate, these bequests had to be reduced proportionately. The Falcidian fourth created a legitimate valuation problem for jurists because many types of bequests did not have observable market values. Because there was not a developed market for life annuities, this was the case for bequests of life incomes. Some method was required to convert bequests of life incomes to a form that could be valued for legal purposes.

In Roman law, a legal solution for valuing bequests with a life contingent component was introduced by the jurist Ulpian (Domitianus Ulpianus, ?-229) who devised a table for the conversion of life annuities to annuities certain, a security for which there was a known method of valuation. Ulpian's Conversion Table is given by Hald (1990, p.117):

Age of annuitant in years

0-19	20-24	25-29	30-34	35-39	40 ... 49	50-54	55-59	60-
30	28	25	22	20	19 ... 10	9	7	5

Comparable Term to maturity of an annuity certain in years

While the stated connection between age and the pricing of life annuities is a fundamental insight of Ulpian's table, this interpretation of the Table may be too generous. For example, Greenwood (1940, p.246) and Hendricks(1853, p.224) both maintain that Ulpian took no account of interest in the calculations, so the reference to 'annuities certain' is misleading. Nicholas Bernoulli (1709, p.27) observes that, in practice, the use of Ulpian's table produced valuations that were often determined by taking the annual value of the legacy, and multiplying this value by the term to maturity of the annuity certain to get the associated legacy value. For example, if the individual was 24 years old and was receiving a life income of £10 per year, then the legacy value according to Ulpian's Table would be £280. Bernoulli correctly identifies the method of multiplying the table value by the size of the payment as faulty due to the omission of the value of interest. Bernoulli (1709, p.41) observes that at the conventional 5% compound interest the value of the legacy would be only £148.93.

While it is tempting to attribute a valuation aspect to Ulpian's table, there appears to be little more in the table than an early attempt to estimate life expectancy. Even this interpretation is generous. For example, the use of a thirty year life expectancy for the 0-19 group ignores the significantly higher mortality in the early years of life in ancient Roman. In addition, as Bernoulli (1709, p.40) discusses, various authors have observed the obvious inconsistency in the Table associated with a person of, say, 24 being given a longer life expectancy (24 + 28) than a person of, say, 26 (26 + 25), despite the 26 year old having already lived longer. Similarly, the possibility of individuals living beyond 65 is not admitted, though it is known that there were septuagenarians and octogenarians in Roman times.

Ultimately, the table was useful for the purpose for which it was designed: to ensure that the legal inheritance rights specified in the Falcidian law were protected. The biases associated with practical application of the table typically result in an overvaluation of the life contingent bequests, thereby ensuring that at least one-fourth of the estate passes to the rightful heir.

Ulpian's table was concerned with life incomes, maintenances and the like, which are not quite the same as life annuities which are traded securities with defined cash flow patterns dependent on life contingencies. The life annuity evolved from the *census* which was a form of investment dating at least to feudal times. The English word annuity is an approximate translation, but annuity does not make the appropriate connection to the source of the *census* return being derived from a "fruitful good" (Noonan 1957, p.155). In Roman law, the *census* was not used, though different types of annuities were available. The various forms of *census* contracts formed a basis for the emergence of government debt issues. The types of securities employed were annuities certain (fixed term annuities), perpetual annuities and life annuities to fund, first, municipal and, eventually, national borrowing.

Census contracts were initially designed, in feudal times, as a type of barter arrangement, present goods for future goods. The contract appears to have originated in Continental Europe, eventually facilitating the evolution of a market for long-term debt (for example, Tracy 1985, pp.7-8). The conventional *census* contract gradually took the form of a modern annuity where cash was received by the seller of the annuity in exchange for an agreement to make a stream of annual payments over time. By the end of the 15th century, the nobility, the church, the state and the landed gentry were all involved as sellers of *census*. Many different variations of *census* were offered: a life *census* in which payments were made over the life of a buyer, or their designee; a perpetual *census*, that had no fixed maturity date; and, a temporary or term *census* that ran for a fixed number of years, similar to a mortgage. A *census* could have conditions that permitted it to be redeemable at the option of either the buyer or seller. Noonan (1957) estimates that credit raised using *census* arrangements may have exceeded that raised through *societas* (business partnerships).

A major impetus to the development of securities markets was provided by the various 'financial revolutions' in government finance. These revolutions started at different times, in different countries, beginning with the Italian city states and, somewhat later, extending to the cities in northern France and Flanders. The key feature of these revolutions was the transition of government debt from the status of a short-term loan to an individual, debt as an obligation of the sovereign, to a long-term loan to a political entity independent of the ruler. The revolutions in government finance transformed government debt operations from the realm of individual borrowing, which was typically short-term and secured by assets, to long-term borrowings which were secured by specific funding sources and were, to varying degrees, independent of the creditworthiness of the monarch.

The earliest forms of 'public debt', issued by Italian city states and the northern European cities and municipalities, were either forced loans on wealthy citizens, for example, the *prestiti* in Venice, or were *rentes* backed by specific revenue sources of the sovereign or town government. Northern European towns favoured annuities or *rentes* secured by urban taxes. As early as 1260, such early issues of *rentes heritables* and *rentes vagieres* (life annuities) appeared in the French cities of Calais and Douai, spreading to the Low Countries and German towns such as Cologne (Tracy 1985, p.13). Between 1275 and 1290 the city of Ghent in Flanders issued *lijfrenten* or life annuities followed by issues of *erfrenten* or redeemable *rentes*. There was a form of guarantee by the sovereign associated with some of these municipal issues, for example, the Court of Flanders 'undertook to see that the city lived up to its promises'. Municipalities, particularly in Holland, Flanders and Brabant, continued to issue life and redeemable annuities leading to increasingly larger stocks of public debt and, ultimately, to repayment difficulties for some towns by the 16th century.

The transition from municipal to national public debt issues was gradual. Though claims could be made for certain German territories, Dutch provinces or the Spanish monarchies, Hamilton (1947) traces the beginnings of national public debt to 16th century France. For some centuries before, the French had a tradition of the monarch selling long-term *rentes*, supported by the income from royal properties. These sales were often sold at deep discounts to royal officials and could not be considered 'public debt' but, rather, were obligations of the sovereign. Hamilton (1947, p.119) marks 1522 as a turning point. Though sovereigns had recognized the importance of the debt market in financing state military ventures for some time, the emergence of the public debt gave the debt market a new status as an instrument of state power.

Despite this claim to first-mover status by the French, there were numerous difficulties with the administration of the French public debt. Large increases in outstanding principal to sustain various military adventures led to periods of suspended interest payments and forced reduction of a portion of the outstanding principal. By the beginning of the 18th century, French national finances were in a sorry condition. The Dutch were decidedly more successful in developing their public debt. The Dutch provincial governments pioneered various innovations in public debt issues during the 16th century, including the development of a 'free market' for provincial *renten* issued in Holland (Tracy 1985, ch.IV). The English were relative late comers in developing public debt issues, with the beginnings of English public debt starting only with the reign of William and Mary in 1688. However, by the mid-18th century the English had assumed front-runner status and the system of English public debt had become a model for European governments.

