# Common Misunderstandings Concerning Duration and Convexity

**Timothy Falcon Crack and Sanjay K. Nawalkha**

*In our experience, most finance students are unnecessarily confused by the roles that duration and convexity play in the traditional textbook plot of bond price versus bond yield. There are two main sources for this confusion: first, the existing literature promotes ambiguity by using several different definitions of duration and convexity without due care; and second, the traditional textbook plot is simply not well suited to explaining the roles of duration and convexity. We demonstrate several common misunderstandings, and we offer a new plot for illustrating the roles of duration and convexity. [JEL: A20, G10]*

■If you draw the traditional textbook plot of a bond's price versus its yield (i.e., Exhibit 1) on a blackboard and ask finance students what the slope is at any given point, it has been our experience they invariably answer that the slope is either the Macaulay duration or the modified duration of the bond–both wrong. Ask them if changing slope (as yield changes) illustrates changes in this duration, and they invariably answer yes–wrong again.

Although Exhibit 1 is a common tool for explaining duration and convexity concepts, these explanations generate substantial confusion. The confusion has two main sources. The first source of confusion is that different books use quite different definitions of duration and convexity, sometimes without due care or clarity. Unfortunately, each definition carries different properties. The second source of confusion is that the traditional textbook plot in Exhibit 1 is simply not well suited to explaining the roles of Macaulay duration and convexity.

To avoid further confusion, we begin with a very clear definition of which duration and convexity measures we are using (these definitions are explored later in the paper). We then demonstrate many misunderstandings that result from the above-mentioned confusions. We conclude by presenting an alternative to the traditional textbook plot for illustrating duration and convexity. We provide tables that summarize all the formulae in our paper, and also present tables and graphs with multiple numerical examples for classroom instruction.

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## I. Duration and Convexity Misunderstandings

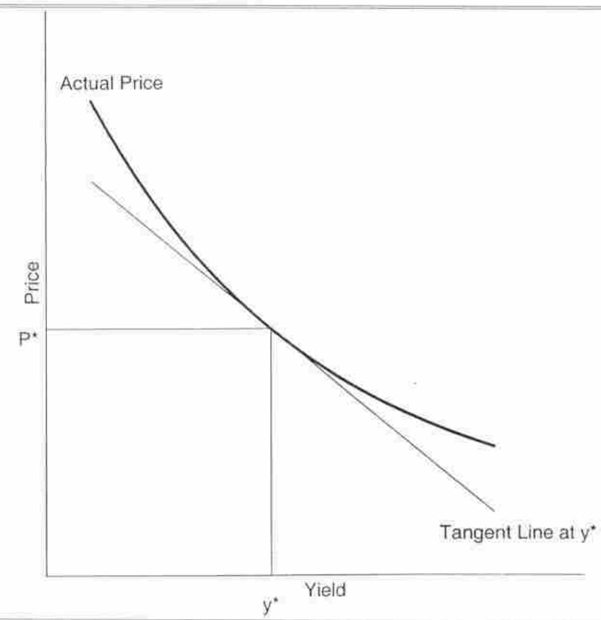
Macaulay duration is often presented as a weighted average time to maturity of a bond. We denote it very explicitly as $D_{mac}$ to avoid any possible confusion with other duration measures:

$$D_{mac} \equiv \frac{\sum_{t=1}^{T} tC_t e^{-yt}}{\sum_{t=1}^{T} C_t e^{-yt}} = \sum_{t=1}^{T} t\omega_t \qquad (1)$$

where $D_{mac}$ is Macaulay duration, T is time to maturity, $C_t$ is bond cash flow at time t, y is continuously compounded bond yield, and

### Exhibit 1. Bond Price/Yield Relationship with Tangent Line

This is the traditional plot of bond price versus yield. The tangent line at the initial yield indicates that restricting your analysis to duration only (a "first order" approximation) is equivalent to assuming the pricing relationship is linear in yield.




$\omega_t \equiv \frac{C_t e^{-yt}}{\sum_{t=1}^{T} C_t e^{-yt}}$ is a weight where $\sum_{t=1}^{T} \omega_t = 1$. The use of continuous yields simplifies our exposition without affecting the main thrust of our arguments (see later discussion). With this notation, we define bond convexity as follows (again, discussed later):

$$C \equiv \frac{\sum_{t=1}^{T} t^2 C_t e^{-yt}}{\sum_{t=1}^{T} C_t e^{-yt}} = \sum_{t=1}^{T} t^2 \omega_t \tag{2}$$

In the familiar case of a zero-coupon bond of maturity T, all weights except $w_T$ are zero, and thus $D_{mac}$=T, and C=$T^2$.

Let us try to interpret $D_{mac}$ and C in relation to the simple plot of bond price versus bond yield, (i.e., Exhibit 1) as is often done in textbooks or in the classroom. It is well known that limiting your analysis to duration only (i.e., ignoring convexity) is equivalent to assuming that the bond pricing relationship is linear in yield and thus that the relationship in Exhibit 1 follows the tangent line. Many other statements can be made about Exhibit 1. We find some or all of the following interrelated statements in each of several different sources[1] in the literature and it is our experience that this is how many people think about Exhibit 1 with respect to our definitions of $D_{mac}$ and C:

1. Duration measures the sensitivity of a bond's price to changes in its yield, and is thus given by the (negative of the) slope of the plot of bond price versus bond yield.
2. Duration decreases (increases) as bond yield increases (decreases)–this property holds for all option-free bonds.
3. Duration is the steepness of the tangent line in Exhibit 1. The steeper the tangent line, the greater the duration; the flatter the tangent line, the lower the duration.
4. Yield-induced changes in duration accelerate (decelerate) changes in prices as yields decrease (increase). This is why absolute and percentage price changes are greater when yields decline than when they increase by the same number of basis points.
5. Bond convexity is a second order measure of the sensitivity of a bond's price to changes in its yield, and is thus given by the curvature (i.e., rate of change of slope) of the plot of bond price versus bond yield.
6. Bond convexity is the rate of change of duration as yields change.
7. Bond convexity decreases (increases) as bond yield increases (decreases)–this property holds for all option-free bonds.

