7. Option Concepts

7.1 Basic Option Properties

Some Distribution Free Properties of Options[1]

The celebrated Black-Scholes option pricing formula depends fundamentally on the assumption: that stock prices are log-normally distributed. While it is not possible to develop an option pricing formula without making a distributional assumption, there are still a number of distribution free properties that can be identified. A number of these basic properties of options have already been developed in previous Chapters. For example, the $max[\cdot]$ function associated with the expiration value of an option describes one distribution free property of options. Put-call parity is another property. For completeness, even those properties already derived will be listed here.

Given the notation listed in Sec. 1.1, unless otherwise stated, the following Properties are stated *for the case of a* ***non-dividend paying stock*** *(or deliverable commodity that does not earn a carry return).* This case is examined for traditional reasons, early statements of the distribution-free properties of options, e.g., Merton (1973), used this case. Extensions to other cases, such as options on dividend paying stocks, spot commodities and futures contracts, are provided when appropriate, either in this Section are in other Sections, e.g., in the discussion of early exercise of American currency options in Section 8.4. The distribution-free properties rely on conventional ***perfect markets assumptions***: no transactions costs, no taxes and riskless lending and borrowing at the constant riskless interest rate.

Property 1: Non-negative prices

$$C[S,\tau,X], \geq 0,\ C_A[S,\tau,X] \geq 0,\ P[S,\tau,X] \geq 0,\ P_A[S,\tau,X] \geq 0$$

This property holds for every $t \leq T$.

Property 2: Expiration date value

$$C_A[S(T),0,X] = C[S(T),0,X] = max[0,\ S(T) - X]$$
$$P_A[S(T),0,X] = P[S(T),0,X] = max[0,\ X - S(T)]$$

As with other Properties stated in this Section, this property ignores transactions costs and other related expenses such as taxes.

Property 3: Non-Negative Value to Exercising:

$$C_A[S,\tau,X] \geq C[S,\tau,X] \qquad P_A[S,\tau,X] \geq P[S,\tau,X]$$

This property can also be expressed as:

$$C_A[S,\tau,X] = C[S,\tau,X] + EEP_C[S,\tau,X] \text{ and } P_A[S,\tau,X] = P[S,\tau,X] + EEP_P[S,\tau,X]$$

where $EEP[S,\tau,X]$ is the early exercise premium with the subscripts C and P referring to calls and puts respectively. Property 3 then implies that $EEP \geq 0$ with equality almost surely for deep out-of-the-money options.

This is because early exercise is only rational if the option is in-the-money and the probability of a deep out-of-the-money option being exercised is negligible. Hence, the prices of European and American calls will be approximately equal for deep out-of-the-money options. For calls and puts with the same exercise value and time to maturity, increasing (decreasing) EEP_C implies decreasing (increasing) EEP_P.

Property 4: No Exercise Arbitrage Profits:

$$C_A[S,\tau,X] \geq S(t) - X \qquad P_A[S,\tau,X] \geq X - S(t)$$

The value of the American option is bounded below by the exercise value. This is ***not*** a property that applies to European options, though it is possible for European options on dividend paying securities to trade below the intrinsic exercise value $S(t) - X$ for calls and $X - S(t)$ for puts (see Sec. 8.4).

Property 5: Non-negative Value to Earlier Exercise

$$C_A[S,\tau_1,X] \geq C_A[S,\tau_2,X] \qquad for\ \tau_1 > \tau_2$$
$$P_A[S,\tau_1,X] \geq P_A[S,\tau_2,X] \qquad for\ \tau_1 > \tau_2$$

The right of earlier exercise has a non-negative value.

Property 6: Exercise Prices and Options Premiums, for $X_1 > X_2$

$$C[S,\tau,X_1] \leq C[S,\tau,X_2] \qquad C_A[S,\tau,X_1] \leq C_A[S,\tau,X_2]$$
$$P[S,\tau,X_1] \geq P[S,\tau,X_2] \qquad P_A[S,\tau,X_1] \geq P_A[S,\tau,X_2]$$

The lower the exercise price the more (less) valuable is the call (put) option.

Property 7: Calls with Zero Stock Prices: $C_A[0,\tau,X] = C[0,\tau,X] = 0$

The right to purchase a stock or commodity with no value has no value.

Property 8: Perpetual American ***Call*** with Zero Exercise Price: $C_A[S,\infty,0] = S$

The perpetual property requires the American early exercise feature. Given this, the spot commodity (e.g., common stock) can be described as a perpetual option with a zero exercise price. This is one sense in which the common stock can be described as an option.[2] This property is a special case of the price for perpetual American call options, the only general case where a pricing formula for American call options is generally available.

Property 9: Put Premium Upper Bound: $X \geq P_A[S,\tau,X] \geq P[S,\tau,X]$

While call option values are unbounded above, put options prices are bounded above by X because the stock price can only fall to zero, due to limited liability. This property of put options is important in distinguishing American put and call options. Unlike American calls, there is an additional incentive to exercise American put options early: when the stock price is sufficiently close to zero, the value of the potential interest on the exercise premium exceeds the expected gain from holding the put option until expiration.

Property 10: Let $PV[r,\tau]$ = the present value of \$1 to be paid in τ days discounted at the discrete annualized interest rate r. In discrete form $PV[\tau]$ can be expressed:

$$PV[r,\tau] = \frac{1}{1 + r\frac{\tau}{365}} = \frac{1}{1 + rt^*}$$

In continuous form, with continuous (annualized) interest rate *r:*

$$PV[r,\tau] = \exp\{-r\frac{\tau}{365}\} = \exp\{-rt*\}$$

where t* is the fraction of the year remaining on the security (see Appendix I).

With appropriate specification of the continuous or discrete *r:*

$$C[S,\tau,X] \geq Max[0, S(t) - X\ PV[r,\tau]]$$

This condition has considerable practical value. Demonstrating this result requires a specific portfolio: the portfolio shorts the spot commodity generating *S(t).* For the short position, in all states of the world at time *T, -S(T)* is the payoff. The portfolio involves taking the money from the short and purchasing a call option with exercise price *X* and τ days to maturity as well as buying a pure discount bond that has τ days to maturity and a par value of *X*. Whenever *S(T)* < *X*, then the call will expire worthless and the portfolio will earn only the maturing value of bonds, *X*, and *-S(T*), the cost of the short. In this case, the terminal value of the portfolio is positive, by construction. When *S(T)* > *X*, the call will have value *S(T)* - *X* which, when combined with the maturing value of the bond and the short gives a value of zero. Hence, because the call option and lending portfolio has a greater value than the short stock position in one state of the world and the same value in the other state, the portfolio must involve a net investment of funds at *t* such that $C + X\ PV[r,\tau] - S \geq 0$. Manipulating and combining this condition with Property 1 gives the desired result.

In practice, this Property can provide a mechanism for "replicating" the return on the stock position: the value of the call plus lending will be close to the value of the stock, providing a viable trading opportunity.[3] In other words, instead of buying the stock and holding for some horizon τ, it is also possible to invest in bonds and buy calls. From put-call parity arbitrage, the cost of this position will exceed the cost of buying the stock by the cost of buying a put with exercise price *X* and time to expiration τ. It follows that a call plus bond portfolio will have the same payoff as an 'insured' stock portfolio. The advantages and disadvantages of this strategy will be discussed in Sec. 9.3. Recognizing that transactions costs for puts and calls will be approximately the same, insofar as execution costs for bonds are lower than for stocks, the call plus bond strategy will be preferred.

A Property 10 condition is available for puts. In this case, the relevant portfolio involves a long stock position combined with buying a put with exercise price *X* and time to maturity τ with the funds obtained by borrowing *X* using a pure discount bond from t until *T*. In all the future states of the world where *S(T)* > *X*, the stock will be worth *S(T),* the put will expire worthless and the borrowing will be worth -*X*. This is positive by construction. In the future state of the world where *S(T)* ≤ *X* the put will be worth *X* - *S(T),* the stock *S(T)* and the borrowing -*X*, for a portfolio value of zero. The implication is that at time t, the value of the put plus the stock minus borrowing must be positive: $P + S - X\ PV[r,\tau] \geq 0$. Manipulating and combining with Property 1 produces:

$$P[S,\tau,X] \geq Max[0, X\ PV[r,\tau] - S(t)]$$

As with the call plus bond portfolio, the implication is that a portfolio of a stock with a put must sell for more than a bond with par value equal to *X*. From put-call parity the difference will equal the amount paid for an appropriate call.

Another practical implication of Property 10, follows from combining this result with Property 3 to get:

$$C_A[S,\tau,X] \geq C[S,\tau,X] \geq Max[0, S(t) - X\ PV[r,\tau]] \geq S(t) - X$$

This result fills in the inequality given in Property 4. Upon some consideration, it is apparent that, except in extreme cases such as described in Property 7, the value of the American call option at any time t will almost surely be greater than the exercise value, $S(t) - X$. Colloquially, ***"the American call on a non-dividend paying stock* is *always worth more alive than dead"***, it will always be worth more to the call option holder to sell an American call rather than to exercise prior to maturity.

***Property 11*:** American and European Call and Put Options On Non-dividend Paying Stocks

Property 10 raises a quandry, if the American call on a non-dividend paying stock will never rationally be exercised early, what value does the right of early exercise have in this case? This leads to the following property:

$$C_A[S,\tau,X] = C[S,\tau,X]$$

This result follows directly from Property 10 because, for $\tau > 0$, $Max[0, S(t) - X\, PV[r,\tau]] \geq S(t) - X\, PV[r,\tau] > S(t) - X$. Recognizing that $S(t) - X$ is the immediate exercise value of the American call, the value of the American call on a non-dividend paying stock is always strictly greater than the exercise value. Hence, the value of the early exercise premium is zero because the call will never be exercised early. Property 11 is specific to calls and does not extend to either American put options on non-dividend paying stocks or to American call options on dividend paying stocks. The latter cases are examined in Sec. 7.3.

Early exercise for American puts can be motivated by examining the European put condition from the discussion in Property 10:

$$P[S,\tau,X] \geq Max[0, X\, PV[r,\tau] - S(t)]$$

For puts that are in-the-money, it is ***not*** the case that $Max[0, X\, PV[r,\tau] - S(t)] \geq X - S(t)$. Because the value of the American put is not supported by portfolio trading involving long the stock, long the put combined with borrowing $X\, PV[r,\tau]$, as the put goes deeper in the money the put price will fall to the early exercise boundary of $X - S(t)$.[4] In this case, if the put is exercised the holder will receive $X - S(t)$ that can be invested at r to provide $(X - S(t))\, exp\{rt^*\}$ on the expiration date. If the put is held to maturity, the holder will receive $max[0, X - S(T)]$.

Comparing $(X - S(t))\, exp\{rt^*\}$ with $max[0, X - S(T)]$ reveals that the early exercise decision for the put depends on the tradeoff between the interest income associated with exercising immediately and the potential gain associated with expected falls in the stock price. In the extreme case $S(t) = S(T) = 0$ and the decision is obvious (for $X > 0$). There is no possibility for further gain from spot price changes and rationality requires immediate exercise of the put option. Once the put price has reached the arbitrage boundary of $X - S(t)$, the general condition for early exercise of the American put on a non-dividend paying stock is: $X(exp\{rt^*\} - 1) > S(t)\, exp\{rt^*\} - E[S(T)]$. It follows that if $S(t)$ is expected to stay the same, then the put will be exercised.

Property 12: Exercise and Call Prices

$$C[S,\tau,X_1] - C[S,\tau,X_2] \leq PV[r,\tau]\, \{X_2 - X_1\} \quad where\ X_2 > X_1$$

This implies that a \$1 increase in the exercise price reduces the value of the option by less than \$1. The proof of this condition is given as an assignment in the end of Chapter questions.

A combination of Properties 8, 7, 2, 10 and 12 can be used to motivate a graphical presentation of call option price behavior. Without Property 10, a plausible relationship between the call option price and the stock price is given in Graph 7.1.[5] While the precise shape of the $C[S;\cdot]$ function will be examined in Sec. 7.2, it is sufficient to know that:

Graphs 7.1-7.2 **Basic Properties of Call Option Prices**

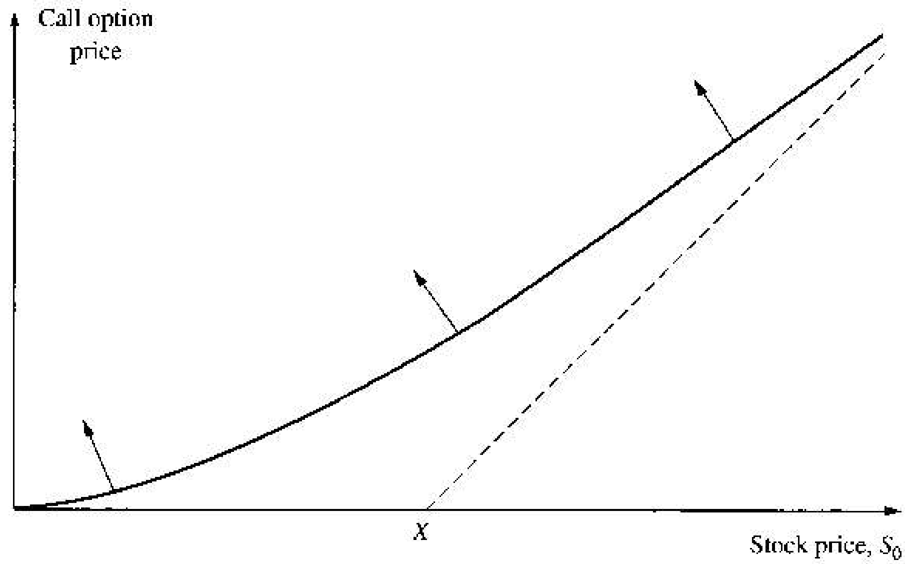


Figure 7.1 Variation of price of an American or European call option on a non-dividend-paying stock with the stock price, S_0.

