Absence of Arbitrage, Detrending and Rational Stock Prices

This paper considers the theoretical foundations for detrending procedures used in empirical tests of rational stock pricing models and tests of related null hypotheses, such as implied variance bounds tests and tests for rational bubbles, e.g., Shiller (1981), Marsh and Merton (1986), Campbell and Shiller (1987), Diba and Grossman (1988), Evans (1991), Campbell and Kyle (1993), Wu (1997), Sarno and Taylor (1999). These empirical studies all invoke some method to detrend price-dividend data, usually to achieve the statistical property of covariance stationarity which, in turn, is exploited to conduct hypothesis tests. It is well known that the method selected to detrend the data contains implicit equilibrium restrictions. Recognizing that absence of arbitrage is a fundamental requirement for security pricing models, this paper utilizes further conditions on the underlying probability distributions to specify a detrending method which does satisfy absence of arbitrage. The theoretical conditions required for determining an appropriate detrending method are used to derive a number of specific closed form examples of detrenders. Detrended data are used to empirically test for the presence of rational bubbles. Tests of the null hypothesis are adapted from the dynamic linear rational expectations approach, e.g., West (1987).

In this paper, arbitrage free detrending is motivated by using the theoretical result that the product of the state price density (deflator) and the observed security price follows a martingale. For empirical testing purposes, exploiting the martingale property involves a substantively different econometric approach than that followed by conventional studies which exploit the properties of a covariance stationary process, e.g., West (1987), Evans (1991). Even though first differences of a martingale process do satisfy the essential requirements for a covariance stationary process, the arbitrage free detrending process imposes further equilibrium restrictions which involve specifying a fully parametric security pricing model. This produces a maximum likelihood estimation problem, where a specific hypothesis can be tested by determining whether certain parameters take particular values. In contrast, conventional studies do not typically make such strong assumptions about the theoretical model generating asset prices. Detrending to covariance stationarity is arguably done to achieve a minimal destruction of statistical information, in order to facilitate the inductive testing of specific hypotheses. While this does avoid the joint hypothesis imposed by arbitrage free detrending, it does raise other questions, e.g., about validity or generality of specific results.

There are many potential methods of detrending prices and dividends to achieve covariance stationarity. The presence of an equivalent martingale measure dictates that there are certain methods of detrending which both satisfy covariance stationarity and are consistent with absence of arbitrage opportunities in security prices. With this in mind, this paper investigates the following types of questions: What are the properties of arbitrage free detrenders? How restrictive are the equilibrium conditions imposed by absence of arbitrage? What are the limitations of using specific detrenders? In the following, Section I provides an introduction to the general problem being examined. Section II states the relevant assumptions and derives two important Propositions concerning arbitrage free pricing and differential restrictions that the equivalent measure must satisfy. Section III provides some closed form examples for specific detrenders. The connection between the form of the detrender and the distributional assumption for the state variables is illustrated. Section IV is concerned with examining the empirical properties of the detrending procedure, using an updated version of the Shiller (1981) data set. Finally, Section V draws some conclusions from the results presented in the paper.

I. The Theory of Rational Stock Pricing

In continuous time, security valuation problems have the general form:

$$p(t) = V(t,T)\ p(T) + \int_t^T V(t,u)\ d(u)\ du \qquad (1)$$

where p(t) is the security price observed at time t, d(t) is the instantaneous dividend or coupon paid at t, and T is the terminal or maturity date for the valuation problem ($T \geq t \geq 0$). Equation (1) is often referred to as a "no arbitrage" condition, e.g., Blanchard and Watson (1982).[1] The security valuation problem is associated with determining the correct form of the valuation operators {V(t)} which obey the fundamental property V(t+s) = V(t)V(s). The valuation operators {V(t)} will, in general, involve both discounting and expected value operations. Two conditions almost always imposed on the {V(t)} are:

$$\lim_{T \to t} V(t,T) = I \qquad \lim_{T \to \infty} V(t,T) = 0$$

The first condition ensures continuity and consistent pricing. The second condition ensures pricing convergence, e.g., Craine (1993).[2] It is well known that the convergence of the operator as $T \to \infty$ is required for the satisfaction of a transversality condition which, in turn, is needed to ensure that the expectational difference equation identified by (1) will not have an infinite number of solutions.