Although appealing, each and every one of the above

[1]See the following: Johnson (1990), Livingston (1990), Kritzman (1992), Fabozzi, Pitts, and Dattatreya (1995), Cole and Young (1995), and Fabozzi (1996).

statements is false for our definitions of Macaulay duration and convexity unless accompanied by additional assumptions or restrictions. We give simple counter examples in the next section. We shall also demonstrate that if the (flawed) intuition behind these statements is applied to bonds with embedded options it creates substantial confusion.

## II. Simple Counter Examples

Consider a five year zero-coupon bond (i.e., a "zero"). The plot of bond price versus yield to maturity for the zero looks like Exhibit 1. A zero has Macaulay duration equal to its maturity, so a five-year zero has $D_{mac}$ equal to five, regardless of its yield. It follows immediately that in the case of the zero, the changing slope of the plot in Exhibit 1 cannot be equal to the negative of the zero's Macaulay duration because the zero's $D_{mac}$ is fixed at five regardless of yield. We also conclude that $D_{mac}$ need not change with changing yield (even though the tangent line in Exhibit 1 flattens out with increasing yield).

The curvature of the plot in Exhibit 1 means that decreases (increases) in yield *are* associated with accelerated (decelerated) changes in price per basis point change in yield. However, in the case of a zero, changes in $D_{mac}$ cannot be the cause. Also, the convexity C of a zero equals its maturity squared regardless of yield. However, the rate of change of the zero's Macaulay duration $D_{mac}$ with respect to its yield is zero (because the zero's Macaulay duration does not change with yield). Thus bond convexity C cannot be simply the sensitivity of duration to changes in yield. We also conclude that convexity (fixed at C=25 for our five-year zero) need not change with changing yield (even though the curvature of bond price in Exhibit 1 decreases with increasing yield). With a little math, it can also be shown that bond convexity C is not the curvature of price with respect to yield. This result appears in the next section along with explanations.

Let us emphasize here that this is not simply a matter of the given statements failing to hold for zeroes or failing to hold when using continuous yields. The problem is deeper than that–it is just that it is easiest to see when looking at zeroes and using continuous yields. The casual statements mentioned in the previous section overlook the following interrelated facts:

1. The slope of Exhibit 1 is not (the negative of) Macaulay duration.
2. The curvature (i.e., rate of change of slope) in Exhibit 1 does not illustrate changing Macaulay duration.
3. The curvature (i.e., rate of change of slope) in Exhibit 1 is not bond convexity.
4. Changing curvature in Exhibit 1 does not illustrate changing convexity.

## III. Explanations

To confuse the slope and curvature of Exhibit 1 with (the negative of) Macaulay duration and with convexity, respectively, leads to a fundamental misunderstanding of these concepts. We need to return to first principles to understand what is going on. We must examine the Taylor series expansion of bond price as a function of yield.

Using continuous yields, the price P of the bond is related to its yield $y$ as follows:

$$P = \sum_{t=1}^{T} C_t e^{-yt} \tag{3}$$

where $C_t$ is the cash flow to the bond at time $t$. A Taylor series approximation truncated at the second term gives:[2]

$$\begin{aligned} \Delta P &\approx \frac{\partial P}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}(\Delta y)^2 \\ &= -P\left[-\frac{\frac{\partial P}{\partial y}}{P}\right]\Delta y + \frac{1}{2}P\left[\frac{\frac{\partial^2 P}{\partial y^2}}{P}\right](\Delta y)^2 \\ &= -PD_{mac}\,\Delta y + \frac{1}{2}PC\,(\Delta y)^2, \end{aligned} \tag{4}$$

where

$$D_{mac} \equiv \left[-\frac{\frac{\partial P}{\partial y}}{P}\right], \text{ and } C \equiv \left[\frac{\frac{\partial^2 P}{\partial y^2}}{P}\right] \tag{5}$$

are Macaulay duration and bond convexity respectively. The definitions in Equation 5, when applied to Equation 3, give the save formulae for $D_{mac}$ and C as in Equations 1 and 2 earlier.