In the late 17th and early 18th centuries, actuarially sound analytical solutions were proposed to the problem of valuing life annuities. Arguably, these analytical solutions represent the most important theoretical contributions to the early history of financial economics. The intellectual preliminaries required to sustain these contributions start around the latter part of the 16th century in Holland where important university mathematicians, such as Simon Stevin, were drawn to solving practical fixed income valuation problems, complementing the work of the commercial algorists, for example, Poitras (2000, ch.4); Hawawini and Vora (1980). Even though the development of discounting and compounding techniques were important for determining the return from partnerships and valuing commonly traded term annuities such as mortgages and lease-purchase transactions, these techniques were not sufficient to value life annuities and other types of securities involving life contingent claims. Such problems were important because, in the absence of pension funds and life insurance, life annuities performed an essential social function.

The life annuity usually was a contract between three parties, the subscriber who provided the initial capital, the shareholder who was entitled to receive the annuity payments and the nominee on whose life the payout was contingent, for example, Weir (1989). Different variations were possible. For example, one person could be subscriber, shareholder and nominee; a parent could be a subscriber and designate a child as the nominee with the shareholder status passing from parent to child as an inheritance; or, joint life annuities could be specified where more than one nominee was designated and payments continued until both nominees died. Similarly, payments on a reversionary annuity (reversion) would be paid to nominee x after the death of nominee y and would continue until the death of x. If x died before y then no payments would be made. Reversions could also have joint features. For example, *x* could be both parents and *y* could be their children.[3] The life annuity was further complicated by the need to establish proof of survival for the nominee prior to each annuity payment date. While it was technically possible to resell most life annuity contracts to third parties, the difficulties associated with verifying the survival and probability of survival for the nominee made resale difficult. Oddly enough, until the 19th century, market practice usually involved governments selling life annuities without taking accurate consideration of the age of the nominee.

Though there were larger and less frequent issues of life annuities by the emerging nation states starting in the 17th century, the typical government issuers of life annuities were municipalities, with prices varying widely from town to town depending on prevailing local interest rates and pricing conventions. Annuity prices were quoted in '***years' purchase***', which is the price of the annuity divided by the annual annuity payment. For a perpetual annuity, years purchase is the inverse of the annual yield to maturity. Nicholas Bernoulli (1709, p.27-8) provides numerous historical examples of single life annuities selling for 6 to 12 years' purchase, without allowance being made for the age of nominee. Reference is also made to the valuation of dowries at ten years purchase. De Witt quotes a 1671 price for a single life annuity in Amsterdam of 14 years' purchase with a 4% interest rate and no allowance for age of nominee; this is compared with a price of 25 years' purchase for a redeemable annuity, effectively a perpetual annuity with an embedded option for the borrower to redeem at the purchase price. Houtzager (1950) quotes a 16th and early 17th century Dutch pricing convention of 1.5 to 2 times the years' purchase for a redeemable annuity to determine the price for single life annuities, i.e., the years' purchase of a life annuity equaled the years' purchase for a redeemable annuity divided by 1.5 to 2. The inefficiency of the practice of selling life annuities without reference to the age of the nominee did not escape the notice of those responsible for government finance. However, solving the problem of determining a correct price was not easy. The first sound solution to this difficult analytical problem was proposed by Jan de Witt.[4]

2. Valuation of Life Annuities by De Witt and Nicholas Bernoulli

Jan de Witt was not a professional mathematician. He was born into a burgher-regent family and attended university at Leiden as a student of jurisprudence, where de Witt lived in the house of Frans van Schooten. While he was formally a professor of jurisprudence, van Schooten was also deeply involved in mathematical studies. Van Schooten encouraged Christian Huygens, Jan Hudde and de Witt in their mathematical studies and published their efforts as appendices to two of his mathematical books. De Witt's contribution on the dynamics of conic sections was written around 1650 and published as an appendix to van Schooten's 1659 exposition of Cartesian mathematics, *Geometria a Renato Des Cartes*. From the perspective of the history of mathematics, de Witt's contribution is an interesting and insightful exposition on the subject but “marks no great advance” (Coolidge 1990, p.127).

Around 1650, de Witt began his career in Dutch politics as the pensionary of Dordrecht. In 1653, at the age of 28, de Witt became the grand pensionary or prime minister of Holland. During his term as grand pensionary, de Witt was confronted with the need to raise funds to support Dutch military activities, first in the Anglo-Dutch war of 1665-67 and later in anticipation of an invasion by France which, ultimately, came in 1672. Life annuities had for many years been a common method of municipal and state finance in Holland and de Witt also proposed that life annuity financing be used to support the war effort. However, de Witt was not satisfied that the convention of selling of life annuities at a fixed price, without reference to the age of the annuitant, was a sound practice. Instead de Witt proposed a method of calculating the price of life annuities which would vary with age.

Poitras (2000) considers this remarkable contribution to be the start of modern contingent claims analysis.

More precisely, aided by contributions from Huygens in probability and Hudde in mortality statistics, in Value of Life Annuities in Proportion to Redeemable Annuities (1671, in Dutch) de Witt provided the first substantive analytical solution to the difficult problem of valuing a life annuity.[5] Unlike the numerous variations of fixed term annuity problems which had been solved in various commercial arithmetics, the life annuity valuation required the weighting of the relevant future cash flows by the probability of survival for the designated nominee. De Witt's approach, which is somewhat computationally cumbersome but analytically insightful, was to compute the value of a life annuity by applying the concept of mathematical expectation advanced by Huygens in 1657.

De Witt's approach involved making theoretical assumptions about the distribution of the number of deaths. To provide empirical support for his calculations, he gave supplementary empirical evidence derived from the register at The Hague for life annuitants of Holland and West Friesland for which he calculated the average present values of life annuities for different age classes, based on the actual payments made on the annuities. This crude empirical analysis was buttressed by the considerably more detailed empirical work of Hudde on the mortality statistics of life annuitants from the Amsterdam register for 1586-90. For the next century, the development of pricing formula for life annuities is intimately related to progress in the study of life contingency tables, a subject which is central to the development of modern demographics and actuarial science.

The solution to the problem of pricing life annuities given by de Witt uses an age interval between 3 and 80. Hence, de Witt is considering the value of a life annuity written on the life of a three year old nominee. Adjusting the solution to price life annuities for older nominees at different ages is straightforward, involving a reduction in the number of terms being summed and an adjustment of the divisor. As the practice up to his time was to sell life annuities at the same price, regardless of the age of the nominee, it was conventional to select younger nominees from healthy families. Based on Hudde's data for 1586-90 Amsterdam life annuity nominees, approximately 50% were under 10 years of age, and 80% under 20 (Alter and Riley, p.33). ***Throughout the following annuities will be assumed to make a payment of 1 unit of currency (florin, dollars, etc.) each period.*** Instead of assuming a uniform distribution where death at each age would be equally likely, De Witt divided the interval between 3 and 80 into four subperiods: (3,53), (53,63), (63,73) and (73,80). Within each subperiod, an equal chance of mortality is assigned. The number of chances assigned to each subgroup is 1, 2/3, 1/2, 1/3. The chance of living beyond 80 is assumed to be zero. While de Witt corresponded with Hudde about mortality data he was collecting and tabulating for the 1586-9 Amsterdam annuitants, these probabilities were assumed and not directly derived from a life table.