Source: Hull (2000, p.176)

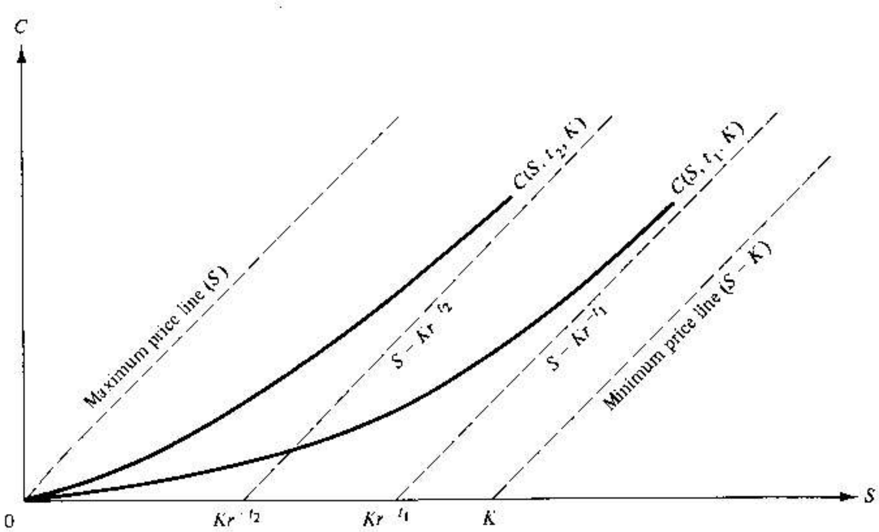


Figure 4-5 Call-Stock Price Diagram

Source: Cox and Rubinstein (1985, p.185)

Graph 7.3 **A Collection of Expiration Date Profit Diagrams**

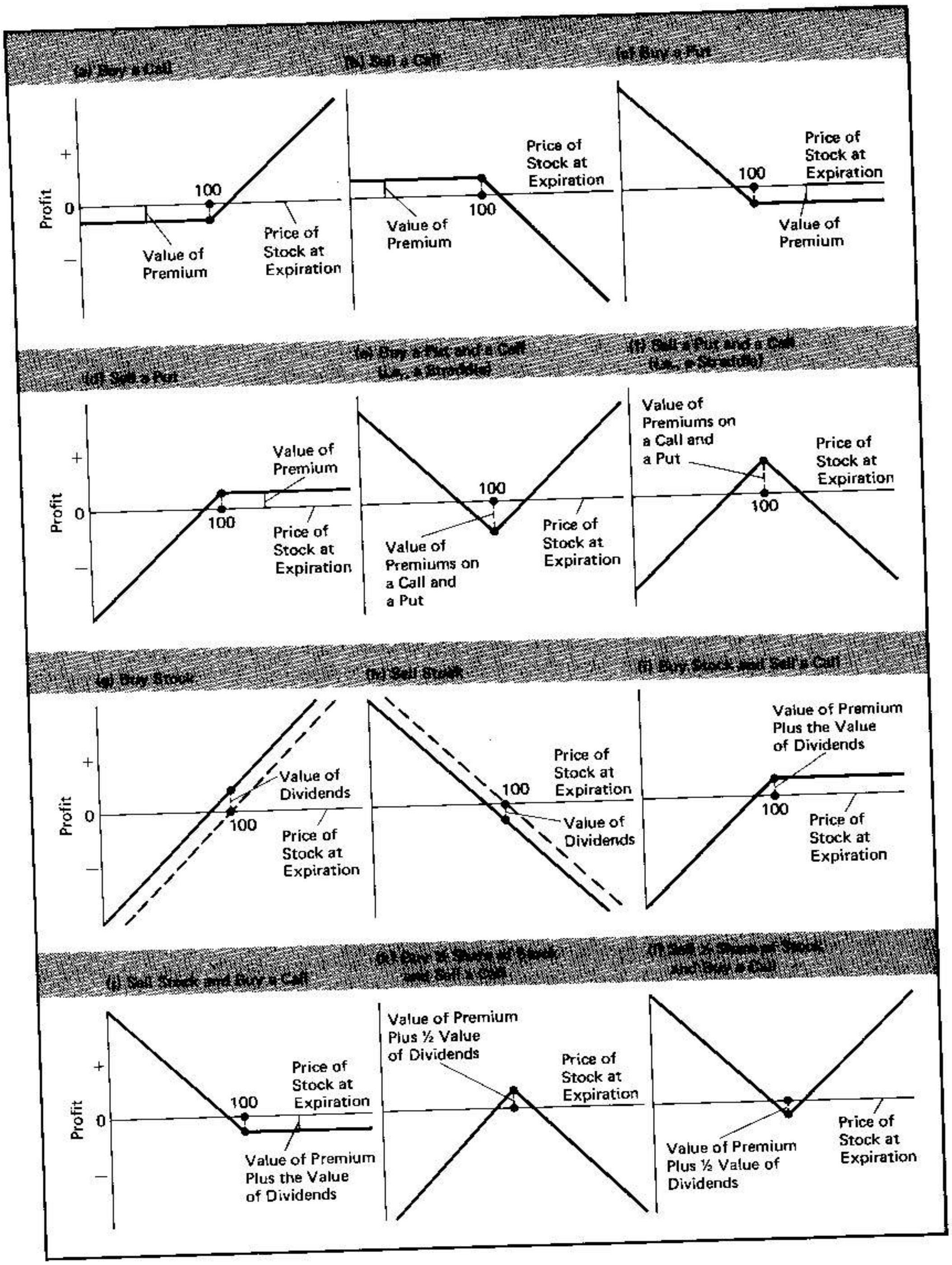


FIGURE 16-7
Profits and Losses from Various Strategies.

Source: Sharpe, Investments (1985, p.490)

$$1 \geq \frac{\partial C}{\partial S} \geq 0 \qquad\qquad \frac{\partial C}{\partial S}\Big|_{S \to \infty} = 1$$

The inclusion of Property 10, combined with the above restriction on $C[\cdot]$, produces Graph 7.2 that illustrates the type of inter-temporal behavior associated with the call option. One of the applications arising from the Black-Scholes formula is the analytical expression that can be derived to capture the precise nature of the call price, as various parameters are allowed to change (see Sec. 9.1).

Expiration Date Profit Diagrams

Exchange traded stock options differ substantively from warrants insofar as exercise of an exchange traded option does not have any implications for the amount of common stock outstanding, i.e., there is no dilution of the common stock associated with exchange traded stock options. For analytical purposes, it is most expedient to assume exchange traded options for illustrating the relevant option payoff functions using the ***expiration date profit diagram*** technique. While the presentation may differ from source to source, the relevant profit function is usually derived by subtracting the option premium paid from the $max[\cdot]$ function that gives the value of the option at expiration. For a call option, the value of the call option, $C(\cdot)$, depends on the expiration date T and exercise price X. At expiration, the value of the call option $C(T,X)$ is:

$$C(T,X) = max[0, S(T) - X]$$

For the put option price, $P(T,X)$:

$$P(T,X) = max[0, X - S(T)]$$

It follows that the profit function for the call purchaser is: $\pi = C(T,X) - \{call\ premium\}$, and for the put purchaser, $\pi = P(T,X) - \{put\ premium\}$. While it is theoretically correct to include an allowance for foregone interest on the premium over the life of the option, this issue is ignored at this point for simplicity.

The combination of puts and calls with writers and purchasers leads to four versions of the expiration date profit diagram for naked option positions. Graph 7.3 is a collection of the various elementary expiration date profit diagrams. Be careful to observe that the expiration date profit is being plotted against the expiration date stock price. The diagrams of the profit functions for the written positions are a mirror image of the diagrams for the purchased positions, because the gains (losses) on written positions are the losses (gains) of the associated purchased positions. (Following the definitions given in Sec. 1.1, written positions are “short” positions and purchased positions are “long”.) Given this, these diagrams are best interpreted as describing the payoff for a $t=0$ option trader planning to hold the position until $t=T$. This suppresses certain essential elements of the option valuation problem, e.g., how the premium is determined or admitting the possibility of early exercise. While perhaps most descriptive of the earlier OTC-style options, the diagrams do illustrate the essential features of naked options: the purchased call (put) has value when the stock price at expiration is *above (below)* the exercise price; the maximum possible loss on the purchased option is loss of premium; possible losses (gains) on a written (purchased) call are unbounded; and, possible loss (gain) on a written (purchased) put is the exercise price minus the premium.

Evaluation of Different Positions

While of limited analytical value when applied to naked positions, expiration date profit diagrams are of greater value when applied to more complicated options positions. For example, consider the case of a ***covered***, as opposed to naked or uncovered, options position. In this case, the option is purchased in conjunction with some spot position, e.g., long (short) IBM stock combined with an IBM options position. To see this, consider the associated expiration date profit diagram for a long position in a ***non-dividend paying*** stock: $\pi = Q\{S(T) - S(0)\}$. Both the long and short spot position expiration date profit diagrams are given in Graph 7.3. If dividend payments are permitted, the profit function for the long (short) will shift up (down) by the amount of the dividends received (paid). As with the payment of interest, this complication will be ignored for present purposes. It is now possible to derive the diagrams for covered positions by geometrically combining the profit functions for the two components of the relevant covered position. Because each of the four uncovered positions can theoretically be combined with either a long or a short cash position, this gives rise to eight different possible scenarios. Graphs 7.4-7.7 provide the four most commonly encountered cases.

Analysis of the expiration date profit diagrams for covered positions reveals some of the fundamental replication properties of options. Consider the standard strategy of using purchased puts to insure the value of a cash position against downside moves, Graph 7.4. Subject to the caveats associated with transactions costs, size of premium, dividends and interest on the cash position and so on, when $X = S(0)$ the payoff on this covered position is identical to the payoff on a purchased ***uncovered call*** position with exercise price X. Similarly, the covered position that combines short-the-cash with a purchased call $(X = S(0))$ is shown to be equivalent to an uncovered purchased put with exercise price X, Graph 7.5. The replication strategies for naked written positions follow appropriately, long-the-cash combined with a written call replicates a written put, Graph 7.6, and short-the-cash combined with a written put replicates a written call, Graph 7.7. The remaining four types of covered positions do not produce replication.[6] Rather, these positions will tend to increase exposure to volatility of the underlying cash position.

The final group of (uncovered) replication strategies to consider involve using options to replicate either a long or a short position in the underlying commodity or stock (Graph 7.8). In order to replicate a long stock position, Graph 7.8, a purchase call is combined with a written put with the same X and time to expiration. Taking the premiums on the call and put at $t=0$ to be $C[0]$ and $P[0]$, the expiration date profit diagram (again ignoring foregone interest on premiums) is:

$$\pi[T] = P[0] - C[0] + max[0, S(T) - X] - max[0, X - S(T)]$$

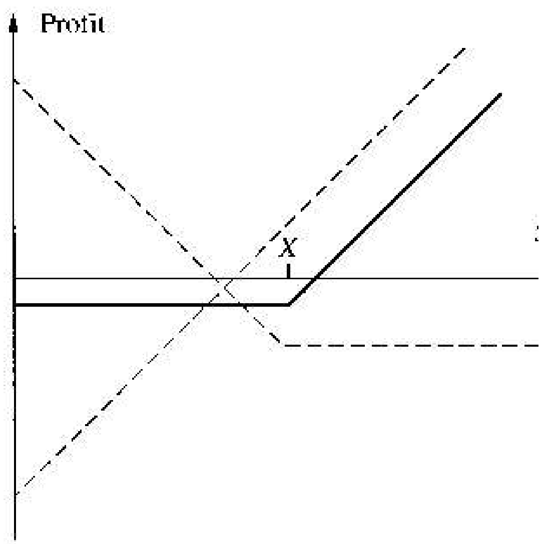
Similarly, for the replicated short stock position that combines a put purchase with cash outflow of $P[0]$ with a written call cash inflow of $C[0]$:

$$\pi[T] = C[0] - P[0] + max[0, X - S(T)] - max[0, S(T) - X]$$

The replicated long stock position is represented in Graph 7.8. The replication of the short stock position is left as an exercise (see end of Chapter Questions). The combination of the $max[\cdot]$ functions provide 45^o lines through X. In interpreting these diagrams, the net premium terms shifts the 45^o line along the horizontal axis. Due to differences in the time premium for puts and calls this operation will not typically produce a payoff that is anchored exactly centered at $S(0)$, ***even when X = S(0) as may appear to be the case in the diagrams***.[7] This point is also important for interpreting these replication strategies when the options involved are not at the money.

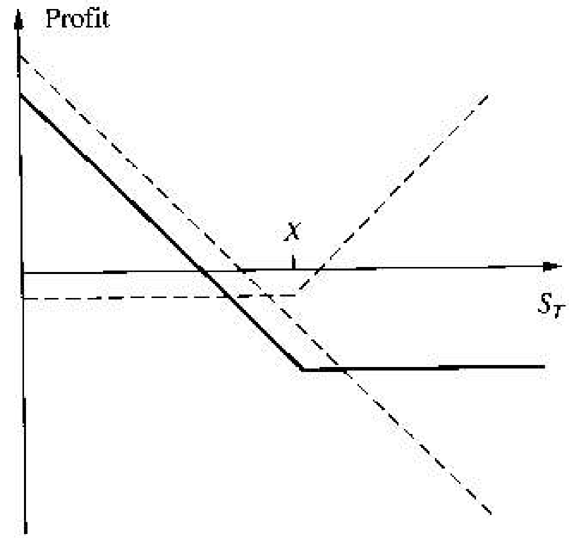
To understand this, consider the nature of the payoffs on a replicated long stock position when $X = 50$ and $S(0) = 60$. In this case, the combination of $max[\cdot]$ functions produce a 45^o line through $X = 50$. The call premium will generate a cash outflow that is a combination of intrinsic expiration value ($60 - $50 = $10) and a time premium. The cash inflow from the put will depend solely on the time premium, as an out of the money put has *zero* expiration value. In this case, as discussed in Sec. 8.1, the time premiums for the put and call will not be equal. Given this, the net cash outflow at $t=0$ will be $10 plus the difference in the call and put time premiums. This will shift the 45^o line to the right by this amount, producing a replicated long stock (spot commodity) position that has a breakeven close to, but not precisely, $S(0)$. Similarly, for the short stock position, with $X = 50$ and $S(0) = 60$, the written call will generate a cash inflow of $10 plus the difference in the call and put time premiums. This will result in a shift of the 45^o line to the right by the amount of the net cash inflow.[8] A similar analysis holds when the put is in the money and the call is out of the money. In this case, the 45^o lines will shift to the left by the amount of the expiration value of the put plus the net time premiums. While, in all cases, a payoff is created that is approximately centered at $S(0)$, as illustrated in Sec. 8.2, there will be substantive differences between the sensitivity to changes in stock prices of the various possible in and out of the money positions.