Dividing Equation 4 through by P we obtain:

$$\frac{\Delta P}{P} \approx -D_{mac}\Delta y + \frac{1}{2}C(\Delta y)^2 \tag{6}$$

It follows from Equation 6 that duration and convexity are directly related to the first two coefficients in a second order approximation of "instantaneous bond

[2]Our Taylor series approach is a mathematical tautology to show the relationship between bond price, duration and convexity. We are not suggesting a return generating process that assumes parallel shifts in the term structure. See Christensen and Sorensen (1994) for a similar exposition of the Taylor series approach.

return" (i.e., $\frac{\Delta P}{P}$) with respect to change in yield. If instead of relating instantaneous bond return to change in yield (as in Equation 6) we relate change in price to change in yield (as in Equation 4), we find that the roles of duration and convexity are "contaminated" by price level. Indeed, this can be shown directly by rearranging the definitions in Equation 5 to arrive at the following equalities:

$$\frac{\partial P}{\partial y} = -D_{mac} P \text{ , and } \quad \frac{\partial^2 P}{\partial y^2} = CP \tag{7}$$

The first equality in Equation 7 says that the slope of the plot in Exhibit 1 is not $-D_{mac}$, but $-D_{mac}P$ (i.e., the negative of what is known as "dollar duration"). The second equality in Equation 7 says that the curvature (i.e., rate of change of the slope) of the plot in Exhibit 1 is not C, but CP (i.e., what is known as "dollar convexity"). That is, when you relate price to yield, price contaminates the roles of duration and convexity. We suspect that many of the earlier statements drawn from the literature refer to dollar duration and dollar convexity respectively.

The definitions in Equation 5 also yield $\frac{\partial D_{mac}}{\partial y} = D_{mac}^2 - C$. Thus the sensitivity of duration to changes in yield is $D_{mac}^2 - C$ and this can be calculated directly by a practitioner who has already calculated both duration and convexity. This latter equality also yields a very nice property for zero-coupon bonds: $C=D^2_{mac}$ (because $\frac{\partial D_{mac}}{\partial y} = 0$ for a zero). It follows that convexity is fixed regardless of yield for a zero, and therefore that convexity does not necessarily change with yield.

It is dollar duration (i.e., the product $D_{mac}P$) that measures the (absolute) dollar change in the price of a bond for a given change in yields. Thus changing slope in Exhibit 1 does not illustrate changing Macaulay duration (or changing Modified duration either for that matter).[3] Rather, changing slope in Exhibit 1 illustrates changing dollar duration and this does not necessarily tell us anything about Macaulay duration $D_{mac}$. Similarly, the curvature in Exhibit 1 illustrates dollar convexity (i.e., CP) and this differs substantially from convexity C. The numerical examples in Exhibits 2-4 show just how different the afore-mentioned concepts can be.

The following simple properties of duration and convexity for standard fixed-rate bonds without embedded options are illustrated in Exhibits 2-3 and are able to be derived easily from Exhibits 10 or 11. Macaulay duration does not change with yield for a zero-coupon bond, but Macaulay duration does decrease slowly with increasing yield for a coupon-bearing bond; dollar duration decreases rapidly with increasing yield regardless of coupon rate; using discretely compounded yields, convexity decreases slowly for increasing yield regardless of coupon rate; using continuously compounded yields, convexity does not change with yield for a zero-coupon bond, but convexity does decrease slowly with increasing yield for a coupon-bearing bond; dollar convexity decreases rapidly with increasing yield regardless of coupon rate; and finally, other things being equal Macaulay duration and convexity decrease with increasing coupon level, but dollar duration and dollar convexity increase with increasing coupon level.

## IV. Application to Callable Bonds

We have demonstrated that many statements about duration and convexity do not hold in the simple case of a zero-coupon bond. Let us now take the more complicated example of a security with an embedded option–a callable zero-coupon bond–to illustrate how misleading these statements can be if applied more generally.

Consider a $100 face value 10-year zero-coupon bond that is callable (European-style) in one year at 80% of it face value. Exhibit 5 plots the bond's price, duration, and dollar duration as a function of yield. The bond price as a function of yield first steepens, and then flattens as yield increases (see Panel A of Exhibit 5). Inferring duration from the slope in Panel A of Exhibit 5 implies *incorrectly* that duration first increases and then decreases as yield rises–whereas, the duration of the callable bond is monotonically increasing in yield (see Panel B of Exhibit 5).[4] The correct inference is that it is dollar duration that first increases and then decreases with increasing yield (see Panel C of Exhibit 5). We conclude that inferring duration from the slope of the price-yield relationship causes substantial confusion in the case of a callable bond.

[3]With continuous yields, there is no distinction between modified duration and Macaulay duration. Although there is a distinction with discrete yields, it is still the case that changing slope in Exhibit 1 is not illustrative of changing modified duration. See Exhibit 10 and Exhibit 11 for theoretical details, and Exhibits 2-4 for numerical examples.

[4]See Nawalkha (1995) for analytical details of the pricing and duration of this callable bond–we are assuming that volatility of returns to the bond is V=0.01.

## Exhibit 2. Numerical Examples–Continuously Compounded Case

For each of three bonds, the slope and curvature of the price-yield curve change substantially more than the Macaulay duration and convexity, respectively as yields change. Although the first two bonds have the same Macaulay durations and the same convexities, their slopes (negative of dollar duration) and curvatures (dollar convexity) differ substantially.