From these assumptions, de Witt constructs a distribution for the number of deaths and calculates the life annuity price as the expectation of the relevant annuity present values. In doing this, de Witt explicitly recognizes that life annuities were paid in semi-annual instalments, requiring time to be measured in half years and for survivors to be living at the end of the half year in order to receive the payment. The 77 year period translates into 154 half years. Using a discount rate of 4% per annum, De Witt uses his assumed chances of mortality in any half year to calculate a weighted average of the present values for the certain annuities associated with each half year. The resulting value is the expected present value of the life annuity which is the recommended price at which the annuity should be sold.

Algebraically, de Witt's technique can be illustrated by defining A_n to be the present value of an annuity certain with a 4% annual rate to be paid at the end of the half year n:

$$A_n = \sum_{t=1}^{n} \frac{1}{(1+r)^t} = \frac{1}{r} - \frac{1}{r(1+r)^n} \qquad where \quad (1+r) = \sqrt{1.04}$$

To evaluate the expected present value of the life annuity, de Witt performs the calculation:

$$E[A_n] = \frac{\sum_{n=1}^{99} A_n + \frac{2}{3}\sum_{n=100}^{119} A_n + \frac{1}{2}\sum_{n=120}^{139} A_n + \frac{1}{3}\sum_{n=140}^{153} A_n}{128}$$

Interpretation of the sums is aided by observing that individuals must be alive at the end of the half year to qualify for annuity payments. For example, dying in the first half year means that no payments will be received. The divisor of 128 is calculated by determining the total number of chances as:

$$(100)1 + (20)\frac{2}{3} + (20)\frac{1}{2} + (14)\frac{1}{3} = 128$$

where the number in brackets is the number of half years in each subgroup. De Witt's solution can be compared to the less realistic case where the distribution of deaths is assumed to be uniform:

$$E[A_n] = \frac{\sum_{n=1}^{153} A_n}{154} = \frac{1}{154}(0) + \frac{1}{154}\frac{1}{1+r} + \frac{1}{154}\sum_{t=1}^{2}\frac{1}{(1+r)^t}$$

$$+ \ldots\ldots + \frac{1}{154}\sum_{t=1}^{153}\frac{1}{(1+r)^n} + 0$$

By assigning less weight to the largest cash flows, de Witt's calculated expected value of 16.0016 florins for ***annual*** payments of 1 florin differs from the expected value of 17.22 florins calculated using a uniform distribution.[6]

Though there is evidence that de Witt's solution to the life annuity valuation problem was known by the intellectual community outside Holland, it does not appear that the results were widely disseminated. For example, Sylla (2005) observes that James (Jacob) Bernoulli 'for several years previous to his death, Bernoulli had been writing to Gottfried Wilhelm Leibniz asking him to supply a copy of Jan De Witt's pamphlet in Dutch on life and term annuities, but Leibniz had mislaid the copy previously in his possession.' Though Nicholas Bernoulli (1709, p.34) does not specifically identify de Witt's contribution, an explicit reference is made to the issue of life annuities that de Witt (1671) was involved in pricing. In addition, the solution to the life annuity valuation given by Bernoulli (1709, p.33) uses a similar approach to that of de Witt, albeit with a different life table and an easier solution methodology.

Before stating the solution to the life annuity valuation, Bernoulli (1709) first explores the possibility of solving the value of a life annuity by using an annuity certain with a term to maturity equal to the expected duration of life from a given starting age x. The difference of such a solution from an $E[A_n]$ valuation was recognized by de Witt, but the point was still of enough interest that Bernoulli (1709) calculated the relevant values for individuals with ages from 6 to 76 (by decades). At 5% compound interest, the values given by Bernoulli are:

Age of annuitant in years

Newborn	6	16	26	36	46	56	66	76
18.2/11.8	20.8/12.7	20.3/12.6	19.4/12.2	17.5/11.5	15/10.4	11.7/8.7	8.3/6.3	5/4.3

Life Expectancy in Years / Value of the annuity certain in years purchase

Having done these calculations Bernoulli states:'I notice that the value of (life annuity) incomes is not correctly calculated by supposing that the return will last as many years as someone is supposed probably to live.' This is followed by a derivation of life annuity values that is in the spirit of that given by de Witt. The values determined by Bernoulli are:

Age of annuitant in years

Newborn	6	16	26	36	46	56	66	76
9.420	10.600	10.593	10.576	10.164	9.457	8.148	6.545	4.558

Value of the life annuity

Though the valuation method provided by Bernoulli appears decidedly simpler than that of de Witt, the only substantive differences between the solution methods are the use of: annual instead of semi-annual payments; seven age groups instead of four; and a maximum life span of eighty-six instead of 80.

Bernoulli provides the following description of his solution procedure:

> For the sake of brevity we will demonstrate with a single example for a youth of sixteen years. From Chapter 2 (of *De Usu*) it was established that there are 15 chances that a youth of 16 years will die in the first decade, 9 for the second decade, 6 for the third, 4 for the fourth, 3 for the fifth, 2 for the sixth and 1 for the seventh; if he should die in the first decade, the just premium is 4.558 (which is the arithmetic mean of the first ten numbers in this Table <of the value of annuities certain for each year>) ...; if he should die in the second decade the premium is 10.519 ... therefore by our general rule the value of this return is:

$$= \frac{15*4.558 \ + \ 9*10.519 \ + \ 6*14.179 \ + \ 4*16.427 \ + \ 3*17.806 \ + \ 2*18.653 \ + \ 1*19.173}{40}$$

$$= \frac{423.720}{40} = 10.593$$

Bernoulli reduces the number of direct calculations done by de Witt by averaging the A_n by decade. For example, the value of 4.558 for the first decade is the arithmetic average of A_1 to A_{10}. However, because the life probabilities are the same across the averaging period there is no substantive difference with de Witt's method. Bernoulli's approach does make the bias introduced by this simplification more apparent, as illustrated by a comparison of the values for 66 and 76 year olds for the annuity certain using the duration of life and for the life annuity.

3. *The Contribution of Edmond Halley*

It is difficult to assess the impact of de Witt's contribution to the practice of pricing life annuities.

Based on his recommendation, in 1672 the city of Amsterdam began offering life annuities with prices dependent on the age of the nominee. However, this practice did not become widespread and by 1694, when Edmond Halley (1656-1742) published his influential paper 'An Estimate of the Degrees of Mortality of Mankind, drawn from the curious Tables of the Births and Funerals at the City of Breslaw; with an Attempt to ascertain the Price of Annuities upon Lives', the English government was still selling life annuities at seven years' purchase, independent of age. Halley's paper is remarkable in providing substantive contributions to both demography and to financial economics. The importance of this paper reinforces the intellectual stature of an individual who is recognized in modern times primarily for his contributions to astronomy. Yet, despite this importance, Bernoulli (1709) makes no reference to Halley's solution for a life annuity, reflecting the speed at which research contributions spread in this era.