The importance of the transversality condition in testing for the presence of rational bubbles in stock prices can be seen by interpreting (1) as a continuous time version of a price-dividend (stock) valuation model, where p(T) is the anticipated (selling) price at time T and d(u) is the continuous dividend paid over $u \in (t,T]$. Progressive substitution for p(T) produces the infinite horizon, discounted-dividend model. This model forms the basis of empirical tests for rational bubbles. Theoretically, a rational bubble is specified by observing that the security price p(t) can be modelled as the sum of two components, a market fundamentals component $p_F(t)$ and a rational bubbles component B(t), e.g., Evans (1991): $p(t) = p_F(t) + B(t)$, where $p_F(t)$ is associated with the infinite sum of the discounted value of expected future dividends and B(t) can be any random variable that satisfies B(t) = V(t,T) B(T). Because p(t) can incorporate both fundamental and bubble information, progressive substitution for p(T) produces restrictions on the bubble component. If **the transversality condition**:

$$\lim_{T \to \infty} V(t,T)\ p(T) = 0$$

is satisfied, then a rational bubble is ruled out. The current price will be determined solely by the discounted value of expected future dividends.

More precisely, in valuation problems where p(T) and d(·) are uncertain at the t=0 decision date, the {V(t)} take the form of expectations operators which also possess a discounting feature. In terms of the price-dividend model, (1) can be expressed:

$$p(0) = E[\ e^{-\{R_T\ T\}} p(T) \mid F_0] + E[\int_0^T e^{-\{R_u\ u\}}\ d(u)\ du \mid F_0]$$

$$= E[e^{-\{R_T\ T\}} p(T) \mid F_0] + \int_0^T E[e^{-\{R_u\ u\}}\ d(u) \mid F_0]\ du = V(0,T)\ p(T) + \int_0^T V(0,u)\ d(u)\ du$$

where F_t is the conditioning information available at time t and R_i is the appropriate stochastic discounting factor for period i. Following Blanchard and Watson (1982) and others, the progressive substitution for $E[p(T) \mid F_0]$ in (1) required for the discounted dividend model to hold involves applying **iterated expectations** over increasing F_t to expected future prices to derive a general solution of the form:

$$p(0) \ = \int_0^{\infty} V(0,u) \ d(u) \ du \ + \ B(0) \ = \ p_F(0) \ + \ B(0) \qquad (2)$$

where:

$$B(0) \ = \ \lim_{T \to \infty} \ V(0,T) \ B(T)$$

In other words, the market price of a security can deviate from its fundamental value without violating absence of arbitrage, providing the expected value of the rational bubble increases at a faster rate than that specified by the {V(t)}. For a rational bubble, B(0) ≠ 0 and the transversality condition is not satisfied. There is a component of the price, the rational bubble, which is not related to the discounted dividends. In order to still be present in the limit, the bubble component is growing at a faster rate than the discounting process.[3]

In Section II it is demonstrated that, by making specific assumptions about the stochastic properties of the state variables, it is possible to precisely specify the expectation embedded in the {V(t)} in a way which is useful for identifying appropriate detrending methods to use in tests of rational bubbles. To this end, the equivalent martingale measure provides a fundamental connection between the stochastic structure of the state variables and the absence of arbitrage in security prices. With "great generality" it has been shown that the existence of an equivalent martingale measure implies an absence of arbitrage opportunities in security prices, though the converse is not necessarily true (Back and Pliska 1991). As a consequence, the conditions required for an equivalent martingale measure provide a relatively weak set of restrictions on the pricing operators {V(t)} required to ensure absence of arbitrage in the {p(t)}. Much sharper restrictions are imposed when starting from an assumed empirical measure which is a diffusion, or strong Markov stochastic structure. An important analytical advantage of using the diffusion approach is that, under appropriate assumptions, it permits the introduction of restrictions which can be used to derive a closed form for the equivalent martingale measure. In turn, this permits the observed prices to be transformed into a martingale process in a fashion which ensures consistency with absence of arbitrage. With further restrictions, the resulting martingale process can be used to conduct empirical tests.