| Continuous Yield | y | 0.00 | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 |
|---|---|---|---|---|---|---|---|
| *Panel A. Five Year Zero-Coupon Bond with Face $10* | | | | | | | |
| Price | P | 10.00 | 7.79 | 6.07 | 4.72 | 3.68 | 2.87 |
| Macaulay Duration | $D_{mac}$ | 5.00 | 5.00 | 5.00 | 5.00 | 5.00 | 5.00 |
| Convexity | C | 25.00 | 25.00 | 25.00 | 25.00 | 25.00 | 25.00 |
| Slope of P-y Plot | $-PD_{mac}$ | -50.00 | -38.94 | -30.33 | -23.62 | -18.39 | -14.33 |
| Curvature of P-y Plot | *CP* | 250.00 | 194.70 | 151.63 | 118.09 | 91.97 | 71.63 |
| *Panel B. Five Year Zero-Coupon Bond with Face $100* | | | | | | | |
| Price | P | 100.00 | 77.88 | 60.65 | 47.24 | 36.79 | 28.65 |
| Macaulay Duration | $D_{mac}$ | 5.00 | 5.00 | 5.00 | 5.00 | 5.00 | 5.00 |
| Convexity | C | 25.00 | 25.00 | 25.00 | 25.00 | 25.00 | 25.00 |
| Slope of P-y Plot | $-PD_{mac}$ | -500.00 | -389.40 | -303.27 | -236.18 | -183.94 | -143.25 |
| Curvature of P-y Plot | *CP* | 2500.00 | 1947.00 | 1516.33 | 1180.92 | 919.70 | 716.26 |
| *Panel C. Five Year 15% Annual-Coupon Bond with Face $100* | | | | | | | |
| Price | P | 175.00 | 142.59 | 116.77 | 96.14 | 79.61 | 66.33 |
| Macaulay Duration | $D_{mac}$ | 4.14 | 4.05 | 3.94 | 3.83 | 3.71 | 3.59 |
| Convexity | C | 19.00 | 18.38 | 17.70 | 16.99 | 16.23 | 15.44 |
| Slope of P-y Plot | $-PD_{mac}$ | -725.00 | -577.08 | -460.45 | -368.37 | -295.58 | -237.95 |
| Curvature of P-y Plot | *CP* | 3325.00 | 2620.30 | 2067.36 | 1633.21 | 1292.08 | 1023.84 |

## Exhibit 3. Numerical Examples–Discretely Compounded Case

For comparison with Exhibit 2, the yields here are Y=exp(y)-1 using the y's from Exhibit 2. Thus the Y's are 0.0000%, 5.1271%, 10.5171%, 16.1834%, 22.1403%, and 28.4025% (but appear rounded below). We see that the Macaulay durations $D_{mac}$ are unchanged from Exhibit 2. The Modified durations $D^*$ fall slightly as yield rises–even for the zeroes. However, the fall in $D^*$ is much less proportionately than the change in slope.

| Discrete Yield | Y | 0.00 | 0.05 | 0.11 | 0.16 | 0.22 | 0.28 |
|---|---|---|---|---|---|---|---|
| *Panel A. Five Year Zero-Coupon Bond with Face $10* | | | | | | | |
| Price | P | 10.00 | 7.79 | 6.07 | 4.72 | 3.68 | 2.87 |
| Modified Duration | $D^*$ | 5.00 | 4.76 | 4.52 | 4.30 | 4.09 | 3.89 |
| Macaulay Duration | $D_{mac}$ | 5.00 | 5.00 | 5.00 | 5.00 | 5.00 | 5.00 |
| Convexity | C | 30.00 | 27.15 | 24.56 | 22.22 | 20.11 | 18.20 |
| Slope of P-y Plot | $-PD_{mac}$ | -50.00 | -37.04 | -27.44 | -20.33 | -15.06 | -11.16 |
| Curvature of P-y Plot | *CP* | 300.00 | 211.41 | 148.98 | 104.98 | 73.98 | 52.13 |
| *Panel B. Five Year Zero-Coupon Bond with Face $100* | | | | | | | |
| Price | P | 100.00 | 77.88 | 60.65 | 47.24 | 36.79 | 28.65 |
| Modified Duration | $D^*$ | 5.00 | 4.76 | 4.52 | 4.30 | 4.09 | 3.89 |
| Macaulay Duration | $D_{mac}$ | 5.00 | 5.00 | 5.00 | 5.00 | 5.00 | 5.00 |
| Convexity | C | 30.00 | 27.15 | 24.56 | 22.22 | 20.11 | 18.20 |
| Slope of P-y Plot | $-PD_{mac}$ | -500.00 | -370.41 | -274.41 | -203.28 | -150.60 | -111.57 |
| Curvature of P-y Plot | *CP* | 3000.00 | 2114.06 | 1489.76 | 1049.81 | 739.79 | 521.32 |
| *Panel C. Five Year 15% Annual-Coupon Bond with Face $100* | | | | | | | |
| Price | P | 175.00 | 142.59 | 116.77 | 96.14 | 79.61 | 66.33 |
| Modified Duration | $D^*$ | 4.14 | 3.85 | 3.57 | 3.30 | 3.04 | 2.79 |
| Macaulay Duration | $D_{mac}$ | 4.14 | 4.05 | 3.94 | 3.83 | 3.71 | 3.59 |
| Convexity | C | 23.14 | 20.29 | 17.72 | 15.42 | 13.37 | 11.54 |
| Slope of P-y Plot | $-PD_{mac}$ | -725.00 | -548.94 | -416.63 | -317.06 | -242.00 | -185.31 |
| Curvature of P-y Plot | *CP* | 4050.00 | 2893.11 | 2069.59 | 1482.80 | 1064.24 | 765.31 |