The Breslau data used in the preparation of Halley's 'Estimate' was much better suited to construction of a life table than the bills of mortality. Thanks to Leibnitz, the data set came to attention of the Royal Society and Halley, the editor of the Society's journal, was selected to analyze the data. From the end of the 16th century, Breslau, a city in Silesia, had maintained a register of births and deaths, classified according to sex and age. For the purposes of constructing a precise life table, only the population size is missing. The paper is primarily concerned with constructing Halley's life table and uses the valuation of life annuities as an illustration of applying the information in the life table. In the process, Halley presents a more general and, arguably, more correct approach than de Witt to the valuation of a life annuity. While this paper was Halley's primary effort in demographics, he did make another contribution to financial economics detailing the use of logarithms in solving present value problems, for example, Poitras (2000, p.155).

General details of Halley's life are available in numerous sources, such as Pearson (1978), Ronan (1978), Poitras (2000, ch.6) Unfortunately, there is so much in the life of Edmond Halley that a conventional historiography quickly becomes overwhelming; the process of sifting out important details becomes unmanageable. For example, Halley had an important relationship with Sir Isaac Newton. Some of the connections between Halley and Newton were immediate, such as Halley being instrumental in getting the *Principia* published: 'There is little doubt that we owe its publication to the good offices of Halley' (Pearson 1978, p.86). This aid came in financial support for publication from both the Royal Society and Halley, as well as 'important editorial aid' (Ronan 1978, p.68) in preparing the manuscript. Newton was a reluctant author, if only because he was not fully satisfied with the results that were being published.

Halley is best known for his work on the periodicity of comet orbits. The naming of Halley's comet was a posthumous recognition for his theoretical and empirical work on a particular bright comet which exhibited a periodicity of seventy-five years. Though Halley's observations were well known to astronomers, 'it was not until the 1682 comet reappeared as predicted in 1758 that the whole intellectual world of western Europe took notice. By then Halley had been dead fifteen years; but his hope that posterity would acknowledge that this return "was first predicted by an Englishman" was not misplaced, and the object was named "Halley's comet"' (Ronan, 1978, p.69). This recognition was a fitting tribute for someone who had contributed to so many fields, from astronomy and mathematics to history and philology.

As was the fashion at the time, Halley's presentation of the life annuity pricing problem was done by presenting mathematical concepts in a verbal format. Halley (1693, p.602-3) provides the following solution for the life annuity valuation:

Use V. On this depend the Valuation of Annuities upon Lives; for it is plane that the Purchaser ought to pay for only such a part of the value of the Annuity, as he has Chances that he is living; and this ought to be computed yearly, and the Sum of those yearly Values being added together, will amount to the value of the Annuity for the Life of the Person proposed. Now the present value of Money payable after a term of years, at any given rate of Interest, either may be had from Tables already computed; or almost as compendiously, by the Table of Logarithms: for the Arithmetical Complement of the Logarithm of Unity and its yearly Interest (that is, of 1,06 for Six per Cent. being 9,974694.) being multiplied by the number of years proposed, gives the present value of One Pound payable after the end of so many years. Then, by the foregoing Proposition, it will be as the number of Persons living after that term of years, to the number dead; so are the Odds that any one Person is Alive or Dead. And by consequence, as the Sum of both or the number of Persons living of the Age first proposed, to the number remaining after so many years, (both given by the Table) so the present value of the yearly Sum payable after the term proposed, to the Sum which ought to be paid for the Chance the person has to enjoy such an Annuity after so many Years. And this being repeated for every year of the person's Life, the Sum of all the present Values of those Chances is the true Value of the Annuity. This will without doubt appear to be a soft laborious Calculation, but it being one of the principal Uses of this Speculation, and having found some Compendia for the Work, I took the pains to compute the following Table, being the short Result of a not ordinary number of Arithmetical Operations; It shews the Value of Annuities for every Fifth Year of Age, to the Seventieth, as follows:

Age	Years' Purchase	Age	Yr's P.	Age	Yrs' P.
1	10	25	12,27	50	9,21
5	13,40	30	11,72	55	8,51
10	13,44	35	11,12	60	7,60
15	13,33	40	10,57	65	6,54
20	12,78	45	9,91	70	5,32

With this in mind, it is possible to reexpress Halley's formula in a more modern mathematical form by observing that the total number of annuities sold on a life starting at year x, ℓ_x, equals the sum of $d_x + d_{x+1} + \ldots\ldots + d_{w-1}$ where d_i is the number of annuities which terminate in period i due to the death of annuitant nominees in that year and that $d_i = 0$ for $x \geq w$. Taking ℓ_{x+t} to be the number of nominees, starting in year x surviving in period $x + t$, it follows that: $d_{x+t} = \ell_{x+t} - \ell_{x+t+1}$ and that the probability of death in any given year j is (d_{x+j}/ℓ_x). Starting with the expected value formulation used by de Witt and Bernoulli, the connection to Halley's pricing formula for a life annuity follows:

$$E[A_n] = \frac{1}{\ell_x}\sum_{m=1}^{w-x-1} A_m\, d_{x+m} = \frac{1}{\ell_x}\sum_{m=1}^{w-x-1} d_{x+m} \sum_{t=1}^{n} \frac{1}{(1+r)^t}$$

$$= \frac{1}{\ell_x}\sum_{t=1}^{n} \sum_{m=1}^{w-x-1} d_{x+m} \frac{1}{(1+r)^t} = \frac{1}{\ell_x} \sum_{m=1}^{w-x-1} \ell_{x+m} \frac{1}{(1+r)^m}$$

The last step in the derivation (which gives Halley's formulation) comes from progressively collecting terms associated with $(1+r)^{-t}$. For example, the $(1+r)^{-1}$ term will appear in each annuity and will, as a result, have coefficients which are the sum of d_{x+1}, d_{x+2}, ... d_{w-1}. Recalling the definition of d in terms of ℓ, this sum returns ℓ_{x+1}. In symbolic form, this is the single life annuity pricing formula presented by Halley (1693).

4. *De Moivre's Approximation for the Single Life Annuity*

In assessing Halley's contribution to the history of financial economics, it is natural to immediately mention Abraham de Moivre (1667-1754), an expatriate Frenchman transplanted to London following the Repeal of the Edict of Nantes. Halley and de Moivre were first acquainted in 1692 and in 1695 de Moivre's first paper contributed to the Royal Society was presented by Halley. Unlike Halley who touched only briefly on the pricing of securities, de Moivre spent much of his productive life studying the practical problem of pricing life annuities. By the time de Moivre undertook his work on life annuities, the basic groundwork had been laid. However, Halley and others recognized that the brute force approach to calculating tables for valuing life annuities would require 'a not ordinary number of Arithmetical operations'. Halley attempted to develop simplifying mathematical procedures, 'to find a Theorem that might be more concise than the Rules there laid down, but in vain.'

In the early history of security analysis de Moivre can be recognized for fundamental contributions involving the application of applied probability theory to the valuation of life annuities. This work laid the theoretical foundation for Richard Price (1723-1791), James Dodson (c.1710-1757) and others to develop the actuarially sound principles required to implement modern life insurance. The immediate incentive for de Moivre was to value the various aleatory contracts which became increasingly popular as the 18th century progressed. Being (together with Laplace) one of two giants of probability theory in the 18th century (Pearson 1978, p.146), de Moivre was singularly well suited to the task of developing the foundations of insurance mathematics. It is one of the quirks of intellectual history that de Moivre's most significant contributions, which lay primarily in the area of probability theory and applied mathematics, contributed little to his personal comfort while his contributions to security analysis and valuation managed to help de Moivre maintain body and soul.