Graphs 7.4 **Long the Stock, Buy a Put**



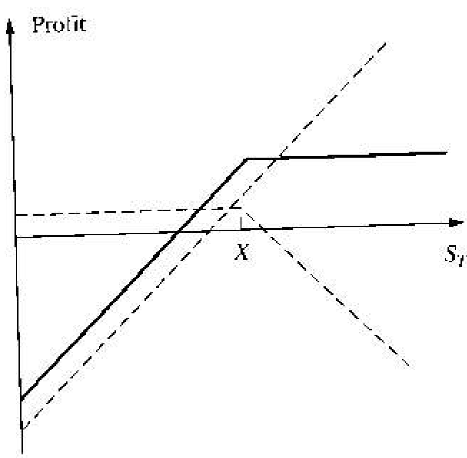
(c)
Long Position in a Stock
Combined with Long Position in a Put

Graph 7.5 **Short the Stock, Buy a Call**



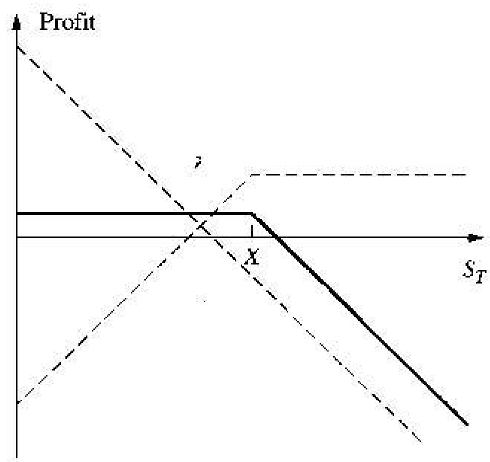
(b)
Short Position in a Stock
Combined with Long Position in a Call

Graph 7.7 **Short the Stock, Write a Put**



(a)
Long Position in a Stock
Combined with Short Position in a Call

Graph 7.6 **Long the Stock, Write a Call**

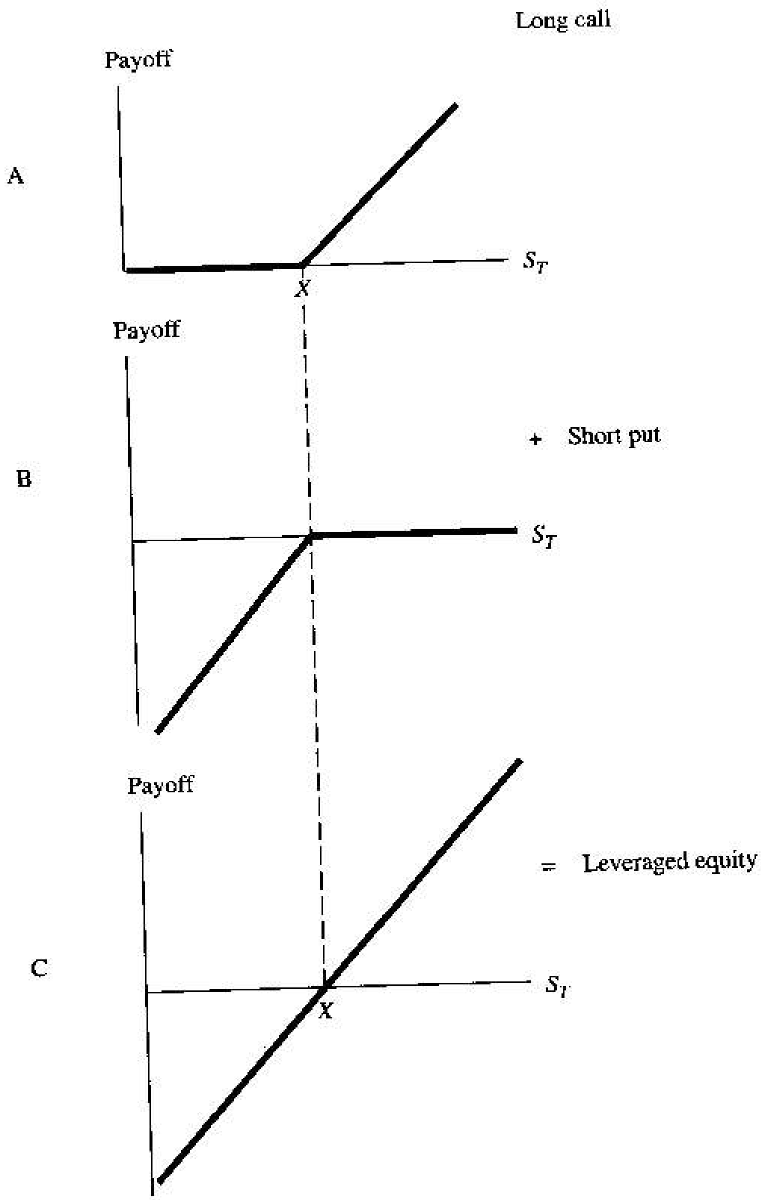


(d)
Short Position in a Stock
Combined with Short Position in a Put

Source: Hull (2000, p.186)

Graph 7.8 **Expiration Date Profit Diagram Approach to Put-Call Parity**

Buy a Call, Write a Put



	$S_T \leq X$	$S_T > X$
Payoff of call held	0	$S_T - X$
Payoff of put written	$-(X - S_T)$	0
Total	$S_T - X$	$S_T - X$

Hedging with Options and Hedging for Options

The expiration date profit diagrams illustrate the wide range of possible payoffs that are constructed by combining options with options or options with spot positions. In the specific context of using options in stock portfolios, Bookstaber and Clarke (1983, p.5) observe: "Potentially, the use of options in combination with a portfolio of stocks can provide the investor with a portfolio containing a wide range of return characteristics. The use of options allows the investor to 'mold' the return distribution of the portfolio to fit a given set of investment objectives. Indeed, the range of returns that can be created through the use of the option market makes the two-dimensional tradeoffs of conventional mean-variance portfolio theory obsolete." (p.5) This observation can be readily extended to commodities other than stocks. In general, the presence of options substantively complicates analysis of the hedging decision, compared to the case where only futures contracts are available.

A significant difference between using options and futures for hedging purposes is that options involve the payment of a premium at $t=0$ while futures do not involve any initial cash flow (ignoring margin and transactions costs). In addition, the sensitivity of options and futures prices to spot price changes also differs. While it is possible to use options to 'replicate' a futures position, this will involve incrementally adjusting (dynamically trading) the size of the options position to account for changes in the spot price. To see this, consider a ***hedge portfolio*** in stocks involving one unit of stock and β units of written call options. The net value of this portfolio, V, will depend on the cash outflow associated with the price of the stock, S, and the cash inflow associated with selling β units of call options at price C. In order to replicate the payoff for a hedge portfolio using options to hedge the stock position, it is necessary to determine β such that $\partial V/\partial S = 0$.

For the option hedge portfolio: $V = S - \beta_c C$. In this equation, V is the net investment in the portfolio and S is the value of the stock position that, for pedagogical purposes, can be taken to be one unit of stock. Observing that C is the price of a call option on the one unit of stock, β_c is the number of ***written*** call options per unit of stock. The value of $\beta_c C$ enters with a minus sign because the written options position generates a cash inflow that can be used to partially offset the cost of buying the stock. Because ΔS is the only source of randomness in the portfolio, the hedge portfolio condition can be used to determine the value of β_c:

$$\frac{\partial V}{\partial S} = 0 = 1 - \beta_c \frac{\partial C}{\partial S} \qquad \Rightarrow \qquad \beta_c = \frac{1}{\dfrac{\partial C}{\partial S}}$$

Because the value of a call option will change as the stock price changes, it is necessary to ***dynamically*** determine β_c, the appropriate number of options to write to maintain the hedge portfolio. One of the useful analytical features of the Black-Scholes option pricing formula is that β can be solved in closed form.

The above discussion involved hedging with options. The inverse problem is encountered by option market makers: how many units of stock to hold in order to hedge a book of traded options positions. As option books usually involve written option positions, the problem of hedging for options involves the value equation: $V = \delta_c S - C$. The choice variable in the hedge is the δ_c, the number of units of stock to purchase (short) to hedge a written (purchased) call option written on one unit of stock. The associated solution for the hedge ratio is:

$$\frac{\partial V}{\partial S} = 0 = \delta_c - \frac{\partial C}{\partial S} \qquad \Rightarrow \qquad \delta_c = \frac{\partial C}{\partial S} = N[d_1]$$

The last equality is solved in Chapter 9 and is included here to illustrate the pedagogical basis for using the hedging-for-options formulation of the riskless hedge portfolio.

Similar conditions can be derived for puts, which can also be used to ***dynamically*** hedge the stock position. In this case, purchased puts are used and the hedge portfolio has the form: $V = S + \beta_p P$, where P is the price of a put option and β_p is the number of puts needed to be purchased in order to hedge one unit of stock. The hedge portfolio condition can now be applied to solve for the number of puts:

$$\frac{\partial V}{\partial S} = 0 = 1 + \beta_p \frac{\partial P}{\partial S} \qquad \Rightarrow \qquad \beta_p = -\frac{1}{\frac{\partial P}{\partial S}}$$

Anticipating the discussion in Chapter 9, the Black-Scholes formula can be used to show that the number of purchased put options is directly related to the number of written call options required to construct the two hedge portfolios. In particular, it will be demonstrated that $\beta_p = \beta_c - 1$.

The dynamic hedging of the options hedge portfolio can be contrasted with the case of the futures hedge portfolio, where the spot position is approximately hedged without further need for dynamic adjustment once a futures position has been established. However, like the hedge portfolio with puts or calls, it could be necessary to dynamically adjust a hedge portfolio created with futures. Using the hedging for futures formulation, the futures hedge has the value function: $V^* = \beta^* S - F$. Assuming that the spot commodity is the same as the commodity underlying the futures hedge, then it is possible to construct a delivery hedge. Using the cash-and-carry arbitrage condition, $F = (1 + ic)S$, it follows that:

$$\frac{\partial V^*}{\partial S} = \beta^* - \frac{\partial F}{\partial S} \qquad \Rightarrow \qquad \beta^* = \frac{\partial F}{\partial S} = (1 + ic) + S\,\frac{\partial ic}{\partial S}$$

The sensitivity of the futures position is not one-to-one with the spot position, though the futures sensitivity is considerably different than the sensitivity of the call option to changes in the stock price, where the ***delta*** of the call determines the size of the hedge. As demonstrated in Chapter 9, the delta of a call option will be bounded between 0 and 1, making β, the inverse of the delta, a positive number greater than 1.

Do Replicated Positions Differ?

The basic replication properties for covered and uncovered option positions can be illustrated using expiration date profit diagrams. Because this technique suppresses accurate accounting for the foregone interest associated with the option premium, it is possible to provide more precise statements of the replication strategies. Heuristically, the basic insights can be illustrated by considering two possible approaches to capturing the payoff of a long (short) spot position. One approach is to directly take a long (short) spot position. The alternative approach to capturing the long spot payoff given by the expiration date profit diagram approach is to buy a call and write a put with the same T and $X = S(0)$. However, the investment required to purchase a stock position is significantly larger than the call plus put alternative. Hence, combining a purchased call plus a written put does not provide accurate replication of a long stock position. (From put-call parity, for $P = C$ it is equivalent to buying stock using 100% margin borrowing).

Invoking an absence-of-arbitrage approach, it is possible to identify the accurate replication trading strategy. More precisely, ***if the call and put options are correctly priced***, investing the remaining balance between the stock price and the net premium on the put and call options in a fixed income security maturing on the expiration date of the options will equate the initial cost of the two approaches. In order for the expiration date payoff on the purchased call plus written put plus fixed income position to be exactly equal the payoff on the long stock position, the maturity value of the fixed income security must be X. This is the basis of put-call parity arbitrage. There are alternative ways of specifying the transactions involved in the arbitrage, such as using the cost of buying the spot position and an appropriate put option for comparison. For absence-of-arbitrage, the cost of purchasing this position is just equal to the price of an appropriate call option combined with an investment of the balance in an appropriately dated fixed income position, with maturity value equal to the exercise price.

Perhaps the most useful application of put-call parity is to specify the relationship between the price of a call and a put. Once the value of the call is determined, the put-call parity arbitrage condition provides the price of the put having the same X and T. As discussed in Sec. 9.3, it is possible to use the put-call parity condition to develop similar but more complicated strategies applicable to replicating the payoff on naked options positions that use ***dynamic hedging strategies*** involving active trading of fixed income and spot positions. One important instance of the dynamic strategies involves replicating the payoff on a long stock position combined with a purchased put, in other words portfolio insurance. This payoff can be replicated by actively trading a portfolio that contains only stocks and bonds, with no derivatives. Because many institutional investors already possess stock/bond portfolios and are often involved in making active trading decisions, the dynamic trading approach is appealing. The advantages of using one particular approach over another will be examined in later Sections.