The notion that the detrending procedure selected can impact the results of empirical tests is not new. For example, Osborn (1995) and Gregory and Smith (1996) are recent contributions to studies on the role of detrending in the measurement of business cycles. Similar, though less detailed, observations have been made about the impact of detrending in tests for stock price rationality, e.g., Grossman and Shiller (1981), Evans (1991), Diba and Grossman (1988b). Despite this recognition of the potential importance of the detrending procedure selected, a number of somewhat different detrending procedures have been employed in studies of stock price rationality. A recent example is Campbell and Kyle (1993) which uses the Standard and Poor's Composite and the associated dividend series both detrended, initially, by the producer price index, to get a "real stock price" and "real dividend". These series are then further detrended by "the mean dividend growth rate over the sample". The resulting series are, under the Campbell and Kyle method, required to be I(1) processes. With a number of minor improvements, this is the detrending procedure followed in Campbell and Shiller (1987). The failures of the less sophisticated detrending procedures used in early studies, e.g., Shiller (1981), contributed significantly to the "econometric difficulties" identified in Campbell and Kyle (1993).

An important feature of this paper is to make a connection between the detrending process and the specification of the indirect utility function for the representative investor. This connection has also been recognized in previous studies. Diba and Grossman (1988a) extend Lucas (1979) to demonstrate theoretically that the marginal utility of consumption will impact the determination of rational stock prices. Olivier (2000) is an excellent recent example of this approach. Scott (1989) also recognizes the importance of the marginal rate of substitution and develops and tests an applicable estimation procedure. In essence, in order satisfy absence of arbitrage, detrended security prices have to be adjusted for some riskless interest rate, with a further adjustment associated with the riskiness of the security. In a representative investor framework, this implies that specification of the detrending procedure requires some equilibrium assumption about risk aversion properties. Following He and Leland (1993) and Heaney and Poitras (1994), risk aversion properties can be modelled by assuming a specific stochastic process for the state variables. This study extends these results by providing a method for deriving a closed form for the detrending procedure using diffusion processes to model the state variables.

II. Absence of Arbitrage with State Variables as Prices

The theory of arbitrage free valuation is voluminous. Numerous results with varying degrees of generality and applicability are available, e.g., Duffie (1992). A number of analytical simplifications are often invoked to facilitate derivation of results. For example, identifying the relevant state variables is a significant practical complication which arises in accessing the equivalent martingale results. Following Harrison and Kreps (1979), it is conventional to assume that the requisite state variables are prices on non-dividend paying securities. In practical applications this structure is, typically, unnatural. Many individual securities pay "dividends" or coupons. For purposes of relaxing the no-dividends condition, it is convenient to work with the cumulative dividend-price process, the combination of the ex-dividend price processes and the associated dividend payouts, e.g., Duffie (1992, Ch.6). This requires specification of the stochastic price processes and a method for handling dividends.

In addition to making security prices the only state variables, another substantive analytical simplification is achieved by working with processes which have been detrended by the locally riskless interest rate process. In continuous time, locally riskless means the sample paths of the process have finite variation. A normalization condition that the process equal one at the pricing decision date, time 0, is imposed together with the assumption that the discounting process is strictly positive. In some cases, the discounting process is handled with the simplification of assuming that interest rates are zero, directly suppressing consideration of issues associated with the numeraire, but this approach is impractical for empirical applications. Introducing detrending by the interest rate process into the definition of the state variables permits the R-N derivative to be used, without further transformation, to derive arbitrage free shadow prices as the product of the state price density and the observed, detrended price process. To make the connection with empirical applications means that security prices and dividends detrended by the interest rate are the relevant state variables.