De Moivre began his close friendships with Newton and Halley around the same time in the early 1690s. The timing of the 1693 publication of Halley's ‘Estimate’ and Halley's subsequent presentation of de Moivre's first paper to the Royal Society in 1695 make it possible, though not likely, that de Moivre played some role in the inclusion of the life annuity valuation problem in the ‘Estimate’. It is more likely that any enlightened interaction between the two on the subject of applying Halley's life table to the important problem of life annuities influenced de Moivre to more closely examine this subject. In any event, de Moivre's primary contribution to pricing life annuities did not appear until much later in the Annuities Upon Lives (1725) with a second edition (1743) which has an improved exposition and correction for some errors identified by Thomas Simpson. Also important is the 1738 edition and 1756 editions of his The Doctrine of Chances. The 1738 edition contains much of the material from the 1725 edition of Annuities with some additional topics being discussed, such as temporary life assurances and a valuation for successive lives. The 1759 edition contains a section titled “A Treatise of Annuities on Lives” together with discussion of the life tables of Halley, Kersseboom, Simpson and Deparcieux.

In Annuities, de Moivre examined a wide variety of the life annuities available in the early 18th century: single life annuities, joint annuities (annuities written on several lives), reversionary annuities, and annuities on successive lives. His general approach to these valuation problems involves two steps: first, to develop a general valuation formula for each type of annuity based on Halley's approach; and, secondly, to produce an approximation to the general formula suitable for calculating prices without the considerable efforts involved in evaluating the more exact formula. In order to implement most of the approximations, de Moivre developed a mathematical formulation of the information contained in the life table using: arithmetically declining life probabilities; geometrically declining life probabilities; or, a piecewise linear approximation to the sequence of life probabilities. The selection of a particular assumption depended on the analytical usefulness for the type of problem being solved.

De Moivre provided an important simplification for the value of a single life annuity under the assumption that the 'Probabilities of Life ... decrease in Arithmetic Progression' or, in other words, are uniformly distributed starting at year x up to some terminal year w, $n = w - x$. Generalizing the uniformly distributed case, de Moivre's result is derived by observing that for the uniform case:

$$E[A_n] = \frac{n-1}{n}\frac{1}{1+r} + \frac{n-2}{n}\frac{1}{(1+r)^2} + \ldots + \frac{n-(n-1)}{n}\frac{1}{(1+r)^{n-1}} + \frac{n-n}{n}\frac{1}{(1+r)^n}$$

$$= \sum_{t=1}^{n}\frac{1}{(1+r)^t}\left(1 - \frac{t}{n}\right) = \sum_{t=1}^{n}\frac{1}{(1+r)^t} - \sum_{t=1}^{n}\frac{t}{n\,(1+r)^t}$$

From this point, in the first edition of <u>Annuities upon Lives</u> de Moivre provides an obscure reference to a general result for series from the <u>Doctrine of Chances</u> that is used to determine the solution. In later editions, a more tedious demonstration depending on the relationship between the properties of logarithms, 'fluents' and 'fluxions' is given. A more modern derivation is provided in Poitras (2000, p.213) where it is observed that:

$$\sum_{t=1}^{n} \frac{t}{n\,(1+r)^t} = \frac{1+r}{n} \sum_{t=1}^{n} \frac{t}{(1+r)^{t+1}} = -\frac{1+r}{n}\frac{dA_n}{d(1+r)}$$

It follows that:

$$E[A_n] = [A_n + \frac{1+r}{n}\frac{dA_n}{d(1+r)}] = \frac{A_n}{n}\left[n + \frac{1+r}{A_n}\frac{dA_n}{d(1+r)} \right]$$

The last term contains the familiar Macaulay duration for the annuity applicable to the longest life.

Substituting the relevant expressions back into $E[A_n]$ and evaluating the derivative gives:

$$E[A_n] = \left[A_n + \frac{1+r}{n}\,\frac{dA_n}{d(1+r)} \right]$$

$$= \left(\frac{1}{r} - \frac{1}{r(1+r)^n} \right) + \frac{1+r}{n}\left[\frac{n}{r(1+r)^{n+1}} + \frac{1}{r^2(1+r)^n} - \frac{1}{r^2} \right]$$

$$= \frac{1}{r} - \frac{1+r}{n}\left[\frac{1}{r}\left\{ \frac{1}{r} - \frac{1}{r(1+r)^n} \right\} \right] = \frac{1}{r}\left\{ 1 - \frac{1+r}{n} A_n \right\}$$

The final right-hand-side expression is de Moivre's approximation to the value of a single life annuity. If only for the computational savings provided, this formula is a considerable advance. From the tedious calculation of a long weighted sum, with weights extracted from the not completely accurate life tables available at his time, de Moivre provides a calculation which could be done in a matter of seconds with or without the aid of an appropriate table for annuities certain. While the derivation provided is not the same as de Moivre's (see Hald 1990, p. 521-2), the connection to the familiar notion of Macaulay duration is instructive for modern readers. Similar to the improvement of the duration measure provided by the introduction of convexity, the accuracy of de Moivre's formula can be improved by considering higher order derivative terms (Pearson 1978, p.150-2). In this case, the higher order terms improve on the inaccuracy associated with the initial assumption of uniformly distributed death rates.

5. Solutions for the Joint Life Annuity Value

Though more difficult to value, joint life annuities – where the annuity payment continues until all nominees have died – have a history as long as the single life annuity. Joint annuities featuring a husband and wife or a father and his sons as nominees were common. Considerable variation in the relationship between the prices of joint and single life annuities has been observed. Bernoulli (1709, p.26-7) reviews a number of historical instances for joint life annuity valuations, including a case where, on scholastic grounds, the pricing convention was for single and joint life annuities to be sold at the same price. Referencing du Moulin, Bernoulli observes: 'If the rent should be purchased on the lifetime of two in its entirety, a considerably more ample premium ... must be constituted'. For individuals of comparable age and health, twelve years' purchase is recommended by du Moulin for an annuity on two lives compared to 10 years' purchase for a single life. Daston (1988, p.121) observes that, starting in 1402, Amsterdam sold municipal annuities at regular intervals typically 'charging flat rates of 9 /11 percent for annuities on two heads and 11 13/17 percent for one, regardless of age'. The 1704 life annuity issued by the English government charged 9 years' purchase for a single life, 11 years' purchase on two lives, 12 years' purchase on three lives and 15 years' purchase on a 99 year annuity certain.