The possibility of using a number of different methods to achieve a given type of payoff raises some practical considerations. Speaking in the context of stock options, many of the relevant issues are captured by Gibson (1991, p.179):

> We cannot ignore the institutional characteristics of the stock, option markets, and trading restrictions that investors in those markets are actually facing. They are often in sharp contrast with the "perfect markets" paradigm of most option pricing models... These models often ignore the transactions costs and bid/offer spreads investors actually incur. In addition, margin requirements,...short-selling restrictions...differences between the lending and borrowing rates as well as tax considerations will also refrain investors from trading as frequently as predicted.

In effect, practical considerations associated with actual execution costs may determine whether a specific replication strategy is feasible. The practical importance of the institutional characteristic should not be underestimated. For example, certain hedge funds, such as LTCM, can be characterised as: funds designed to exploit inefficiencies that arise in the market valuation of either side in a specific replication strategy. Firms making markets in exotic and mark-to-model OTC derivatives are also attempting to profit by spanning market inefficiencies.

7.2 Put-Call Parity

European Put-Call Parity without Dividends[9]

The most important of all the option replication strategies is the ***put-call parity arbitrage***. There a two possible pedagogical approaches to demonstrating the supporting arbitrage transactions. One approach, which has a history predating the development of the Black-Scholes formula, involves demonstrating that the payoffs on two different portfolios are the same in all future states of the world. If the prices of the two portfolios differ, sell the overpriced portfolio and buy the underpriced portfolio. Hence, the two portfolios must sell for the same price. The alternative approach is more modern and involves specifying an arbitrage portfolio in which there is no net investment of funds. Much as in the discussion of arbitrages for forward and futures contracts, the equilibrium requirement that there be no arbitrage opportunities provides the restriction needed to specify the put-call parity condition. Figures 7.1 and 7.2 illustrate that both approaches provide identical results.

The two portfolio approach demonstrates that the payoff functions for two different portfolios are equal for all possible future outcomes. The arbitrage portfolio approach combines the two portfolios into a single portfolio and observes that the value of the portfolio is zero in all future states of the world. Assuming perfect markets, it follows that the cost of purchasing the two positions should be equal to avoid arbitrage profit opportunities. For a ***European option on a non-dividend paying stock***, at $t=0$, the two portfolios are given in Figure 7.1. In what follows, $PV(\tau)$ is a discounting function, which can be expressed in continuous $(exp\{-rt^*\})$ or discrete $(1/\{1 + r t^*\})$ form where r is annualized and $t^* = \tau/365 = [(T-t)/365]$, the fraction of the year remaining to expiration. Portfolio A is the call option position and Portfolio B is the ***replicating*** portfolio.

Figure 7.1 **The Two Portfolio Approach**

***Strategy A*:** –Buy a call for one unit of stock, with exercise price *X* and days to expiration *T*

Strategy B: - Borrow *X PV(τ)*
–Buy one unit of stock
--Buy a put for one unit of stock, with same *X* and *T* as for the

In the arbitrage portfolio approach, described in Figure 7.2, there is only one portfolio, which satisfies the condition that there is no net investment of funds in the position.

Figure 7.2 **The Abitrage Portfolio Approach**

The Short Arbitrage Portfolio

Strategy : -- Buy a call for one unit of stock, with exercise price *X* and days to expiration *T*
-- Invest *X PV(τ)*
-- Short one unit of stock
-- Write a put for one unit of stock, same *X* and *T* as for the call.

The Long Arbitrage Portfolio

Strategy : -- Write a call for one unit of stock, exercise price *X* and days to expiration *T*
-- Borrow*X PV(τ)*
-- Buy one unit of stock
-- Buy a put for one unit of stock, same *X* and *T* as for the call

Observing that the only admitted source of randomness is the stock price, what is significant about portfolios A and B is the associated payoffs at time *T*, which are given in Figure 7.3. Because there is only one random variable in this world, the stock price, all possible futures states of the world have been taken into account. Figure 7.3 demonstrates that the two strategies have equal expiration date values. Because market equilibrium requires that portfolios with identical payoffs will sell at the same price, the relationship between the market price of puts and calls is established. This condition is specific to the particulars of the security involved in the arbitrage, which is assumed to pay no dividends. It is also specific to the type of option that is being traded, which is a European option. Hence, the condition which is derived is referred to European put-call parity for non-dividend paying securities.

Figure 7.3 **Two Portfolio Approach**

All Possible Values of the Strategies At Expiration

	$X \geq S(T)$	$X < S(T)$
A:	0	S(T) - X
B: Stock	$S(T)$	$S(T)$
Put	$X - S(T)$	0
Loan	$-X$	$-X$
Total B:	0	$S(T) - X$

The same outcome is achieved with the arbitrage portfolio.

Figure 7.4 **Arbitrage Portfolio Approach**

All Possible Values of the Long Strategy At Expiration

	$X \geq S(T)$	$X < S(T)$
Call	0	$S(T) - X$
Stock	$-S(T)$	$-S(T)$
Put	$-(X - S(T))$	0
Zero Bond	X	X
Total	0	0

All Possible Values of the Short Strategy At Expiration

	$X \geq S(T)$	$X < S(T)$
Call	0	$-(S(T) - X)$
Stock	$S(T)$	$S(T)$
Put	$(X - S(T))$	0
Zero Bond	$-X$	$- X$
Total	0	0

The put-call parity condition can now be expressed as:

European Put-Call Parity for Non-Dividend Pay Securities

$$C(S,\tau,X) = P(S,\tau,X) + S(t) - [X\ PV(\tau)]$$

This relatively simple formula has a number of significant practical implications. Taking the case where the options are at the money $(S = X)$, then the call will sell for more than the put because $S = X > X\ e^{-rt^*}$. The *rhs* of the put-call parity condition is sometimes referred to as a ***synthetic call***. It is possible to rearrange the put-call parity condition such that any one of P, S, Xe^{-rt^*} or C appears on the lhs. In each of these cases, the *rhs* is referred as the synthetic position. For example, a ***synthetic put*** would be short the stock, buy a call and invest in bonds. Similarly, a synthetic borrowing would be long the stock, long a put and write a call.

Examining the actual execution of the arbitrage condition, when the call is expensive relative to the synthetic portfolio or, put differently, the actual call is over priced relative to the synthetic call:

$$C(S,\tau,X) + [X\ PV(\tau)] > P(S,\tau,X) + S(t)$$

In this case, the arbitrager would sell the call and borrow money, using these funds to purchase the put and the stock. This particular trade, which is aimed at exploiting call option mis-pricing, is referred to as a ***conversion***.[10] Assuming the "perfect markets" paradigm, the conversion would create an arbitrage profit on the expiration date if the call is overpriced.[11] When the inequality is reversed, the arbitrager would sell the put and short the stock, using the funds to buy the call and invest the balance in zero coupon (pure discount) fixed income securities with maturity date T days ahead. This strategy is known as a ***reverse conversion*** or reversal. The importance of the practical qualifications involved in trading alluded to previously, such as transactions costs, is an important element in determining the profitability of conversions and reversals.

Consideration of the actual trading mechanics involved in the arbitrage reveals the importance of the issues alluded to by Gibson (1991). For example, allowing for significant differences between lending and borrowing rates reduces the equality in put-call parity to a pair of inequality restrictions:

$$S(t) + P(S,\tau,X) - X\ PV(\tau)_B \geq C(S,\tau,X) \geq S(t) + P(S,\tau,X) - X\ PV(\tau)_L$$

The upper boundary refers to the long arbitrage case considered in the last paragraph. The lower bound is associated with the short arbitrage. Another important source of discrepancy that can arise between the short and the long arbitrages, is short-selling costs. This tends to weaken the applicability of the lower boundary. For many types of options, liquidity may be a consideration, affecting the bid/offer spread for the various transactions involved in trading debt, stocks, and both the put and call options. Together with other factors such as margin requirements, marking-to-market, and tax, there is considerable scope for the distance between the put-call parity boundaries to be substantial.[12]

European Put-Call Parity with Dividends

The discussion to this point has focused on European options on non-dividend paying stocks. When discussing the extension of European put-call parity to options on stocks, assets or commodities that pay dividends, it is conventional to assume that the option contract does not allow for dividend payout protection. Such protection could be accomplished in a number ways, including delivering the stock *plus* all dividends paid during the life of the option if the option is exercised. Another method would be to adjust the exercise price for any dividends paid. The absence of dividend payout protection benefits the option writer, who is entitled to receive any dividend payouts during the life of the option. Prior to the advent of the CBOE, it was common for options contract to include dividend payout protections. As option market makers typically write options, it is not surprising that the

CBOE chose to use option contracts without dividend payout protection.

Extending the specification of put-call parity to allow a known future dividend on the underlying stock, or more generally a discrete carry return on the underlying commodity, is not a significant complication, as it only involves adjusting the cash flows to account for the known dividends. More formally, if D is taken to be the present value of all known dividends to be paid between the current date and the options expiration date, T:

$$S(t) + P(S,\tau,X) = X\, PV(\tau) + D + C(S,\tau,X)$$

In this case, the return on a position that is long (short) the stock combined with a purchased (written) put option can be exactly financed by writing (purchasing) a call and borrowing (investing) $X\, PV(\tau) + D$. At maturity, the value of the stock plus accumulated dividends plus the expiration value of the put will just equal the maturity value of the fixed income position, $X + D\, FV(\tau)$, plus the expiration value of the call position, where $FV(\tau)$ is the future value function that corresponds to $PV(\tau)$.

In contrast to known dividend payments, introducing either the early exercise provision of American options or unknown dividend payments undermines the ability to precisely determine the arbitrage transactions. Formally, these features violate requirements needed for ***path independence*** of the option price. For example, at any time $t=0$, if prices are measured continuously, there are a theoretically infinite number of possible paths that the spot price can take. Path independence requires that the payoff on the trading strategy does ***not*** depend on the specific future time path that the spot price actually takes.[13] Otherwise, the strategy is ***path dependent*** and will give a payoff that is uncertain at the decision date, $t=0$. For example, path dependence occurs with the American put because some spot price paths will be sufficiently close to zero that it is profitable to exercise early. As demonstrated in Sec. 7.3, even when future dividend payments are known, early exercise can occur in certain situations, e.g., just before the last ex-date for an American call. Early exercise will also occur when the time premium is negative. Because the occurrence of these events is not known at $t=0$, the associated strategies are path dependent. Similarly for stock options, unknown dividends on the underlying stock means that the payoff on the strategy cannot be determined at $t=0$ and again path dependence emerges.

European Put-Call Parity for Options on Forward and Futures Contracts

Put-call parity for options on forward and futures contracts retains all the features associated with options on spot commodities, with the exception that the net investment in the position is different because, in perfect markets, the forward or futures contract is theoretically costless to create. This means that the net pure discount bond position in the replicating portfolio is equal to the difference between the put and call premiums. There is no longer funds required for investment in the spot. For replicating portfolios involving the spot commodity, the portfolio always involved a pure discount loan if the spot position was long and a pure discount bond purchase if the spot position was short.

For options on forwards and futures contracts, the net investment is determined by the difference between $F(t,T)$ and X, which determines the relative size of the call and put premiums. If $F(t,T) > X$, then the call is in-the-money and the put is out-of-the-money. Using the two portfolio approach, the small put premium has to be incremented by a discount bond purchase to offset the size of the call premium. Similarly, if $F(t,T) < X$, then the call is out-of-the-money and the put is in-the-money. The call premium in this case is small requiring the cost of the large put premium to be financed by discount bond borrowing. Using the arbitrage portfolio approach the argument is messier to explain. The long arbitrage portfolio will require a discount bond purchase when $X > F(t,T)$ and a discount bond borrowing when $F(t,T) > X$.

A minor complication in the specification of the replicating portfolios using forward and futures contracts occurs with the specification of the expiration date. For example, many futures options involve delivery of a futures contract on the expiration date, preventing the same T to be used for the delivery date of the future and the expiration date of the option. Option expiration date settlement involves delivery of the relevant futures contract together with a cash payment reflecting the difference between the options expiration date futures price and the option exercise price $(max[0, F(T,T+1) - X])$. For other futures options, the expiration date of the futures and options is the same. In this case, the option payoff is a cash payment that reflects the difference between the futures expiration date price and the option exercise price *(max[0, F(T,T) - X])*. For ease of illustration, forward contracts with the same T as for the option will be assumed for the $F(t,T) > X$ case and forward contract with different T from the options for the $F(t,T) < X$ case. With appropriate modification, ***the replicating portfolio argument also applies to the other cases as well as for futures contracts.***

Figure 7.5 **Two Portfolio Approach to Put-Call Parity for Options on Forward Contracts**

F(t,T) > X, Call in-the-money, Put out-of-the-money

Strategy A: Purchase a call option on a forward contract with option exercise price X and expiration date T and forward contract delivery date T involving delivery of one unit of spot commodity.

Strategy B**:** --Lend $(F(t,T) - X)\ PV(\tau)$

--Establish a long forward position at price $F(t,T)$,requiring delivery of one unit of the spot commodity.

--Buy a put on the same forward contract with the same X and T as for the call.

For the $F(t,T) > X$ case, the call is in-the-money and the put is out-of-the-money. The relevant theoretical portfolios are described in the following schematic. As before, Portfolio A is the call option position and Portfolio B is the ***replicating*** portfolio. After allowing for the cost of paying the put premium, equating the values of the portfolios means that there will be money left over to lend in Portfolio B. Using the result that $F(T,T) = S(T)$, the resulting payoffs at expiration are determined.

Figure 7.6 **Two Portfolio Approach for Options on Forward Contracts, *F(t,T) > X***

All Possible Values of the Strategies At Expiration

	$X \geq S(T)$	$X < S(T)$
A:	0	$S(T) - X$
B: Forward	$S(T)-F(t,T)$	$S(T) - F(t,T)$
Put	$X - S(T)$	0
Lending	$F(t,T) - X$	$F(t,T) - X$
Total B:	0	$S(T) - X$

Figure 7.7 **Two Portfolio Approach to Put-Call Parity for *F(t,T+1)***

F(t,T+1) < X, Call out-of-the-money, Put in-the-money

***Strategy A*:** Purchase a call option on a forward contract with exercise price X and expiration date T, requiring delivery of one unit of a forward contract with delivery date $T+1$

Strategy B: --Borrow $(X - F(t,T+1))\, PV(\tau)$

--Establish a long forward position, requiring delivery of one unit of the spot commodity, at $Ft,T+1)$

--Buy a put on the same forward contract with the same X and T as for the call.