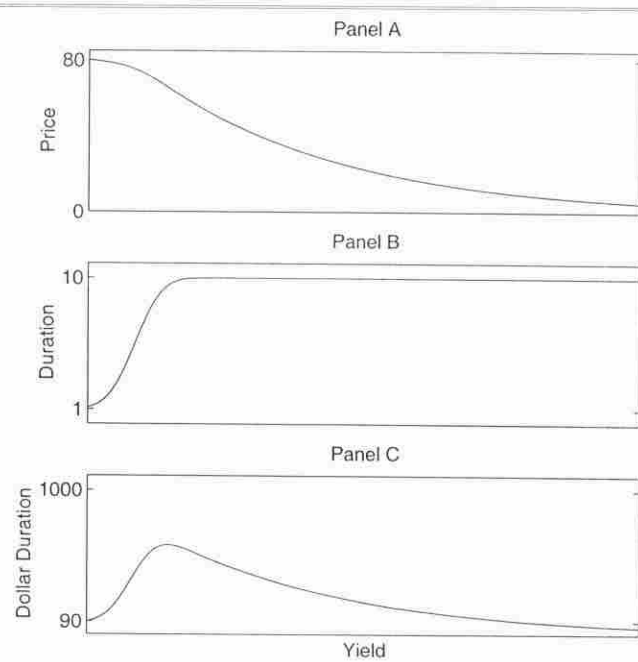
## Exhibit 4. Numerical Examples–Discretely Compounded Case

For ease of calculation, and for comparison to Exhibits 2, and 3, this table uses discrete yields at intervals of 5%.

| Discrete Yield | Y | 0.00 | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 |
|---|---|---|---|---|---|---|---|
| *Panel A. Five Year Zero-Coupon Bond with Face $10* | | | | | | | |
| Price | P | 10.00 | 7.84 | 6.21 | 4.97 | 4.02 | 3.28 |
| Modified Duration | $D^*$ | 5.00 | 4.76 | 4.55 | 4.35 | 4.17 | 4.00 |
| Macaulay Duration | $D_{mac}$ | 5.00 | 5.00 | 5.00 | 5.00 | 5.00 | 5.00 |
| Convexity | C | 30.00 | 27.21 | 24.79 | 22.68 | 20.83 | 19.20 |
| Slope of P-y Plot | $-PD_{mac}$ | -50.00 | -37.31 | -28.22 | -21.62 | -16.74 | -13.11 |
| Curvature of P-y Plot | *CP* | 300.00 | 213.20 | 153.95 | 112.78 | 83.72 | 62.91 |
| *Panel B. Five Year Zero-Coupon Bond with Face $100* | | | | | | | |
| Price | P | 100.00 | 78.35 | 62.09 | 49.72 | 40.19 | 32.77 |
| Modified Duration | $D^*$ | 5.00 | 4.76 | 4.55 | 4.35 | 4.17 | 4.00 |
| Macaulay Duration | $D_{mac}$ | 5.00 | 5.00 | 5.00 | 5.00 | 5.00 | 5.00 |
| Convexity | C | 30.00 | 27.21 | 24.79 | 22.68 | 20.83 | 19.20 |
| Slope of P-y Plot | $-PD_{mac}$ | -500.00 | -373.11 | -282.24 | -216.16 | -167.45 | -131.07 |
| Curvature of P-y Plot | *CP* | 3000.00 | 2132.04 | 1539.47 | 1127.81 | 837.24 | 629.15 |
| *Panel C. Five Year 15% Annual-Coupon Bond with Face $100* | | | | | | | |
| Price | P | 175.00 | 143.29 | 118.95 | 100.00 | 85.05 | 73.11 |
| Modified Duration | $D^*$ | 4.14 | 3.86 | 3.59 | 3.35 | 3.13 | 2.92 |
| Macaulay Duration | $D_{mac}$ | 4.14 | 4.05 | 3.95 | 3.85 | 3.76 | 3.66 |
| Convexity | C | 23.14 | 20.35 | 17.95 | 15.87 | 14.07 | 12.49 |
| Slope of P-y Plot | $-PD_{mac}$ | -725.00 | -552.63 | -427.50 | -335.22 | -266.16 | -213.79 |
| Curvature of P-y Plot | *CP* | 4050.00 | 2916.71 | 2135.51 | 1587.36 | 1196.40 | 913.37 |

## Exhibit 5. Callable Zero-Coupon Bond

The figure is for a $100 face value 10-year zero-coupon bond. The bond is callable (European-style) after one year at an exercise price of $80. Panel A plots bond price versus yield; Panel B plots duration versus yield (see Nawalkha (1995) for analytical details); Panel C plots dollar duration versus yield. Note that although the plot in Panel A at first steepens and then flattens, the duration (in Panel B) is monotonically increasing.



## V. A New Graph

There are two problems with the traditional plot of bond price versus yield (Exhibit 1). The first problem, as discussed above, is that the slope of the tangent line (at the initial yield) is not duration and the change in slope with respect to yield is not convexity. The second (and related) problem is that the traditional plot shows *dollar* changes in bond prices for changing yields. However, a \$2 change in a \$100 bond is not the same as a \$2 change in a \$50 bond, the plot compares apples and oranges. Two ways to reduce these problems come immediately to mind: either plot log price against yield, or plot instantaneous return (i.e., $\frac{\Delta P}{P}$) against yield.

Suppose we follow the first approach and plot log price against yield (Bierwag, Kaufman, and Latta, 1988; Campbell, Lo, and MacKinlay, 1997, and our Exhibit 6). In this case the slope of the plot is easily shown to be $-D_{mac}$, where $D_{mac}$ is the duration of the bond.[5] The change in slope can be shown to be $C - D_{mac}^2$. The vertical change along the plot from the initial point is the log of 1 plus the instantaneous return on the bond.[6] The plot is unbounded below (because log P is unbounded below as $P \rightarrow 0$). In the special case of a zero-coupon bond (where $C = D^2_{mac}$ so the change in slope is zero), the plot of log price versus yield is a straight line (see Exhibit 6).