While the development of theoretical solutions for the value of joint life annuities parallels that for single lives, the details do differ. In the Value of Life Annuities (1671), Jan de Witt only examines the solution for single lives. Based on correspondence during 1671 between Hudde and de Witt reported in Hendricks (1853), it appears that a solution for the joint life annuity was also actively discussed and attempts at appropriate calculations executed. As the joint life solution was decidedly more complex than for a single life, after a number of false starts the solution settled on an ingenious method involving the use of a binomial expansion to specify a weighted average of (Hendricks 1853, p.109):

> 8 young lives (that number being given in order to avoid here too great a complication) and who are found to have lived as follows – the first to have become defunct 7 full years from the well-established date at which ... has been bought ... a life annuity; the second life 15 years; the third 24 years; the fourth 33 years; the fifth 41 years; the sixth 50 years; the seventh 59 years; the eighth 68 years.

Using an equally weighted average of the annuity certain values for each of the life spans, de Witt determines a single life annuity value (with 1 florin annual payment) of 17.22 florins. Solutions for joint life annuities for 2, 3, 4 up to eight lives are to be determined by using a weighted average with weights determined as binomial coefficients.

De Witt provides the following table for the coefficients in the weighted average:

Years from Purchase to Death of Nominee

7	15	24	33	41	50	59	68
	1	2	3	4	5	6	7
		1	3	6	10	15	21
			1	4	10	20	35
				1	5	15	35
			Weights		1	6	21
						1	7
							1

To apply these coefficients, de Witt gives worked examples for 2, 3 and 4 lives. For the two life annuity case de Witt gives the weighted average as:

> the value of a life annuity upon 2 lives ... is rightly and precisely equal to the value of a life annuity upon one life of a class of 28 lives, of which one life has lived 15 complete years; 2 each 24 years; 3 each 33 years; 4 each 41 years; 5 each 50 years; 6 each 59 years; and 7 each 68 years.

The solution of 20.76 florins (for a 1 florin annual payment) is determined by using a weighted average with 28 total chances. From the table the 3 life case would have 56 total chance and values of 1 for 24 years, 3 for 33 years, 6 for 41 years, 10 for 50 years, 15 for 59 years and 21 for 68 years. Solutions for a joint life annuity on 4, 5, 6 and 7 lives follow appropriately with the value for 8 lives being equal to the annuity certain value for 68 years (23.26 florins). While not directly stated in the correspondence, based on the maximum 80 year life span used in The Value of Life Annuities, the underlying assumption is that the lives involved are initially all 12 years old. For simplicity, de Witt uses only 8 lives in calculating the solutions. However, reference is made to Hudde using calculations with 80 lives which would make for more complicated calculations but would avoid the degenerate solution given by de Witt for 8 lives.

Though the correspondence between de Witt and Hudde only examined the uniformly distributed case, it is not difficult to see that the binomial coefficients in the weighted average could be adjusted to reflect the theoretical life table used in The Value of Annuities. However, being available only in correspondence that was not uncovered until Hendricks (1853) in the middle of the 19th century, this ingenious method of solving the joint life annuity faded into history leaving the more cumbersome and computationally intensive approach proposed by Halley to influence the development of solutions for the joint life annuity value. While recommending the usefulness of logarithms for 'facilitating the Computation of the Value of two, three, or more Lives', Halley does not provide completely worked solutions for any joint life annuity values. Though the proximate reason given by Halley is that the Breslau life table was not based on 'the Experience of a very great Number of Years', de Moivre (1925, p.ii) recognizes that:

> even admitting such as Table could be obtained as might be grounded on the Experience of

> a very great Number of Years, still the Method of applying it to the Valuation of several Lives would be extremely laborious, considering the vast Number of Operations that would be requisite to combine every Year of each Life with every Year of all the other Lives.

What Halley does provide is a novel geometric analysis of the solution method for determining the individual terms in the sum that determines the joint life annuity for two and three lives. That Halley spent considerable time and effort on the solution to the joint life annuity value is apparent in Halley (1693) where less than two pages is dedicated to single life annuities with about four and one-half pages to joint life annuities, concentrating almost exclusively on the valuation for two and three lives.

While being computationally demanding, the approach to joint annuity valuation proposed by Halley does has the desirable feature of allowing for nominee lives that are unequal. Halley (1693, p.604) verbally describes the brute-force method for solving the joint life annuity on two lives:

> for the number of Chances of each single Life, found in the [Breslau life] Table, being multiplied together, become the chances of Two Lives. And after any certain Term of Years, the Product of the two remaining Sums is the Chances that both Persons are living. The Product of the two Differences, being the numbers of the Dead of both Ages, are the Chances that both the Persons are dead. And the two Products of the remaining Sums of one Age multiplied by those dead of the other, shew the Chances that there are that each Party survives the other: Whence is derived the Rule to estimate the Value of the Remainder of one Life after another. Now as the Product of Two Numbers in the Table for the Two Ages proposed, is to the difference between that Product and the Product of the two numbers of Persons deceased in any space of time, so is the value of a Sum of Money to be paid after so much time, to the value thereof under the Contingency of Mortality. And as the aforesaid Product of the two Numbers answering to the Ages proposed, to the Product of the Deceased of one Age multiplied by those remaining alive of the other; So the Value of a Sum of Money to be paid after any time proposed, to the value of the Chances that the one Party has that he survives the other whose number of Deceased you made use of, in the second Term of the Proposition.

This explanation is followed by a numerical illustration of the relevant calculations for one term in the sum. Recognizing the difficulty involved in understanding the various calculations, Halley then provides a geometric motivation for these calculations (see Figure 4.1):

INSERT FIGURE 4.1

The whole area of *ABCD* (= $\ell_x \ell_y$) is the total number of chances for two lives at $t=0$. The area of the inner rectangle *HGIB* (= $\ell_{x+m} \ell_{y+m}$) is the total number of chances that both are alive at $t=m$. The two side rectangles *IDGE* ($= \ell_x (\ell_y - \ell_{y+m})$) and *HAFG* ($= \ell_y (\ell_x - \ell_{x+m})$) are the chances that one nominee is alive while the other is dead leaving the upper rectangle *EGFC* to be the chances that both nominees have died. Subtracting the ratio of *EGFC to ABCD* from one determines the weight applicable to the discounted cash flow associated with $(1 / (1 + r))^m$.

Formally, because no allowance is made for the portion of joint annuity payments that would be made when there are no chances that the eldest nominee will be alive, this geometric argument only fully covers the situation where the two nominees have equal ages.[7] Given this, Halley extends the geometric analysis for a joint life annuity on two lives to three lives (see Figure 4.2).

INSERT FIGURE 4.2

For Halley (1693, p.606) the discussion for the two dimension case 'is the Key to the Case of Three Lives'. By extending to analysis to three dimensions, it is apparent that the fraction of total volume associated with the volume where all three nominees are dead is associated with the cube *KLMNOPAI*. Unlike the two dimensional case, Halley does not provide numerical calculations. It is apparent that the calculations involved are too numerous and tedious to warrant a complete resolution. Having laid this rough foundation, Halley never returned to published efforts on this subject, leaving the next series of developments to Abraham de Moivre.

De Moivre provided numerous approximations relevant for joint life annuities using different theoretical assumptions about the life table. Some of de Moivre's approximations were more successful than others and Simpson expended considerable effort showing that direct calculation making use of actual life tables was substantially better for pricing the joint life annuity (Hald 1990, p.532). Starting with de Moivre and continuing with Simpson a direct approach to the price of a joint life annuity, $E[A_{mn}]$, for two lives was employed. This approach involved the price of two single life annuities $E[A_n]$ and $E[A_m]$ and an annuity for joint life continuance which makes payments only when both nominees are alive $E[{}_mA_n]$. Given that the pricing problem for single life annuities was solved, the joint life annuity problem involved solving for $E[{}_mA_n]$.