For the $F(t,T) < X$ case, the put is in-the-money and the call is out-of-the-money. In order to equalize the value of the two portfolios it will be necessary to borrow money in Portfolio B. The relevant theoretical portfolios are described in the following schematic. As before, Portfolio A is the call option position and Portfolio B is the ***replicating*** portfolio. Because expiration date (T) delivery of this option involves a forward contract with delivery at $T+1$, the resulting payoffs at expiration are determined.

All Possible Values of the Strategies At Expiration

		$X \geq S(T)$	$X < S(T)$
A:		0	$F(T,T+1) - X$
B:	Forward	$F(T,T+1)-F(t,T+1)$	$F(T,T+1) - F(t,T+1)$
	Put	$X - F(T,T+1)$	0
	Invest	$F(t,T+1) - X$	$F(t,T+1) - X$

The relationship between the price of puts and calls for forward and futures contracts is given in the following. The two strategies have equal value on the expiration date:

European Put-Call Parity Condition for Options on Forwards

$$C(S,\tau,X) = P(S,\tau,X) + [F(t,T) - X]\, PV(\tau)$$

This is the put-call parity condition for forward and futures contracts.

American Put-Call Parity

Derivation of the condition for put-call parity on American options requires the use of a number of the distribution free properties, especially Properties 10 and 11. One implication of Property 11 is that American call options on ***non-dividend paying stocks*** will not be exercised early. It will always be better to sell the American option than to exercise it. Hence, because there is no rational reason to exercise the call early, the value of the European and American call options will be the same. This is not the case for American call options on dividend paying stocks or for any type of American put option, whether the underlying commodity pays a dividend or not. So, for the case of options on non-dividend paying stocks, it is not possible to specify an equality relationship between the price of put and call options. Rather, all that can be derived are upper and lower boundary conditions on the put price provided by arbitrage transactions involving calls.

The lower boundary on the put follows from Property 3 which states that, if there is a possibility of early exercise, then the value of the American option will be greater than the value of the European option. Because the values of the European and American calls are equal for a non-dividend paying security:

$$P_A[S,t^*,X] \geq C_A[S,t^*,X] + X\,PV[t^*] - S(t)$$

To derive the upper bound on the put option, let Portfolio A be long the put with price $P_A[S,t^*,X]$ and let Portfolio B be long the call at $C[S,t^*,X]$, short the stock at $S(t)$ and invest X in a bond earning $r(t,T)$ with maturity date T.

To evaluate the put-call parity conditions for American options requires the expiration date payoff to be examined, together with the value of the payoff if exercise is initiated prior to the expiration date. The expiration date payoff for the two portfolios is given in the following schematic. Observing the $1/PV[t^*] = exp\{rt^*\} > 1$, in all future states of the world Portfolio B will have a higher payoff than Portfolio A. Similarly, for all times prior to maturity, the same schematic applies, with the proviso that T is changed to the exercise date. This produces the upper bound condition for the American put:

$$C_A[S,t^*,X] + X - S(t) \geq P_A[S,t^*,X]$$

Figure 7.8 **American Option Put-Call Values**

All Possible Values of the Strategies At Expiration

		$X \leq S(T)$	$X > S(T)$
A:		0	$X - S(T)$
B:	Stock	$-S(T)$	$-S(T)$
	Call	$S(T) - X$	0
	Invest	$X/PV[t^*]$	$X/PV[t^*]$
Total B:		$X\,(1/PV[t^*] - 1)$	$X/PV[t^*] - S(T)$

Combining these two conditions produces the upper and lower put-call parity boundaries for the American put.

Extending the put-call parity condition to American options on dividend paying stocks follows much the same procedure as for Europeans. It is assumed that the dividend payment streams are known with certainty, which permits the present value of future dividend payments (D) to be determined. It is left as an exercise to demonstrate that the arbitrage boundaries will be:

$$C_A[S,t^*,X] + X + D - S(t) \geq P_A[S,t^*,X] \geq C_A[S,t^*,X] + X\,PV[t^*] - S(t)$$

In practice, because most traded options are American, it is unfortunate that the put-call parity conditions do not provide the same type of sharp restrictions on American put and call prices as are provided for European options.

Using Put-Call Parity to Estimate the Early Exercise Premium

In practice, almost all exchange traded options have the American feature. Yet, closed form pricing formulas for options are usually only available for options with the European feature. As a consequence, information about the empirical behavior of the early exercise premium (*EEP*) is of considerable interest. For example, Zivney (1991) suggests that empirical estimates of the *EEP* can be used as inputs to the Hull and White (1988) control variate technique to numerically obtain American option prices. There are various possible methods of estimating the *EEP*. For example, examining foreign currency options Jorion and Stoughton (1989) derive comparative statics for the *EEP* under the assumption that the spot exchange rate follows a diffusion. Critical exercise values associated with specific values of the spot exchange rate are identified for both puts and calls. The comparative static conditions are then estimated empirically using regressions with the difference between the American and European prices for the same underlying spot exchange rate. Unfortunately, the reported empirical results were not impressive indicating that the methodology is not the best technique to use in evaluating empirical *EEP* behavior.

Direct estimates of the *EEP* are not essential to obtaining American option prices. There are various methods available for numerically estimating American prices, e.g., Hull (2000, Chapter 16). Once the American option price is obtained, a value for the *EEP* can be estimated by differencing the estimated American price and the European price obtained from Black-Scholes. Estimated over a range of times to expiration and exercise prices, it is possible to obtain an estimate for the early exercise boundary. Bodurtha and Courtadon (1995) is an excellent example of how the early exercise boundary can be directly estimated using numerical methods. While useful, such methods are not readily accessible and results depend on the accuracy of the pricing models selected. For example, Bodurthat and Courtadon (1995) report that not all the options are exercised that the estimated early exercise boundary predicted would be exercised.

The direct approach to evaluating EEP involves examining the difference in the prices of European and American options written on the same commodity. A major difficulty with the direct approach is the relative absence of European option prices. Zivney (1991) and de Roon and Veld (1996) have used a creative approach based on the European put-call parity condition. Recall that, for spot commodities paying dividends, that:

$$S(t) + P(S,\tau,X) = X\,PV(\tau) + D + C(S,\tau,X)$$

It follows that: $C - P = S - XPV - D$. Using the result that $C_A = C + EEP_C$ and $P_A = P + EEP_P$ it follows that:

$$C_A - P_A = C + EEP_C - P - EEP_P = S - XPV - D + (EEP_C - EEP_P)$$

$$\rightarrow \quad EEP_C - EEP_P = C_A - P_A - (S - XPV - D) \qquad (7.1)$$

d

The difference between the *EEP* for calls and puts is now represented by values that are obtainable from American option prices and observable cash market values.

An interesting development on this approach is provided by de Roon and Veld (1996) where the DAX index traded on the American Stock Exchange was used in place of the S&P 100 index options used by Zivney (1991). The DAX index is a performance index where dividends paid on the underlying securities in the index are reinvested, unlike the S&P 100 index options where dividends are not paid. In this case, the index call option will not be exercised early and any deviation in the rhs of (7.1) can only be due to EEP_P. Zivney (1991) found: the early exercise premium for puts was greater than for calls, *ceteris parabus*; and, *EEP* increases with time to expiration, moneyness and the riskless rate of interest. Similar results were found by de Roon and Veld (1996), though the time to expiration results did differ. Both studies suggest that American option pricing models may fail to capture all the nuances of the early exercise decision.

7.3 Spread Trades and Strategies[14]

Straddles, Straps and Strangles

Two general types of speculative trading strategies for options can be identified.[15] The first type relies on option mis-pricing; effectively, deviations of observed options prices from theoretical values as determined by an applicable option pricing formula or a theoretical arbitrage boundary condition. Examples of these types of trades are conversions, reversals, ratio spreads and box spreads. Due to transactions costs and difficulties in obtaining real time option quotes, these strategies are difficult to execute profitably for non-exchange traders. The amount of mis-pricing is typically of a small enough order that access to cheap transactions and execution costs, associated with exchange member trading, is required. The other types of strategies are natural extensions of the speculative trades examined in Chapter 3. The profitability of these trades require a view on the direction or volatility of spot prices, or some other variable such as interest rates. The associated trades, and their profit functions, will be the central concern of this Section. For want of a better description, these trades will be referred to as directional trades.

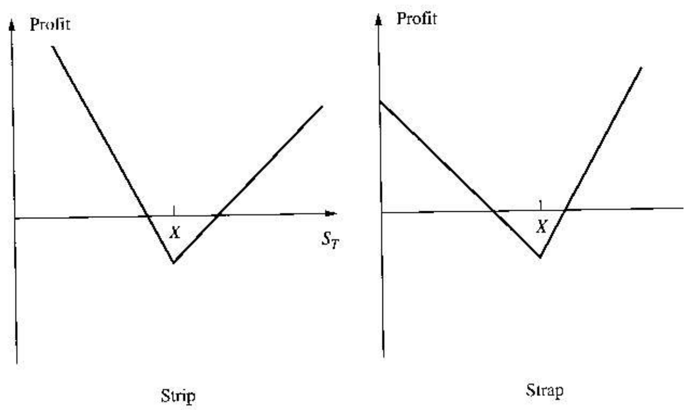


Figure 8.11 Profit patterns from a strip and a strap.

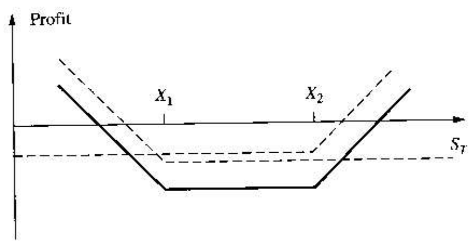


Figure 8.12 A strangle.

Two general types of directional trades can be identified: ***spreads*** and ***combinations***. Combinations involve taking positions in both puts and calls on the same security, as in a straddle, while a spread traded involves taking positions in 2 or more options of the same type, i.e., 2 (or more) calls, two (or more puts), as in a butterfly or vertical spread. At a basic level, spreads involve the simultaneous purchase of one option and the sale of another, usually on the same spot commodity, with the two options differing in exercise price and/or time to expiration. If the options differ in exercise price, but have the same expiration date, the trade is a ***vertical spread***. If the options differ in expiration date, with the same exercise price, the trade is a ***horizontal spread*** or time spread. If the options combine difference in both time and exercise price, the trade is a ***diagonal spread***. As with futures, an intra-commodity spread involves the same spot commodity for both options, while the inter-commodity spread will use different commodities. Unlike spreads, combinations involve having the same type of position, either long or short, in the relevant options. A ***straddle*** involves combining a put and a call, on the same commodity, with the same exercise price and time to maturity (see Graph 7.9). When the exercise price of the put is less than the exercise price of the call, the position is called a ***strangle*** (see Graph 7.10 Variations on the straddle include the ***strap***, 2 calls and one put, and the ***strip***, 2 puts and one call (see Graphs 7.11).

Using the assumptions that interest on premiums is ignored, the expiration date profit diagram can be used to illustrate the terminal payoffs associated with spreads and combinations. Prior to expiration more advanced valuation techniques are required. These complications will be explored in Chapter 8. Consider the case of a purchased straddle, with both options having exercise price X-- not necessarily at the money--and expiring on the same date (see Graph 7.9). Selling the straddle, again with the same X and expiration date for both options, will be the negative of the purchased straddle.[16] The importance of stock price volatility to the profitability of these trades should be apparent. A purchased straddle requires that spot commodity prices move a greater amount, either up or down, than is implied in the option premium (plus the foregone interest that is being ignored by assumption). Selling the straddle is the reverse, the volatility of the spot commodity will be less than implied by the option premium (less the interest). The key feature of these trades is the reliance on volatility to determine trade profitability. The direction of spot prices does not matter, only the movement will be greater (less) than some amount for the purchased (sold) straddle.

In certain cases, the speculative trader may have notions both about the direction of prices and about volatility. For example, there may be a potential takeover bid emerging for a company. If the bid is announced the stock price will increase substantially. However, if the bid collapses, then prices may fall substantially. The weight of evidence may be in favor of either alternative. In either case, volatility will increase substantially. In these types of cases, if it is more likely that the bid will successfully materialize, it is possible to purchase a strap (see Graph 7.11). Similarly, if the bid is viewed unfavorably and prices are felt to be more likely to fall than rise, then a strip could be purchased (see Graph 7.11). It is also be possible to exploit this type information by selling straps and strips. For example, instead of purchasing a strap (strip), it is also possible to write a strip (strap). However, as was the case with the straddle, the written positions have a bounded expiration date profit function, versus the purchased positions where the profit functions are unbounded. Hence, for the written position, straps and strips allow the seller to generate additional premium income based on a prediction about the direction of spot prices. Variations on straps, strips and straddles can be developed based on where the exercise price of the options is relative to the value of the spot commodity, i.e., whether the option is in-the-money, out-of-the-money or at-the-money.

Vertical and Horizontal Spreads

While there are a number of similarities between speculative trading strategies in futures and options, there are notable differences in the nature of the payoffs. This is the case with option spread trades where, despite combining purchased (long) and written (short) positions, the futures/options payoffs differ significantly. Three basic types of intra-commodity spread trades can be identified: ***vertical spreads***, where the options being combined have different exercise prices; ***horizontal spreads***, where the expiration dates differ; and, diagonal spreads, which involve combining options with different exercise prices and time to expiration.[17] Much as in the futures discussion in chapters 3 and 5, it is possible to extend these notions to inter-commodity trading, but this development will not be explored here. In addition, it is possible to use various types of replication strategies to approximate one or both of the options used in the spread position as well as to replicate the spread payoff function directly. This point will be developed more fully in Sec. 9.2 where the concepts of synthetic security design will be developed more precisely. Other extensions of spreads include the butterfly trade, which has a substantively different payoff function than for the futures case. In turn, the butterfly trade can be used to provide restrictions on the premiums for options with different exercise prices.