The final point of reference concerns the use of continuous time to specify the stochastic structure. More precisely, the security market being considered contains K interest rate detrended prices, S, which follow the Kx1 vector **diffusion** price process:

$$dS = \alpha(S,t)\ dt + \sigma(S,t)\ dB$$

This compact notation requires that dS, dB and $\alpha(\cdot)$ are a Kx1 vectors while $\sigma(\cdot)$ is a KxK matrix. The $\sigma(\cdot)$ notation is somewhat misleading because Σ, the variance-variance matrix for the dS, is equivalent to $\Sigma = \sigma\sigma'$. Arbitrage free pricing requires the specification of K risk premia (λ) of the form: $\lambda = \alpha(S,t) + D$ where D is the interest rate detrended dividend and $\alpha(\cdot)$ is the drift coefficient for the diffusion process. This stochastic structure requires some method for handling D. To avoid having to introduce additional state variables, it is assumed that D is a continuous differentiable function of the stock price process, denoted D(S,t). This assumption includes deterministic dividends (including D=0) and Merton proportional dividends. Introducing distinct dividend processes produces complications similar to including other stochastic non-price state variables, such as stochastic volatility. This case is examined, only by example, in Section III.

Given these preliminaries, the basic analytical structure for diffusions requires a filtered probability space $(\Omega,\mathbf{F},\{F_t\},P)$ where $0 \leq t < T$, and the filtration $\{F_t\}$ satisfies the usual conditions: F_0 contains all the null sets of P; and $\{F_t\}$ is right continuous, meaning that $F_t = \bigcap_{s>t} F_s$ (Dothan 1990, Sec.7.2). S(t) denotes a K-dimensional vector of $\{F_t\}$ measurable state variables defined on $[0,\infty)\times\Omega$. Restrictions imposed on $\alpha(S,t) \in R^K$ and $\sigma(S,t) \in R^K \otimes R^K$ are fundamental to interpreting the applicability of the results. While it is possible to develop more general conditions, it is assumed that both $\alpha(\cdot)$ and $\sigma(\cdot)$ are F_t measurable and defined on $[0,T]\otimes R^K$ and satisfy the requisite Lipschitz and growth conditions. S(t) is assumed to be a continuous real-valued diffusion process that takes on values on R^K. Given that B(t) is a K-dimensional vector of independent Brownian motions defined on $(\Omega,\mathbf{F},\{F_t\},P)$, assume there exists Q, a martingale measure on $(\Omega,\mathbf{F})$ equivalent to P. **The security market is assumed to be complete** which implies that Σ will be of full rank (Jarrow et al. 1999).

The method of constructing Q from P depends fundamentally on the R-N derivative, dP/dQ. More precisely, for any $A \in \mathbf{F}$:

$$\int_A dP = \int_A \frac{dQ}{dP}\, dP \equiv \int_A Z(T)\, dP$$

The equivalence of P and Q implies that the necessary and sufficient conditions for the general existence of a measurable Z(T) transformation are provided by the Radon-Nikodym Theorem. The existence of the R-N derivative serves to define a strictly positive martingale {Z(S,t)} = {Z(t)} where:

$$Z(t) = E[Z(T) \mid F_t] = \frac{dQ}{dP}\Big|_{F_t} \qquad \forall \qquad 0 \le t \le T$$

where Z(0) = 1. Dothan (1990) descriptively refers to {Z(t)} as the Likelihood-Ratio process. Further properties for the change of measure in security pricing situations can be derived from Girsanov's Theorem. Under the assumptions provided Z_T will be unique due to market completeness (Jarrow et al. 1999).

Considerable effort has been given to establishing that the presence of an equivalent martingale measure is sufficient to ensure absence of arbitrage in security prices. For this purpose, the R-N derivative and the associated {Z(t)} provide the appropriate transformation. Arbitrage free pricing requires that S(t) and D(t), the price and dividend processes, be such that the cumulative dividend-price process:

$$S(t) + \int_0^t D(u)\, du$$

is a martingale under Q.[4] The precise role played by Z(T) in achieving this result can be formalized in the following adaption of Girsanov's theorem:

Proposition 1: **Arbitrage Free Shadow Pricing**

Assuming Q exists, then there exists a positive martingale {Z(t)} on (Ω,**F**,{F_t},P), such that for the cumulative dividend-price process, the transformed process:

$$Z(t)S(t) \ + \int_0^t Z(u) \ D(u) \ du$$

is a martingale on (Ω,**F**,{F_t},P) such that for t,k $\geq$ 0:

$$Z(t) \ S(t) \ = \ E^P \ [\int_t^{t+k} Z(u)D(u) \ du \ + \ Z(t+k) \ S(t+k) \ | \ F_t] \qquad (3)$$

To model current prices, it is appropriate to let t = 0 and observe that Z(0) = 1.