The de Moivre approach to solving a joint life annuity written on two lives involved the relationship:

$$E[A_{mn}] = E[A_n] + E[A_m] - E[{}_mA_n]$$

This result follows from observing that the probability of having survival of at least one of the two lives at time t is $1 - (1 - Prob[x,t])(1 - Prob[y,t]) = Prob[x,t] + Prob[y,t] - Prob[x,t]Prob[y,t]$ where $Prob[x,t]$ is the probability of x (y) surviving at time t which can be related to ℓ_{x+t}/ℓ_x in the $E[A_n]$ formula given previously. Multiplying by $(1+r)^{-t}$ and summing gives the required result which holds for any life table. From this point de Moivre used two approaches to solve for approximations to $E[{}_mA_n]$. The approach initially used in the first edition of Annuities Upon Lives (1925) takes $Prob[\cdot]$ to be geometrically declining. More precisely, having already provided a simple solution for the single life annuity values, using geometrically declining life probabilities de Moivre is able to show that the value of the joint continuance is:

$$E[{}_mA_n] = \frac{E[A_n]\ E[A_m]\ (1 + r)}{(E[A_n] + 1)(E[A_m] + 1) - (E[A_n]\ E[A_m]\ (1 + r))}$$

While this approach leads to a less complicated result for $E[A_{mn}]$, later in the first edition de Moivre provides a more complicated, but presumably more exact result, for $E[A_{mn}]$ using arithmetically declining life probabilities. A numerical example for two lives aged 40 and 50 is provided and, with $r = 5\%$, the value of $E[{}_mA_n]$ for the arithmetically and geometrically declining cases are solved as 14.53 and 14.55 years' purchase, respectively.

Compared to Halley's obscure and incomplete effort, the simplicity of de Moivre's solution for the joint life annuity on two lives is impressive. The extension to three lives follows appropriately:

$$E[A_{mnq}] = E[A_n] + E[A_m] + E[A_q] - E[_mA_n] - E[_qA_n] - E[_mA_q] + E[_{mn}A_q]$$

where $E[A_{mnq}]$ is the value of the joint life annuity on (the longest of) three lives and $E[_{mn}A_q]$ is the value of an annuity paid when all three nominees are alive. Having already provided approximations to solve all but one term, using geometrically declining life probabilities de Moivre gives the result:

$$E[_{mn}A_q] = \frac{(E[A_n] + E[A_m] + E[A_q])\ (1 + r)^2}{(E[A_m] + 1)\ (E[A_n] + 1)\ (E[A_q] + 1) - \ (E[_{mnq}A]\ (1 + r)^2)}$$

While no results or calculations are given for the three life joint annuity using arithmetically declining life probabilities, de Moivre does provide the solution methodology for 'finding the Values of as many *joint* Lives as may be assigned'. The simplification provided by assuming that the nominees' lives are equal, and the use of the binomial expansion to solve the coefficients in this case, is also recognized.

A detailed examination of de Moivre's contributions to joint life annuity valuation would be incomplete without considering <u>The Doctrine of Annuities and Reversions</u> (1742, 2nd ed. 1775) by Thomas Simpson. As evidenced in preface of the 1743 edition of <u>Annuities on Lives</u>, the contents of <u>The Doctrine</u> (1742) incensed de Moivre. Without identifying Simpson by name, de Moivre makes the following observations that were clearly directed at Simpson: 'he mutilates my Propositions, obscures what is clear, makes a Shew of new Rules, and works by mine, in short confounds in his usual way, every thing with a croud of useless Symbols'. This attack by de Moivre compelled Simpson to include in future printings of <u>The Doctrine</u> an additional 'Appendix containing some remarks on Mr. Demoivre's book ... with answers to some personal and malignant misrepresentations, in the preface thereof'. In this Appendix, Simpson claims:

> It is not my design to expatiate on the unseemliness of this gentleman's usage, not to gratify a passion, which insinuations so gross must naturally excite in a mind that looks with contempt on such unfair proceedings; but only to offer a few particulars to the consideration of the public, with no other view than to clear myself from a charge so highly injurious, and do justice to the foregoing work.

In the Appendix, Simpson makes a strong case for the credibility of his contributions that is difficult to deny. In addition, it is likely that the results and comments in The Doctrine led de Moivre to make some substantive changes to the 1743 edition, and later versions, of Annuities on Lives.

At the time The Doctrine (1742) appeared, Thomas Simpson was still in the early stages of an academic career that was to produce a number of seminal advances, primarily in mathematics, earning the eponym Simpson's Rule for a method of approximating an integral using a sequence of quadratic polynomials.[8] His appointment as the head of mathematics at the Royal Military Academy at Woolwich – a position that provided Simpson with sufficient security to pursue academic interests – was not to occur until 1743. Being the son of a weaver with little formal education, Simpson had for some years made a living as an itinerant lecturer teaching in the London coffee houses. Around this time, certain coffee houses functioned as 'Penny Universities' that provided cheap education, charging an entrance fee of one penny to customers that drank coffee and listened to lectures on topics specific to that coffee house, for example, Poitras (2000, p.293-7). Popular topics were art, business, law and mathematics. It is well known that de Moivre was a fixture at Slaughter's Coffee House in St Martin's Lane during this period.

In support of his activities at the penny universities and continuing at Woolwich, Simpson started producing a successful string of textbooks beginning with a text on the theory of fluxions (A New Treatise of Fluxions) in 1737. His next book, The Nature and Laws of Chance (1740), bears a strong similarity to de Moivre's Doctrine of Chances (1718, 1738 2^{nd} ed.). In both The Nature and Laws of Chance and The Doctrine, Simpson makes a grateful reference to de Moivre in the preface but no references in the body of the text despite obvious similarities in the content. De Moivre's incensed reaction to The Doctrine can be viewed as a response to Simpson's continuing plagiarism, though this is not the only possible interpretation. A precise explanation requires information that is unavailable. The seemingly damning evidence that Simpson was also accused of plagiarism by a number of others can, initially, be attributed to a desire to build a reputation required to sustain his livelihood and, over time, to a lack of sympathy for formal academic procedure. Even after his appointment at Woolwich and election to the Royal Society (in 1745), Simpson continued to be accused of plagiarism, though not by persons of the academic stature of de Moivre.[9]

Given that much of the seminal work on joint (and single) life annuities had been accomplished by de Moivre, assessment of Simpson's contributions has a significant subjective element. Hald (1990, p.511) identifies three important new contributions: '(1) a life table based on the London bills of mortality; (2) tables of values of single- and joint-life annuities for nominees of the same age based on this life table; and (3) rules for calculating joint life-annuities for different ages from the tabulated joint-life annuities". This assessment would appear to be based largely on the content of The Doctrine and ignores the practical content of A Supplement to the Doctrine of Annuities and Reversions (1752, 2^{nd} ed. 1971) which appears as Part VI of Select Exercises for young Proficients in the Mathematicks. Judging from Richard Price's references to Simpson, de Moivre and Halley in the seminal Observations on Reversionary Payments (1772) – one of the outstanding intellectual achievements of the 18^{th} century – it was the practical content of Simpson's numerous worked problems and examples that had the most relative impact. While recognizing the importance of de

Moivre's approximations, Price dedicates Essay II of Observations to the unacceptable errors that assuming geometrically declining life probabilities has on the calculated values for joint life annuities on two and three lives.