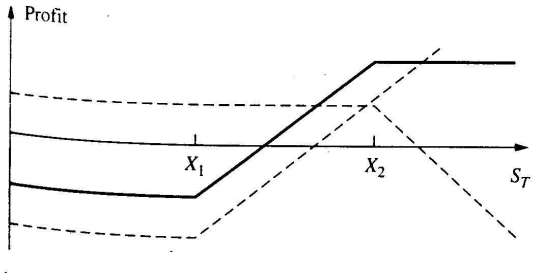
Consider a vertical spread trade that combines a purchased call at X_1 with a written call at X_2, where $X_2 > X_1$ (Figure 7.9). This is sometimes referred to as a "bullish" vertical spread. The effect of the written call position is to tradeoff a portion of the upside potential of the purchased call for a reduction in the (net) premium paid. Similarly, a "bearish" vertical spread would involve writing a call at X_1 and purchasing a call at X_2 (Figure 7.10). In this case, the trader is seeking to eliminate the unbounded nature of the written call, in exchange for some reduction in (net) premium income received. Vertical spreads can also be established for puts (Figures 7.11 and 7.12). In one case, the purchased put is at X_2 with a written put at X_1, where $X_2 > X_1$. This reduces the (net) premium paid to purchase the put. Where the written put is at X_2 and the purchased put at X_1, (net) premium income is reduced in order to bound the payoff function. By recalling the replication strategies described previously, various extensions are possible. For example, the "bullish" vertical spread. The purchased call position can be replicated with long the cash plus buy at put at X_1. Hence, long the stock, combined with at purchased put at X_1 and a written call at X_2 will 'replicate' the payoff on a vertical spread. At this point, the basis connection to caps, collars and floors should be apparent.

While not difficult conceptually, horizontal, or time, spreads are problematic to depict using expiration date profit diagrams. This is because the expiration date can only apply to one of the options, the value of the other option on the expiration date of the first option must be estimated. To see this, consider a horizontal in GM options, observed on Oct. 20 with a current GM stock price of $54 7/8 and an exercise price of $55.

Dec. 2 7/8 Mar 4 3/4 June 5 1/4

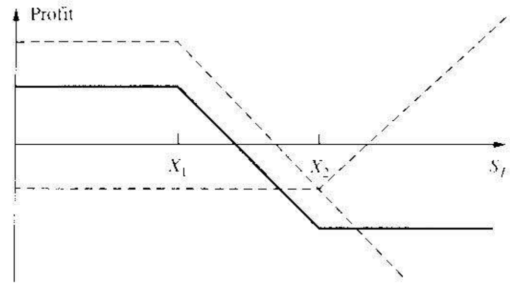
If the horizontal spread involves purchasing the March option and selling the December, then the "basis" on the trade is -1 7/8. This is the net premium income that has to be paid to establish the spread on Oct. 20. Consider the payoff on this trade on the expiration date of the December option given in Figure 7.9. In deriving this payoff profile, it was necessary to ***estimate*** the value of the Mar 55 option that would prevail on the expiration date of the Dec. 55 option.

Figure 7.9 **Bullish Vertical Call Spread**



Buy a call at X_1 Write a Call at X_2

Figure 7.10 **Bearish Vertical Call Spread**

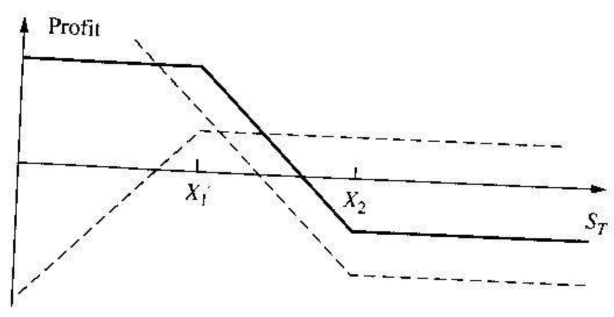


Write a call at X_1 Buy a Call at X_2

Figure 7.11 **Bullish Vertical Put Spread**

Buy a put at X_1 Write a put at X_2

Figure 7.12 **Bearish Vertical Put Spread**



Buy a put at X_1 Write a put at X_2

Figure 7.13 **Time Spread**

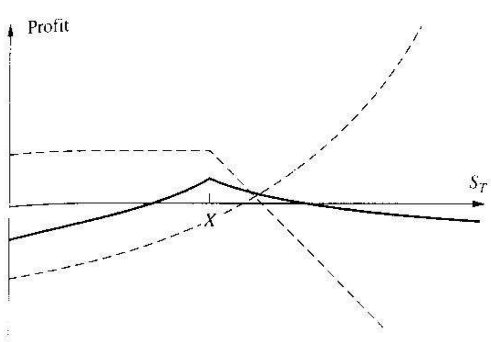
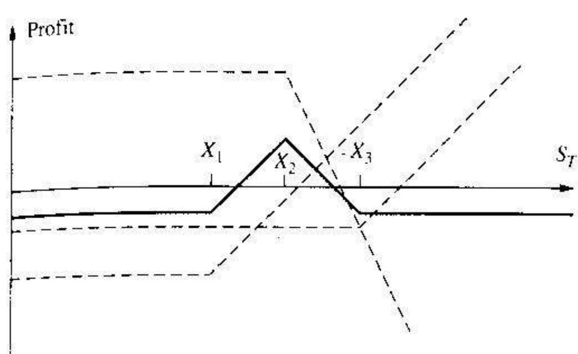
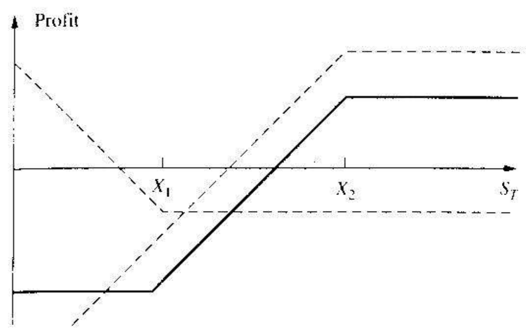


Figure 7.14 **Sandwich Spread**




Profit
X_1
X_2
S_T

Butterflies, Sandwiches and Other Trades

The expiration date profit diagram for the time spread associated with the GM example is given in Figure 7.13. Because of the need to estimate the price of one of the options, the motivation for doing a horizontal spread is closely connected to the problem of how options are priced, more precisely how time decay affects options with different expiration dates. This problem will be addressed analytically in Sec. 9.1. Given this, the final spread trade to considered is the intra-commodity ***butterfly*** that involves taking positions in options with three different exercise prices. Specifically, for $X_1 < X_2 < X_3$, a butterfly involves selling one option at both X_1 and X_3 and buying two options at X_2. While similar in construction to the its futures counterpart, the shape of the expiration date profit diagram for the options butterfly trade is responsible for the name attached to this trade (see end of chapter questions). Conceptually, a butterfly can be viewed as a bounded straddle. Because of the different shape of the expiration date profit diagram, the opposite trade that involves purchasing options at both X_1 and X_3 and writing at X_2 is sometimes referred to as a ***sandwich*** (see Figure 7.14). Much as in the futures case, the small potential profits associated with the butterfly restricts the usefulness of this trade primarily to exchange members.

The list of trades considered to this point covers trades that depend on predicting the direction or volatility of some random variable, usually the spot price. Option strategies also include trades based on mispricing, that are primarily of interest to traders on the option exchanges. The mechanics of trades are more complicated, as evidenced in an examination of two important trades aimed at exploiting mis-pricing: the ***ratio spread*** and the ***box spread***. A ratio spread is a trading strategy aimed at simultaneously selling an overpriced (correctly priced) option and buying a comparable correctly priced (underpriced) option on the same stock. The comparable option could differ either in time to expiration or exercise price. Execution of the ratio spread requires some method for determining the appropriate number of contracts to initiate in the two options in order to be properly hedged. As discussed in Chap. 9, the ratio of the derivatives of the Black-Scholes formula with respect to the spot price for the two options, the ***option deltas***, provides the relevant hedge ratio.

A box spread combines bullish (bearish) vertical spreads using calls with bearish (bullish) vertical spreads using puts, using the same $X1$, $X2$ and T, e.g., Billingsley and Chance (1985), Hemler (1997). The trade is designed to exploit mis-pricing across exercise prices. To see this, consider the expiration date payoff on a box spread. The bullish vertical spread using calls involves buying at $X1$ and writing at $X2$ $(X1 < X2)$ to give: $max[0, S(T)-X1] - max[0, S(T)-X2]$. Similarly, the bearish vertical spread using puts gives: $max[0, X2-S(T)] - max[0,X1-S(T)]$. If $S(T) > X2$, then the calls positions payoff $X2-X1$ and both puts expire worthless. If $X1 < S(T) < X2$, then the calls payoff $S(T)-X1$ and the puts provide $X2-S(T)$ or, combining the two positions, $X2-X1$. Finally, if $S(T) < X1 < X2$, then the calls expire worthless and the puts provide $X2-X1$. Hence, in all future states of the world, the box spread will payoff $X2-X1$. Mis-pricing occurs when the discounted value of $X2-X1$ does not equal the net premium generated from the box spread for $t < T$. If the discounted value of $X2-X1$ is greater than the net premium, the bullish call spread and bearish put spread are purchased. If the value is less than the net premium, a bearish call spread combined with a bullish put spread is purchased. This case involves generating ***positive*** net premiums at t that can be invested for t^* to earn a greater amount than the payout of $X2-X1$ at T.

Caps, Floors and Collars

Even though caps, collars and floors are available for a number of different commodities, these instruments are most widely used in adjusting cash flows originating from floating rate debt securities. There is an active OTC market in medium-to-long term caps, collars and floors, collectively known as the ***cap market***. An interest rate cap is an agreement between two parties: the provider of the cap, typically a large financial institution; and, a cap purchaser, often, though not always, a borrower in the floating rate debt market. The cap agreement specifies a par value, a reference rate, typically LIBOR, a term (e.g., 5 years) and a cap level or ceiling rate, often specified as some number of basis points above current LIBOR. If the reference rate goes above the cap level, the provider agrees to make payments, based on the par value, sufficient to keep the cap purchasers interest payments at the cap level. If the reference rate stays below the ceiling rate, no payments are made. In exchange for entering into the cap agreement, the cap purchaser agrees to pay a premium to the provider. When incorporated into a debt issue, a cap provision adds some basis points to the cost of the borrowing.

A floor is the reverse of a cap. The floor is an agreement where the floor provider, often a borrower in the floating rate debt market, agrees to make payments to the purchaser when the reference rate falls below the stated floor. Again, the size of the payments depend on the par value and the number of basis points the reference rate is below the floor. Adding a floor to a floating rate borrowing reduces the cost of the borrowing by some basis points. A collar is a combination of a cap and a floor. This agreement effectively limits the interest rate payments on a floating rate borrowing to a band, determined by the cap and floor rates. An advantage of a collar over a cap is that the premium received for the floor will offset the cost of the cap, making the interest rate hedge less expensive. The resulting cash flows from the debt instrument are a hybrid possessing features of both floating rate and fixed rate debt. In the limit, adjusting a floating rate debt issue by selling a floor and buying a cap, with a reference rate equal to the current interest rate, will transform the floating rate debt into fixed rate debt. Similarly, adjusting fixed rate debt by buying a floor and selling a cap, has the opposite affect.

The ability to combine debt issues with caps and floors to transform floating rate debt into fixed rate debt, and vice versa, implies that there is a direct connection between the pricing of interest rate swaps and the pricing of caps and floors. For the case of single cash flows, there is a direct connection between caps, floors and options. In practice, a cap can be priced as a sequence of put options on individual cash flows, known as ***caplets***. Consider what a caplet offers, if the reference interest rate rises above the ceiling rate, then a payment is made based on the size of the difference and the par value. The connection to options now follows from taking t^* to be the floating rate reset interval, X to be the present value of the exercise price, discounted at the ceiling rate c, $X = exp\{-ct^*\}$, and B to be the present value of a zero coupon bond discounted at the observed reference rate r, $B = exp\{-rt^*\}$. The payoff on the caplet can now be viewed as a put option: $P = max\{0, X - B\}$. In discrete time, which is the way the swap payout is determined in practice: $P = max\{0, ((r-c)t^*)/(1 + rt^*)\}$. Briys, et al. (1991) demonstrate the approximate equivalence of these two formulations. The floor can be similarly interpreted as a sequence of call options on individual cash flows, with the exercise price determined by the floor rate, and payments on the individual options made on the reset dates. The purchased collar can be interpreted as a long position in the cap, at exercise price $X1$, and a short position in the floor, at exercise price $X2$ $(X2 > X1)$.

7.4 Real Options, Insurance and the Demand for Put Options

Real Options

Options occur in many other forms than the exchange traded variations. In some cases, the embedded option is apparent, as with a callable or convertible bond. A floating rate loan with an interest rate cap, collar or floor is another example. Less obvious examples are insurance and common stock. In response to shortcomings in the traditional net present value decision rule, corporate finance and real estate economics have developed valuation models that incorporate the numerous types of ***real options*** which arise in those areas, e.g., Dixit and Pindyck (1994), Trigeorgis (1996), Brennan and Trigeorgis (2000). Real option notions have also had a significant impact in modeling the economics of capital investment. A list of the most important of these options includes: the option to defer, the option to re-develop; the time-to-build option, the option to alter operating scale, the option to abandon or mothball, the option to switch, and growth options. Like conventional options, real options have some exercise 'price' and may be exercised if the option is in-the-money, but exercise is not required if the option is out-of-the-money. The exercise decision is irreversible. Unlike most conventional options, real options are usually long-dated and are often valued as perpetuals, e.g, Capozza and Li (1994).