The plot of log price versus yield has substantially less curvature than the traditional plot of price versus yield, and, in the case of a zero, it has no curvature at all. There is a simple economic reason for this reduced curvature; the absolute value of the slope in Exhibit 6 is duration, and duration does not change very much with changing yield. Contrast this with the traditional plot (Exhibit 1) where the slope is $-D_{mac}P$ which does vary a lot with changing yield, because P is varying a lot.

Unfortunately, even though the slope of the log price versus yield graph is equal to $-D_{mac}$, the curvature (i.e., change in slope) of the graph does not give convexity. To address this shortcoming, we recommend a second approach that plots instantaneous return against yield (see Exhibit 7). The instantaneous bond return is the instantaneous price change divided by initial price, $\frac{(P-P^*)}{P^*}$, say. The slope of the tangent line *at the initial yield* is easily shown to be $-D_{mac}$, where $D_{mac}$ is the duration of the bond. The rate of change of slope *at the initial yield* is easily shown to be C, where C is convexity.[7] Thus duration and convexity are first and second order measures of the sensitivity of a bond's instantaneous return to changes in its yield (as in Equation 6). Given that the (absolute value of the) slope and its rate of change are $D_{mac}$ and C respectively, this plot is more attractive than plotting log price versus yield (it also avoids the potential confusion arising from log price being unbounded below).

For different initial yields, the curve in Exhibit 7 "slides" sideways (see Exhibit 8). The change in slope of the tangent line for increasing yield is minor compared to that in the traditional plot of price versus yield (with no change at all for a zero). The reason is the same as that given above for the log price plot: the slope here is $-D_{mac}$ and $D_{mac}$ does not change much with yield, whereas for the traditional plot, the slope is $-D_{mac}P$ which changes a lot because P changes so much with changing yield. Finally, compared to the initial point, the vertical change along the plot in Exhibit 7 is the instantaneous rate of return on the bond.

Exhibit 9 presents the plot of instantaneous return versus yield for the callable zero-coupon bond that we discussed earlier. It can be seen that the duration (absolute value of slope at initial yield) increases for increasing initial yield–as in Panel B of Exhibit 5. This is because increasing yield decreases the likelihood that the embedded call will be exercised. This, in turn, lengthens the expected maturity of the callable zero and increases its duration. This differs from Exhibit 8 in which duration decreases with increasing yield (the case of a non-callable coupon bond). The monotonically increasing relationship between the duration of a callable zero-coupon bond and its yield cannot be inferred directly from the traditional price-yield graph such as Panel A of Exhibit 5, thus illustrating the importance of our new plot.

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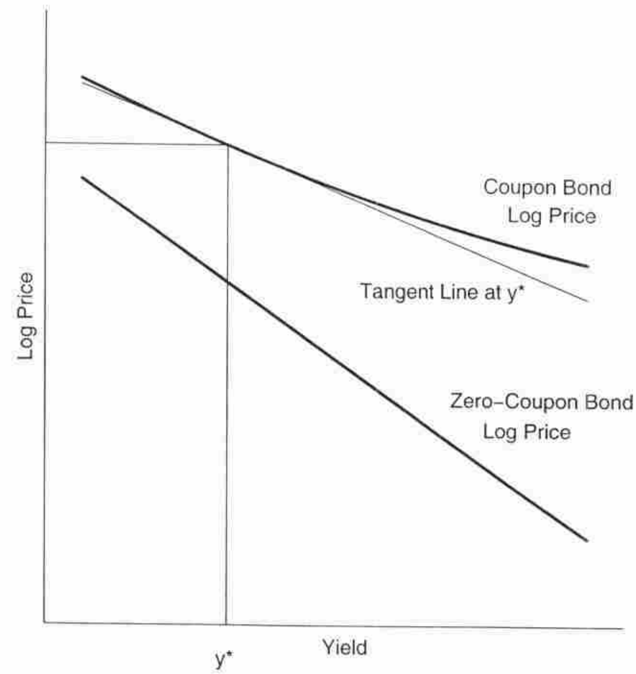
[5]Since $\frac{\partial \log P}{\partial y} = \frac{1}{P}\frac{\partial P}{\partial y}$ is the slope, it follows from the definition of $D_{mac}$ that the slope equals $-D_{mac}$.

[6]Consider an initial price $P^*$ at yield $y^*$ and a new price P at a new yield y. The vertical distance is $\log P - \log P^* = \log\left[1 + \frac{P - P^*}{P^*}\right]$ as stated.