Even casual reading of The Doctrine reveals the close connection to Annuities on Lives. The presentations are similar and various results are, more or less, the same. This said, Simpson does not mimic de Moivre's analytical approximation agenda. The results are adapted to a practical ends. To see the implications of this, consider the solution for the joint life annuity on two unequal lives given in terms of two equal lives provided by Simpson:

$$E[A_{mn}] = E[A_{nn}] + \frac{\frac{1}{2}\, E[A_{nn}]\ (E[A_{mm}] - E[A_{nn}])}{E[A_{mm}]} \qquad \textit{where} \quad m < n$$

This solution, which does not appear in Annuities on Lives (1725) reduces the unmanageable practical problem of preparing tables for the value of joint annuities on two unequal lives to be solved using a manageable table for joint annuity values on two equal lives, table which Simpson provides. On balance, Simpson's contributions to life annuity valuation are sufficient to warrant closing this discussion of the pioneering contributions to life annuity valuation with his work. Building on the initial efforts of James Dodson, Richard Price was about to pioneer a new route for the subject of life contingency valuation by refocusing intellectual efforts onto the core subjects of actuarial science: insurance and pensions. But the story of these pioneers will have to wait for another time.

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FIGURE 4.1
Halley's Method for a Joint Annuity on Two Lives

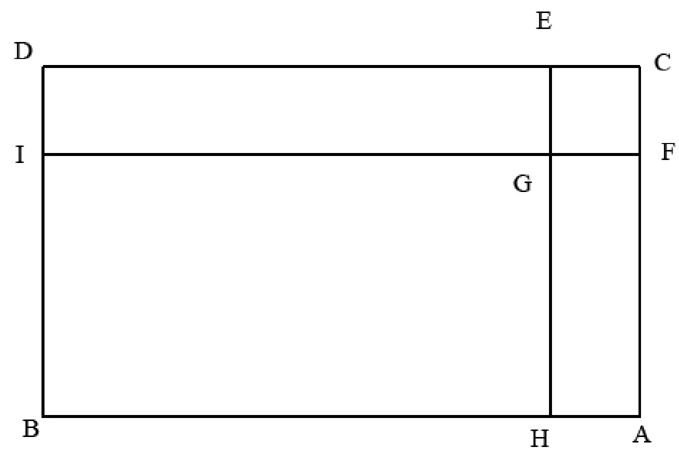
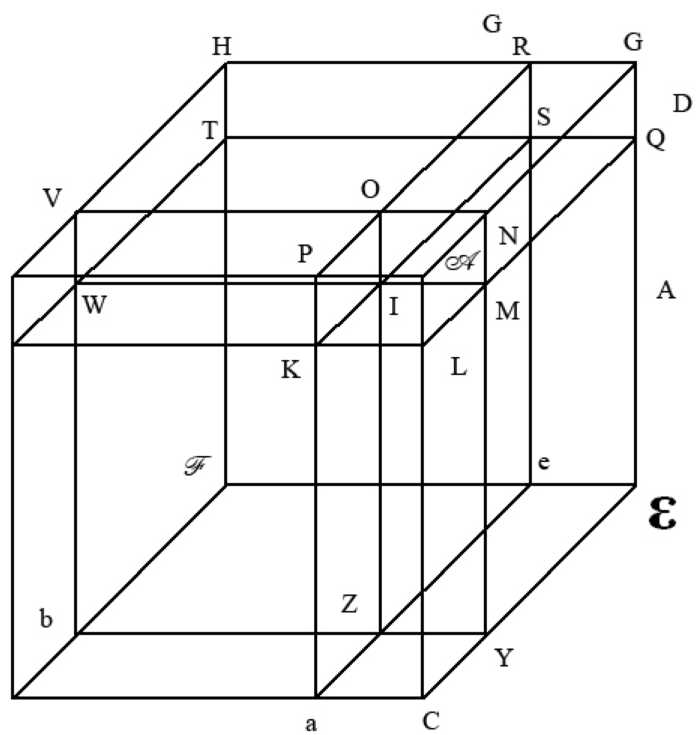


FIGURE 4.2
Halley's Method for a Joint Annuity on Three Lives



NOTES

1. This section and section 2 are based on Poitras (2000, ch.6) and Poitras (2005, sec.2.1).

2. An usufruct is the right of temporary possession, use or enjoyment of the advantages of property belonging to another, so far as may be had without causing damage or prejudice to the property.

3. Reversions and other types of life contingent claims such as copyholds and advowsons could arise in situations other than a formal life annuity contract. For example, reversions could arise in marriage settlements and wills and be of interest for legal reasons of establishing an estate value.

4. It was common at this time for a number of spelling variants to be used, all of which can be considered correct spelling. The spelling Jan de Witt is found in Hald (1990), Coolidge (1990) and Pearson (1978). Hald also gives the variant Johan de Witt while Pearson reports John de Witt. Heywood (1985) uses Johannes de Wit while Hendricks (1852-3) uses John de Wit. In the Valuation, the author is listed as 'J. de Wit'.

5. Karl Pearson, who had strong views on a number of individuals involved in the history of statistics, depreciates de Witt's work by claiming: 'the data are uncertain and the method of computation is fallacious' (Pearson 1978, p.100). This is at variance with Hald (1990), Alter and Riley (1986) and others. Pearson (1978, p.702) also appears to have been unaware of Hudde's contribution, 'I was unaware that (Hudde) had contributed to the theory of probability.'

6. Not long after submitting his Value of Life Annuities to the States General, de Witt's life came to a tragic end. The invasion of the Dutch Republic by France in 1672 led to a public panic which precipitated de Witt's forced resignation and his replacement by the Stadholder William III. However, the demand for public retribution for the Grand Pensionary's perceived failings did not end with his resignation. Later in 1672, de Witt was set upon by a mob and shot, publicly hanged and his body

then violated.

7. Hald (1990, p.140)) provides a helpful exposition of the geometric argument for two lives but does not recognize this point.

8. It is likely that Simpson's Rule was not originated by Simpson but, rather, by Newton; a point that was acknowledged by Simpson. However, Ypma (1995) demonstrates that the Newton-Raphson technique for solving non-linear equations is most appropriately credited to Simpson.

9. Sources on Simpson's life are limited, the most detailed being Clarke (1929). Pearson (1978, p.166) has a discussion of other sources sympathetic to Simpson that rely on Hutton's 1792 introduction to the 2nd edition of Simpson's Select Exercises. In his typical flourishing style Karl Pearson observes: 'to me he is a distinctly unpleasant and truculent writer of cheap textbooks, not a great mathematician like De Moivre'. Pearson (1978, p.176-185) takes great pains to document this view.