Real options also play a role in firm capital budgeting and real asset investment decisions. The traditional net present value (NPV) rule can be stated: in the absence of capital rationing, invest in projects that have NPV greater than zero, e.g., Brealey and Myers (1992). Despite being central to capital budgeting and the macroeconomic theory of investment, this rule fails in many important situations due to the presence of real options that can induce a firm to forego investments with positive NPV. Important features of investment decision problems where the rule fails contain some combination of irreversibility, timing and uncertainty. Irreversibility means that an investment has an element of sunk costs. For example, in real estate, a decision to tear down an existing structure and build a more expensive structure designed to generate more rents is irreversible. The decision to redevelop the property involves exercising the option to defer development to a later time, when expected rentals may be higher. In mining, a decision to abandon a mine is partially irreversible, because the costs associated with startup are similar to the cost of starting a new mining operation. Mine closure is an exercise of the real option to abandon, which depends on the quality of the ore being produced and the expected price of the metal.

Closely related to the real options, such as the development option, are executive stock options. Unlike real options, which are largely of theoretical interest and deal with the return on real assets, executive stock options (ESOs) are non-traded securities which have to be valued in order to satisfy GAAP. ESOs are an important component of modern corporate finance, with roughly 3/4 of larger corporations having such plans, with unexercised ESOs accounting for roughly 13% of the number of shares outstanding. To address the important accounting implications of ESOs, FAS 123 (FASB 1995) deals with "Accounting for Stock-Based Compensation", providing a modified Black-Scholes model (see Chapters 8 and 9) to determine fair value of ESOs for use in financial disclosure statements. Unlike exchange traded options, ESOs are typically subject to numerous restrictions including forfeiture when an employee leaves the company and the inability to sell or hedge the ESO position. These restrictions make it difficult to obtain a precise value for an ESO. The FASB model uses empirical values for expected forfeiture and exercise behavior to determine an estimated option value. The adjustments result in a lower value for an ESO relative to an exchange traded option.

Considerable debate still surrounds the question of whether FAS 123 is the appropriate method to value ESOs, e.g., Huddart and Lang (1996), Carpenter (1998), Soffer (2000). There does not seem to be a consensus on whether FAS 123 results in option values which are too high or too low. For example, Cuny and Jorion (1995) argue stock price performance and employee turnover are highly correlated. In assuming these two variables are uncorrelated, the FAS 123 methodology will undervalue ESOs. Hemmer et al. (1994) argue that by using the expected time to exercise instead of actual time to exercise, FAS 123 uses overstates the value of ESOs. Some options will be exercised earlier and some options will be exercised later than the expected exercise date. Yet, the option value is a concave function of the time to expiration, resulting in an overvaluation of the option value by using the expected time to expiration. The FAS 123 methodology also has supporters. For example, working within a more sophisticated model which incorporate numerous elements not included in FAS 123, e.g., employee risk aversion, Carpenter (1998) demonstrates that, despite the imperfections, FAS 123 does provide a reasonable estimate for the value of ESOs.

Insurance and Option Pricing

At least since Merton (1977) and Smith (1979), it has been recognized that insurance valuation is a potential application of the options pricing methodology discussed in Chapters 8 and 9. Though other methods of obtaining premium payments are available, it is possible to successfully price insurance premiums using options pricing methods, e.g., Doherty and Garven (1986), Cummins (1988), Shimko (1992), Phillips et al. (1998). The extension of options pricing to insurance is not as easy as might appear. It is possible, for example, to conceive of insurance as a form of put option. For example, car insurance involves the payment of a premium in exchange for the right to sell the car back to the insurance company (or at least the damaged part) under conditions laid out in the insurance policy. Similarly, a life insurance premium is made in exchange for the right to receive a cash payment in the event that the death of a specific person occurs in the period up to when the next premium payment is due. Precisely how options pricing methodology could be used in these instances is not always apparent.

Shimko (1992, p.229) identifies three reasons that insurance policies complicate option pricing: "1. For many lines of insurance, a policy holder may submit multiple claims. 2. Policies are typically written with prescribed deductibles and maximum coverage limits. The non-linear nature of these loss-sharing rules creates aggregation problems similar to those encountered in the valuation of portfolios of options. 3. The size and frequency of losses may vary systematically, which requires the calculation of a risk premium that is also affected by the option-like characteristics of the insurance policies." Further complications for option pricing arise when it is recognized that most insurance companies write policies in a number of lines of business, e.g., automobile, property, general liability, and are subject to default risk ((Phillips et al. 1998). Multiple line insurance companies hold equity in a common pool, where the different lines are subject to different risks. Option pricing methods can be used to determine how to allocate equity capital to the different lines of business.

Skewness Preference and the Demand for Put Options

Sec. 2.1 identified the maximization of expected utility for end of period wealth as an appropriate objective for risk management decision making. At various points, optimal hedge ratios have been derived, using the mean-variance moment preference function and other forms of expected utility functions. These techniques can be applied to determine the optimal demand for options, as well as futures and forward contracts. Yet, the payoffs on futures and forward contracts are linear, while option payoffs are non-linear. Linear payoffs work directly on the variance. For example, a full hedge with no basis risk effectively reduces the variance of spot prices to zero. Options work by altering the shape of the return distribution. If desired, options can be used to reduce the dispersion of the return distribution, e.g., using a purchased put to truncate negative returns associated with a stock price falling below the exercise price. However, the non-linear payoff on options will, of necessity, impact the higher moments of the return distribution, particularly the skewness. Bookstaber and Clarke (1983) have numerous useful graphs of the various distributional shapes associated with option usage.

As risk management products, options are much like insurance. As discussed in Sec. 2.2, insurance theory is concerned with pricing the risks associated with situations involving loss or no loss. As such, the main objective of insurance is to reduce the possibility of (extreme) negative returns. This is equivalent to saying that insurance is used to reduce negative skewness in the return distribution. If the firm is long the spot commodity being hedged, then purchased put options would be used. As such, the use of put options is equivalent to buying insurance of the spot commodity return. (The connection between purchased calls and insurance is less obvious.) Because options act on the skewness in the return distribution, it is natural to ask whether an improved estimate for the optimal hedge ratio can be obtained by adding a skewness term to the moment preference function as discussed in Sec. 2.1, where $U''' > 0$, i.e., positive skewness preference, is assumed. Presumably, because put options act to reduce negative skewness, explicitly identifying skewness preference in the objective function will result in an increase in the optimal demand for put options, compared to the mean-variance case where only the put option impact on variance is taken into account. As it turns out, this is not the case.

In perfect markets, options will be priced on an actuarially fair basis implying that the expected return on purchasing an option will be zero. Hence, the mean part of the mean-variance and mean-variance-skewness objective functions will not enter the optimal solutions. Derivation of the optimal solutions now requires the variance and skewness for the terminal wealth function to be identified. While it is possible to do this in general, it is more instructive to consider a practical example. In particular, the choice of a stylized farmer regarding the optimal amount of privately issued crop insurance to purchase has attractive features. The stylized farmer plants one crop which is subject to both price and yield uncertainty. To hedge this uncertainty, the farmer only has access to crop insurance. Three kinds of crop insurance schemes are possible:

> *Quantity Insurance*: where the physical yield is restricted from falling below some minimum amount, usually set as some percentage of historical yields. This case is consistent with many traditional crop insurance plans.
> *Price Insurance*: where the crop delivery price is restricted from falling below a minimum amount. This type of insurance can be accomplished using put options.
> *Mixed or Revenue Insurance*: where the total revenue is restricted from falling below a minimum amount; this case is consistent with farm income stabilization and, to a lesser extent, disaster relief programs.[18]

The farmer is able to select only one of these types at a time.

Because, in practice, there is no stylized farmer and it is useful to provide some context about how crop insurance schemes work in practice. In the US, crop insurance has undergone a substantial changes. The crop insurance program that is federally administered is now managed by the Risk Management Agency of the USDA (www.act.fcic.usda.gov). Where the farmer traditionally only had access to a federal crop insurance scheme that was effectively quantity-based insurance scheme with premiums priced at a subsidy, there are now an area of crop insurance plans offered by private insurance companies, e.g., www.cropinsurance.org, as well as a wider range of plans available from the federal governments. Virtually all the plans have restrictions on the fraction of the crop which can be insured. In addition to pure crop insurance programs, disaster relief programs are also available. The totality of income support, crop insurance and other programs targeted at farmers is complicated to model. In Canada, all three types of insurance have been offered as alternatives under the GRIP program. Various schemes are offered in other countries (e.g., Hazell, et al. 1986).

In the absence of crop insurance, the farmer's terminal wealth function with hedging is given in Sec. 2.1. While the terminal wealth functions for the other forms of crop insurance (price and yield) follow appropriately, some motivation is required. In particular, in the absence of crop insurance and hedging, there is a natural minimum on R. Either a complete crop loss where $Y_{t+1}=0$, or a spot price of zero at time $t+1$ corresponds to the case $(1+R)=0$. Significantly, unlike revenue insurance, neither yield insurance nor price insurance by itself can guarantee a higher minimum return when $(1+R)$ equals zero. For example, price insurance guaranteeing $\$K$ a bushel $(P_{t+1} > K)$ cannot prevent a 100% crop loss due to drought, nor can quantity insurance providing for, say, $\underline{Y}$ bushels an acre $(Y_{t+1} > \underline{Y})$ prevent the future spot price falling to zero. However, both price and quantity insurance do reduce the probability of the total return attaining low values and, as a result, alter the distribution for terminal wealth. As it turns out, there are substantive differences in how price and yield insurance accomplish this result.

The Farmer's Terminal Wealth Function

To see the formal implications of admitting insurance, it is necessary to derive the terminal wealth functions for the price, yield and revenue forms of crop insurance (see Sec. 2.1). For example, if in addition to hedging with futures the farmer is assumed to buy revenue insurance against the full value of the crop $(AP_{t+1}Y_{t+1})$, the terminal wealth function can be specified:

$$\begin{aligned} W_{t+1}^{i} &= W_t\,\{(1+r) + x[\max\{\underline{R},R\} - s - r] - HR_f\} \\ &= W_t\,\{(1+r) + x[(R - r) + \max\{0,\underline{R}-R\} - s] - HR_f\} \\ &= W_t + \pi_{t+1} + W_t\,\{x[\max\{0,\underline{R}-R\} - s]\} \end{aligned}$$

where: s equals $(S\,A)/C(A)$ with S being the price (insurance premium) per acre for the revenue insurance and $\underline{R}$ is the income floor specified in the insurance plan. It can be seen from the terminal wealth function that the effect of adding revenue insurance to the risk management problem depends on assumptions made about both the pricing of the insurance premium (S) and the requirement that the full value of the crop be insured.

It can also be seen from the terminal wealth function that the effect of adding revenue insurance to the risk management problem is to increase terminal wealth by x times the purchase price adjusted payout on a "put option" written on the return R, with exercise "price" $\underline{R}$. If the insurance (put option) is priced as an acutarially sound expected indemnity then insurance will not change the farmer's expected wealth for the selected production level. All insurance does is limit downside risk. Whether this will affect the production level is discussed in Poitras (1993). The conclusion about the effect on expected wealth does not change if the farmer is permitted to choose the fraction of acreage insured, i.e., where the terminal wealth function is given by:

$$W_{t+1}^{i} = W_t\,\{(1+r) + x(R - r) + x\theta\,\max\{0,\underline{R}-R\} - s] + HR_f\}$$

where θ is the fraction of the total planted acreage insured under the revenue insurance scheme.

In many respects, the yield and revenue insurance cases are identical. As in the revenue insurance case, it is the number of acres to insure that is the decision variable:

$$W_{t+1}{}^{y} = AY_{t+1}P_{t+1} + (W_t - C(A))\,(1 + r)$$
$$+ Q_f(f_{t+1} - f_t) + Q_y(P_{t+1}\ \max[0,\underline{Y}-Y_{t+1}] - L)$$

where L is the price (insurance premium) per acre for the crop insurance, Q_y is the number of planted acres covered by the physical yield insurance and $\underline{Y}$ is the yield floor provided by the insurance plan. In practice, $\underline{Y}$ is set based on a percentage (<100%) of relevant historical physical yield averages. While the price used would actually depend on a specific method of price election selected by the farmer, taking the price elected to be the harvest period cash price (P_{t+1}) is not unrealistic (e.g., FCIC 1989).

Assuming that $Q_y = A$, i.e., all planted acres are insured, leads to the following:

$$W^{y}_{t+1} = W_t\{(1+r) + x[(R-r) + \max[0, \frac{P_{t+1}\underline{Y}A - P_{t+1}Y_{t+1}A}{C(A)}] - l] + HR_f\}$$

where l equals $(LA/C(A))$. Observing that the expression inside the max function involves the difference between two random variables illustrates the primary analytical difference between the different forms of the terminal wealth function. This distinction depends crucially on assuming that both price and quantity are uncertain. Observing that it may be unrealistic to assume that all acreage is insured, allowing the farmer to choose the acreage insured leads to:

$$W^{y}_{t+1} = W_t\{(1+r) + x[R-r] + HR_f + x\lambda[\max[0,\underline{RR}-R] - l]\}$$

where λ is the fraction of the total planted acreage insured under the physical yield crop insurance scheme and $\underline{RR} = \{P_{t+1}\ \underline{Y}\ A\}/C(A)$. As with revenue insurance, fair pricing requires insurance to impact the decision problem through its effect on downside risk.