[7]The slope of the plot at the initial yield $y^*$ equals $\frac{1}{P^*}\frac{\partial P}{\partial y}\Big|_{y=y^*} = -D_{mac}$. The curvature (i.e., change in slope) of the plot at the initial yield $y^*$ equals $\frac{\partial^2\left[\frac{P-P^*}{P^*}\right]}{\partial y^2}\Bigg|_{y=y^*} = \frac{1}{P^*}\frac{\partial^2 P}{\partial y^2}\Big|_{y=y^*} = C$

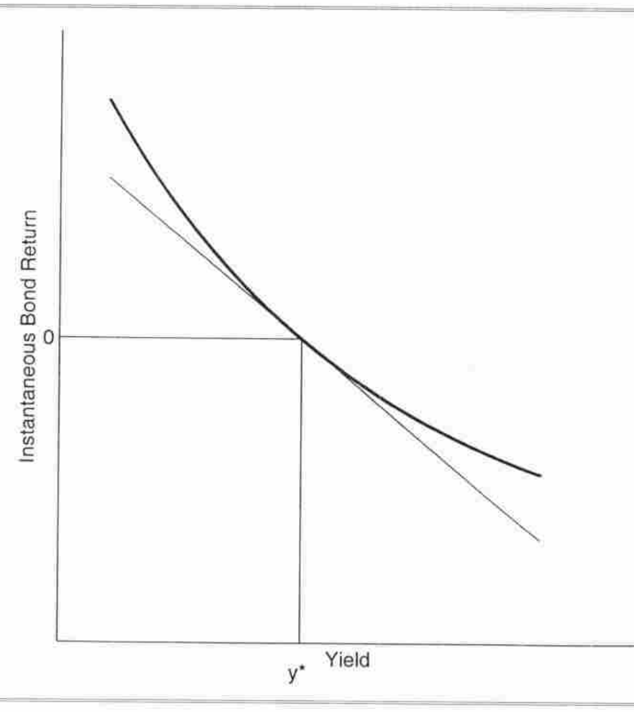
### Exhibit 6. Long Price versus Yield

The natural logarithm of bond price is plotted against yield for both a coupon and a zero-coupon bond. The slope of the tangent line at the initial yield is -D, where D is duration (and thus the plot is a straight line in the case of the zero-coupon bond). The vertical price change along the plot from the initial point is the log of 1 plus the instantaneous return on the bond.



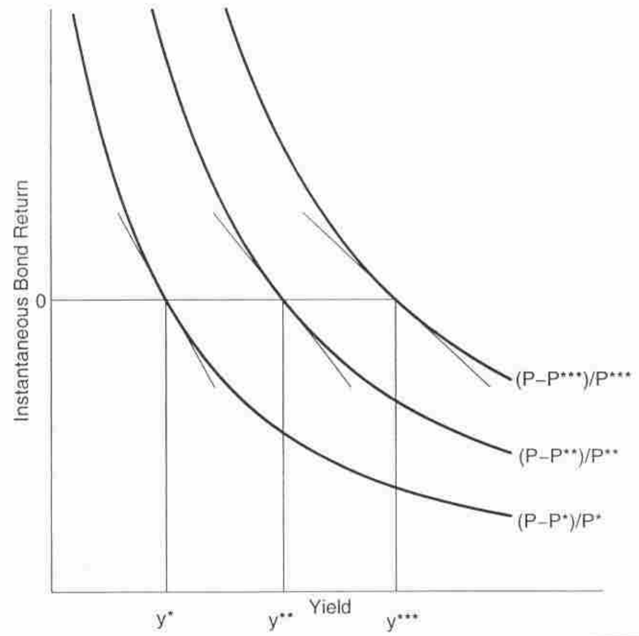

### Exhibit 7. Instantaneous Return versus Yield

The instantaneous return $\frac{(P-P^*)}{P^*}$ is plotted against yield for a bond. The slope of the tangent line at the initial yield y* is -D, where D is duration. The rate of change of slope at this point is C, where C is convexity. The vertical change along the plot from the initial point is the instantaneous return on the bond.



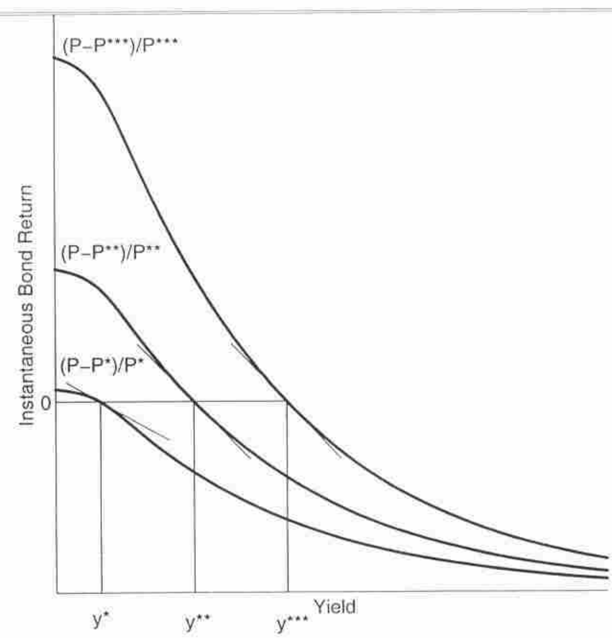
## Exhibit 8. Instantaneous Return versus Yield for Different Initial Yields

The instantaneous return $\frac{(P-P^*)}{P^*}$ is plotted against yield for a regular coupon bond. The slope of each curve at the initial yield is the negative of the duration of the bond. The curvature (or change in slope) of each curve at its initial yield is the convexity of the bond. The absolute values of these slopes and curvatures decrease as the initial yield increases. However, if this were for a zero-coupon bond, the slopes and curvatures would be identical as initial yield changes.




## Exhibit 9. Instantaneous Return versus Yield for Callable Zero

The figure is for a \$100 face value 10-year zero-coupon bond. The bond is callable (European-style) after one year at an exercise price of \$80. The slope of each curve at its initial yield is the negative of the duration of the callable zero. The curvature (or change in slope) of each curve at its initial yield is the convexity of the callable zero. Duration (absolute slope) increases as the initial yield increases. Convexity (curvature) goes from negative to zero to positive as initial yield increases.