Examining the price insurance case involves introducing put options (written on the futures price). This leads to:[19]

$$\begin{aligned}
W^{z}_{t+1} &= AY_{t+1}P_{t+1} + (W_t - C(A))(1+r) \\
&\quad + Q_f(f_{t+1} - f_t) + Q_z(\max[0,K-f_{t+1}] - z) \\
&= W_t\{x(1+R) + (1-x)(1+r) \\
&\quad + HR_f + \frac{Q_z f_t}{W_t}[\max[0,\frac{K-f_{t+1}}{f_t}] - \frac{z}{f_t}]\} \\
&= W_t\{x(1+R) + (1-x)(1+r) \\
&\quad + H\ R_f + \gamma(\max[0,-R_f] - \frac{z}{f_t})\}
\end{aligned}$$

where K is exercise price on the put option <u>that is assumed to be "at the money" (i.e., $K = P_t$)</u>, z is the price per unit of output of the put, Q_z is the number (in output units) of puts purchased, with the ratio γ being the value of the option position divided by initial wealth.[20] As in the yield and revenue insurance cases, if the put option is "fairly priced" then the expected value of the last term in the terminal wealth function is zero and insurance only has relevance for maximizing the expected utility of wealth insofar as it limits downside risk. However, price insurance has a direct impact on the distribution for R_f and not R as in the other two cases.

Mean-Variance-Skewness and the Optimal Demand for Put Options

Following the discussion in Sec. 2.1, the terminal wealth function provides essential information required to derive conditions from the maximization of an appropriate expected utility (*EU*) function. Using the moment preference approach, the presence of options argues for optimizing a mean-variance-skewness *EU* function because the non-linear payoffs on options are designed to impact the higher moments of the return distribution. Poitras and Heaney (1999) use a mean-variance-skewness moment preference function to derive relatively robust results about both the problem at hand, the optimal amount of put option, e.g., crop insurance, to purchase, and the more general issue of modeling optimal decisions involving options using moment preference objective functions.

The results in this section make use of results derived in Section 2.1 for the mean-variance-skewness expected utility function and the wealth process for a long spot position combined with a put option. This combination is of interest because options alter the skewness of the return distribution, so, modeling the decision problem with an objective function that values skewness in addition to mean and variance seems intuitively desirable. Yet, pursuing this approach soon leads to counter-intuitive results. Proposition 7.1 deals with the case of put options that are being used to reduce the negative skewness in the distribution of asset return. The simplest application would be to crop insurance, where a farmer wants to protect against flood, drought, disease or some other outcome that reduces crop yield. Presumably, introducing skewness preference into the expected utility problem would increase the optimal hedge ratio for the put option. Proposition 7.1 demonstrates that this is not the case (Poitras and Heaney 1999).

The results require assuming that: the risk manager optimizes a moment preference approximation to a general expected utility function of the form, $EU_{MVS} = U\{E[W_{t+1}]\} - b\ var[W_{t+1}] + c\ skew[W_{t+1}]$, where b and c are measures of the sensitivity of *EU* to changes in $var[\cdot]$ and $skew[\cdot]$, with $b,c > 0$; and, that all options/insurance premiums are "fairly priced". It is also assumed that R has a negative skewed probability density function, where $skew[R] < 0$.[21] The following now applies:[22]

Proposition 7.1: **The Optimal Demand for Put Options**[23]

Assuming that the risk manager optimizes a moment preference objective function, defined over the mean, variance and skewness of the farmer's terminal wealth function that includes put options, then the optimal demand for a put option with payoff depending on ***yield*** is:[24]

$$\lambda^* = -\frac{\sigma_{Rq}}{\sigma_q^2} + x\,\frac{3c\ W_t}{2b\ \sigma_q^2}\{cosk[\lambda;\underline{RR}]\}$$

where: the subscript q corresponds to the random variable $max[0, \underline{RR} - R]$ and the subscript R refers to the rate of return on the asset. The coskewness term, $cosk[\lambda;\underline{RR}]$, has the interpretation:

$$\begin{aligned} cosk[\lambda;\underline{RR}] \equiv cosk[\lambda] &= E\{\lambda^2\ (\max[0,\underline{RR}-R] - E[\max[0,\underline{RR}-R]])^3 \\ &\quad + (\max[0,\underline{RR}-R] - E[\max[0,\underline{RR}-R]])\ (R - E[R])^2 \\ &\quad + 2\ \lambda\ (\max[0,\underline{RR}-R] - E[\max[0,\underline{RR}-R]])^2\ (R-E[R])\} \\ &\equiv \lambda^2\ skew[q] + \sigma_{qR^2} + 2\{\lambda\ \sigma_{q^2R}\} \end{aligned}$$

The mean-variance solution (λ_{MV}) is given by ignoring the second term on the *rhs* of λ^*. The associated closed form solution is:

$$\lambda^* = \frac{(\sigma_q^2 - 2A\ \sigma_{q^2R}) - \sqrt{(\sigma_q^2 - 2A\ \sigma_{q^2R})^2 - 4A\ skew[q](A\ \sigma_{R^2q} - \sigma_{qR})}}{2A\ skew[q]}$$

where $A = x(3c\ W_t)/2b$.

The stated closed form solution is one of two roots of the quadratic equation in λ.. It is possible to verify by differentiating the first order condition that the stated closed form solution corresponds to a maximum, while the other root corresponds to a minimum (Poitras and Heaney 1999).

Closer analysis of the Proposition can be used to identify conditions for which the optimal put option demand derived from the mean-variance-skewness objective, λ^*, is less than the mean-variance optimal demand, λ_{MV}. A result such as $\lambda_{MV} \geq \lambda^*$ is interesting because it is seemingly counter-intuitive since the mean-variance-skewness function explicitly values positive skewness and the put option is a security that reduces the negative skewness in asset returns. However, the λ that maximizes skewness of W is typically less than the λ that minimizes the variance of W, making the solutions considerably more complicated than simple intuition would suggest.[25] For example, from the Proposition it is apparent that negative $cosk[\cdot]$ at the λ^* optimum is required for $\lambda_{MV} \geq \lambda^*$ to apply. Observing that $cosk[\cdot]$ is a quadratic function of λ leads to consideration of three points associated with $cosk[\cdot]$ function: the two roots of the quadratic that are associated with $cosk[\cdot] = 0$; and, the minimum point of the $cosk[\cdot]$ function. Various methods can be used to show that $cosk[\cdot] < 0$ at the minimum. It follows that the addition of the skewness term to the moment preference function results in a reduction of the optimal demand for put options.

Questions

1. What is a speculative bubble? Can the use of derivative securities lead to speculative bubbles? What role did derivative securities trading play in the Dutch tulipmania?

2. From Section 7.2, algebraically derive the profit functions for: a bearish vertical put spread; a bullish vertical put spread; a time spread; and a butterfly spread.

3. On Friday June 16, 1989, three call options for IBM trading on the CBOT, all expiring in October 1989, sold for the following prices:

Exercise Price	***Option Price***
105	9
115	3.75
125	1.06

Consider a "butterfly spread" with the following positions:

Buy 1 call at 105, Sell (write) 2 calls at 115, Buy 1 call at 125

What would be the values at expiration of such a spread for various prices of IBM at the time? What investment would be required to establish the spread? Given information about the prices of the \$105 and \$125 options, what could you predict about the price of the \$115 option?

4. A short stock position can be "protected" by either selling a put or buying a call. Determine the profit functions for these alternative strategies and determine the breakeven stock price at expiration together with the maximum and minimum profits.

5. Derive the expiration-date profit diagrams for the following trades: straddle, strap, vertical spread, and horizontal time spread. Verify the replication strategies for a written put, a written call, a purchased put, a

purchased call, and a long cash position.

6. The discussion of replication trades in Sec. 7.1 uses the two portfolio approach to derive the relevant conditions. Reconstruct these arguments using the arbitrage portfolio approach.

7. "A call option benefits from increases in the stock price and these increases can be very large. A put option benefits from stock price declines, but the stock price can only fall to zero. Therefore, if we have a put and a call on the same stock with the same terms, the put must sell for less than the call." Do you agree or disagree? Explain making sure that you identify relevant restrictions on the underlying arbitrage.

8. Establish the relationship between caps, collars and floors with a bearish vertical spread that is long the put and short the call.

9. In Sec. 7.1, the derivation of the put-call parity boundary conditions for American options was left as an exercise. Using the two portfolio approach, verify the boundary conditions given in that Section. In addition, use the most recent put and call option prices for IBM together with the current stock price, interest rate and estimated dividends, to estimate the size of the difference between the upper and lower boundaries for options on that stock.

10. Draw the expiration date profit diagram for the put-call combination that replicates a short stock position. Be sure to consider the three scenarios: $X > S(0)$; $X < S(0)$; and $X = S(0)$.

NOTES

1. The seminal source of this material is Merton (1973). Gibson (1991) and other general sources give a comprehensive treatment.

2. In the following, the spot commodity will be assumed to be common stock. In general, dividend payments relate to carry returns on the spot commodity. When the option is written on a futures contract, there is no carry return by construction.

3. This point is explored in more detail in Sec. 9.2.

4. Precisely when this will happen depends on a combination of factors such as the time to expiration, the size of $X - S(t)$, the volatility on the stock, liquidity in the option and so on.

5. It is important to recognize that Graph 7.1is not an expiration date profit diagram, i.e., the horizontal axis measures the current stock price, $S(t)$, while the vertical axis measure the call price.

6. These combinations are: short the stock and write a call; short the stock and buy a put; long the stock and buy a call; and, long the stock and write a put.

7. However, this difference between $S(0)$ and the breakeven stock price for the replication strategies will not usually be large. For this reason, the diagrams have been drawn as though $E = S(0)$ produces a payoff centred at $S(0)$.

8. Shifting both 45^o lines in the same direction, even though one strategy involves a cash inflow and the other a cash outflow, is due to the different slopes, i.e., the short position has slope -1 while the long position has slope +1.

9. There is a literature on put-call parity that predates the introduction of the Black-Scholes formula, e.g., Stoll (1969, 1973), Merton (1973). Many studies of put-call parity have related the issue to market efficiency, e.g., Klemkosky and Resnick (1979, 1980, 1991), Finucane (1991), Nisbet (1991). Other studies have been concerned with developing the put-call parity conditions for specific instruments, e.g., Goodman (1985), Chance (1987). A recent development has been the use of put-call parity conditions to identify the early exercise premium, e.g., Zivney (1991), de Roon and Veld (1996).

10 The use of conversions in options trading has a long history, e.g., Poitras (2000, Ch.9).

11. To see this, observe that if $S(T)=E$ then the stock is sold to pay off the maturing value of the loan, both options expiring worthless. If $S(T) > E$, the put expires worthless and $C(S(T),0,E) = max[0, S(T) - E]$ plus E, the maturity value of the loan, equals $S(T)$, the value of the stock that is to be sold to settle the position. If $S(T) < E$, the call expires worthless and $P(S(T),0,E) = max[0, E - S(T)]$ plus $S(T)$ equals E that is the maturing value of the loan. In all cases, the additional balance that was raised initially due to $C + E\ PV > P + S$ is the residual arbitrage profit.

12. A number of articles have explored the validity of the put-call parity condition, e.g., Klemkosky and Resnick (1979).

13. Goldman, et.al. (1979), Kemna and Vorst (1990), and Wilmott (1998) are useful sources on path dependent options.

14. Much of the literature on the various types of options trading strategies is somewhat dated, with much of the material available in trade publications and textbooks. Examples of early studies include Gombola, et al. (1978) and Ritchken and Salkin (1981). There is also a substantial literature on the performance of covered call option writing strategies, e.g., Yates and Kapprasch (1980), Mueller (1981).

15. It is also possible to devise strategies that involve using options to generate premium income. This income can be used to finance various types of speculative positions in other assets. These types of strategies will not be explored here except where the positions are directly related to arbitrage trades.

16. A variation on the straddle is the ***strangle*** where the exercise prices for the put is below the exercise price for the call positions. In this case, the expiration date profit diagram is flat over the region between the exercise prices. The advantage of a strangle over a straddle is that it is cheaper to purchase. For the writer, while premium income is lower, there is a wider range of expiration date stock prices that generate a profit.

17. These descriptions correspond to the method in which option price quotes are typically observed in the financial press. Differences in prices across exercise prices are recorded vertically, while price differences across expiration dates occur horizontally.

18. Total revenue is the realized income from planting a given crop. Given that various types of farm income stabilization programs are possible, e.g., where payments are made prior to planting on the condition that farmers do not plant certain crops, the revenue insurance schemes being examined here are not fully descriptive of all possible plans.

19. Given the ability to replicate call positions by combining puts and futures, little is gained by introducing a term for a call option. Similarly, while a straddle position would have an interesting random variable distribution, this would add little when futures are present.

20. It is also possible to specify the put option using cash prices. However, this significantly complicates the analysis. In addition, exchange traded options are typically written using futures prices, so the present construction is potentially more realistic.

21 This assumption involves somewhat more than is stated. More precisely, this assumption requires that the put option pay-outs will occur in states where the returns are low. Cases where put option pay-outs occur and revenue is high are excluded to avoid having to consider pathological cases.

22 If the option is not fairly priced, in practice it will probably be underpriced, due to government subsidies. In the crop insurance context, underpricing will produce an increase in the usage of insurance.

23 The optimal demand for the put option depends fundamentally on parameters in $cosk[\lambda]$ that may be unfamiliar, such as $cov[q^2,R]$. Interpretation of these terms by numerical example is considered by example in Section 5.

24 While almost identical, the optimal demand for a yield put option does differ from the price put option in that $\gamma^*/x = [Q_z P_t]/C(A)$ *and* $\lambda^* = Q_y/A$. These decision variables have a somewhat different interpretation. For the put option based on prices, it is the fraction of the initial dollar value of investment in the risky asset that is of interest, while for the yield put option it is the fraction of the physical size of the asset. The optimal demand for put option on revenue is the same as that for yield insurance, with the proviso that the actual value of the various parameters, i.e., variances, covariances and the like, will have different values.

25 The underlying moment preference objective function requires that $dEU/d\ skew[W] > 0$. Using the optimal solution it is now possible to do comparative static analysis on specific parameters, e.g., to determine that $\{d\lambda^*/d\ skew[q]\} < 0$.