## VI. Conclusion

In our experience, finance students invariably fail to understand the relationship between Macaulay duration and convexity on the one hand, and the traditional textbook plot of bond price versus bond yield on the other (i.e., Exhibit 1). We identify many common misunderstandings concerning duration and convexity. The confusion is attributed to ambiguity in the literature and to the traditional textbook plot itself. We show that if you relate bond price to bond yield, the roles of Macaulay duration and convexity are contaminated by the price level. It is argued that plotting the instantaneous bond return against the yield (see Exhibits 7, 8, and 9) is the most sensible graphical illustration of the concepts of duration and convexity. The features of this plot are: 1) vertical changes along the new plot show instantaneous rate of return (instead of dollar price change); 2) the slope of the new plot at the initial yield is $-D_{mac}$, where $D_{mac}$ is duration (instead of $-D_{mac}P$); and 3) the curvature (i.e., rate of change of slope) of the new plot at the initial yield is C, where C is convexity (instead of *CP*). The new plot has the following advantages over the traditional plot of bond price versus yield: property 1) allows sensible evaluation of the impact of yield changes on bond prices; and properties 2) and 3) allow direct interpretation of the plot in terms of duration and convexity–thereby avoiding the existing confusion in the literature.■

## Appendix

This appendix is a pedagogically useful summary of definitions and formulae appearing in the paper. We provide details for both the discrete and continuous interest rate cases. The formulae appearing in Exhibits 10, and 11 are evaluated for several illustrative bonds in Exhibit 2, 3, and 4.

### Exhibit 10. Summary of Formulae in Continuously Compounded Returns Case

The bond pays $C_t$ for t = 1,…,T and has continuously compounded yield *y*.

Bond Price

$$P = \sum_{t=1}^{T} C_t e^{-yt}$$

Macaulay Duration

$$D_{mac} \equiv -\frac{\frac{\partial P}{\partial y}}{P} = \frac{\sum_{t=1}^{T} tC_t e^{-yt}}{P} = \sum_{t=1}^{T} t\omega_t$$

$$\text{where } \omega_t \equiv \frac{C_t e^{-yt}}{P} \;\&\; \sum_{t=1}^{T} \omega_t = 1.$$

Bond Convexity

$$C \equiv \frac{\frac{\partial^2 P}{\partial y^2}}{P} = \frac{\sum_{t=1}^{T} t^2 C_t e^{-yt}}{P} = \sum_{t=1}^{T} t^2 \omega_t$$

$$\text{where } \omega_t \equiv \frac{C_t e^{-yt}}{P} \;\&\; \sum_{t=1}^{T} \omega_t = 1.$$

Slope of Price-Yield Curve

$$\text{SLOPE} = \frac{\partial P}{\partial y} = -D_{mac} P$$

Curvature of Price-Yield Curve

$$\text{CURVATURE} = \frac{\partial^2 P}{\partial y^2} = CP$$

Taylor Series

$$\Delta P \approx \frac{\partial P}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}(\Delta y)^2 = -D_{mac} P(\Delta y) + \frac{1}{2} PC(\Delta y)^2$$

### Exhibit 11. Summary of Formulae in Discretely Compounded Returns Case

The bond pays Ct for t = 1,...,T and has discretely compounded yield $Y$.

| | |
|---|---|
| Bond Price | $P = \sum_{t=1}^{T} C_t (1+Y)^{-t}$ |
| Modified Duration | $D^* \equiv -\frac{\frac{\partial P}{\partial Y}}{P} = \frac{\sum_{t=1}^{T} t C_t (1+Y)^{-(t+1)}}{P} = \frac{1}{(1+Y)} \sum_{t=1}^{T} t\omega_t$ where $\omega_t \equiv \frac{C_t (1+Y)^{-t}}{P}$ & $\sum_{t=1}^{T} \omega_t = 1.$ |
| Macaulay Duration | $D_{mac} = D^*(1+Y) = \sum_{t=1}^{T} t\omega_t$ where $\omega_t \equiv \frac{C_t (1+Y)^{-t}}{P}$ & $\sum_{t=1}^{T} \omega_t = 1.$ |
| Bond Convexity | $C \equiv \frac{\frac{\partial^2 P}{\partial Y^2}}{P} = \frac{\sum_{t=1}^{T} t(t+1) C_t (1+Y)^{-(t+2)}}{P} = \frac{1}{(1+Y)^2} \sum_{t=1}^{T} t(t+1)\omega_t$ where $\omega_t \equiv \frac{C_t (1+Y)^{-t}}{P}$ & $\sum_{t=1}^{T} \omega_t = 1.$ |
| Slope of Price-Yield Curve | $\text{SLOPE} = \frac{\partial P}{\partial Y} = -D^* P = -\frac{D_{mac}}{(1+Y)} P$ |
| Curvature of Price-Yield Curve | $\text{CURVATURE} = \frac{\partial^2 P}{\partial y^2} = CP$ |
| Taylor Series | $\Delta P \approx \frac{\partial P}{\partial Y}\Delta Y + \frac{1}{2}\frac{\partial^2 P}{\partial Y^2}(\Delta Y)^2 = -D_{mac} P(\Delta Y) + \frac{1}{2} PC (\Delta Y)^2$ |

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