The significance of Proposition 1 can be clarified by comparing the {Z(S,t)} with the valuation operators {V(t)} arising from (2). Proposition 1 reveals that {Z(t)} corresponds to the {V(t)} after adjusting for the interest rate detrending of S and D. The expectation in (3) is appropriate for empirical testing purposes because it is taken with respect to the empirical P measure. Assumptions made about the specific functional form of the state variable processes represent a nested null hypothesis under which a specific {Z(t)} is the appropriate, arbitrage free detrender for prices and dividends. In other words, from the dependence of Q on S and D, elements in the process {Z(t)} will also depend on S and D (because D depends on S). However, in order to be of practical value, a method is required to derive specific closed forms for Z(S,t). Because Z(S,t) is a function of a random variable, the precise method for doing this is not obvious. A potential and intuitive approach would be to invoke Ito's lemma to specify partial differential equations for Z(s,t) which can then be solved to determine closed forms for {Z(t)}. This is the approach used here.[5]

For purposes of empirically testing price dividend models, Proposition 1 suggests a general outline for an arbitrage free detrending procedure. Observed prices and dividends, p and d, are initially detrended by the relevant interest rate. The resulting S and D series are then multiplied by the Z(s,t) applicable to the valuation problem at hand. The specific functional form for the Z(s,t) used will depend on both the parameters of the underlying state variable processes and the associated state variable risk premia. To be practical, implementation of this detrending procedure depends crucially on having a closed form for determining the elements of {Z(t)}. The problem at hand involves starting from the S(t) as diffusions and developing further restrictions needed to derive practical closed forms for Z(S,t). Following Heaney and Poitras (1994), it is possible to do this by making the following assumption:

Assumption 1: Z(S,t) is twice differentiable in the state variables, and once differentiable in time for each t in the interval $T \geq t \geq 0$.

Recalling that {Z(t)} is a martingale, if Z(t) is assumed to obey these differentiability requirements then it follows by Ito's lemma that it satisfies a set of partial differential equations which can then be solved to get a specific Z(s,t).

Assumption 1 is a functional restriction on Z which also embeds the relatively weak diffusion assumptions on dS required to derive Proposition 1. These conditions provide for the derivation of useful restrictions on the coefficients in the diffusions:

Proposition 2: **Differential Properties of Z**

Under Assumption 1 and the conditions required for Proposition 1, Z(s,t) obeys the K+1 first order partial differential equations:

$$\frac{\partial}{\partial s_i} Z(s,t) = h_i(t)\ Z(s,t) \qquad (4a)$$

$$\frac{\partial}{\partial t} Z(s,t) = f(t)\ Z(s,t) \qquad (4b)$$

where:

$$h = -\Sigma^{-1}\lambda$$

$$\Sigma = \sigma\sigma'$$

$$f = -(\alpha' h + \frac{1}{2}h'\Sigma h + \frac{1}{2}Tr\ (\Sigma\frac{\partial h}{\partial s}))$$

and Σ is a full rank K×K variance-covariance matrix associated with the Brownian motions of the security price diffusions. Tr denotes the trace (the sum of the diagonal elements) of the matrix in brackets and $\partial h/\partial s$ is a KxK matrix with components $\partial h_j/\partial s_i$ where s_i is a defined point for the random variable $S_i(t)$.

The specification of f in Proposition 2 reveals that the assumption of differentiability on Z(S,t) also requires that the diffusion coefficients $\sigma(\cdot)$ and $\alpha(\cdot)$ be differentiable. This is substantively stronger condition than the Lipschitz and growth conditions required to get Proposition 1. Given the implied restrictions on Σ and λ, conditions (4) can be integrated to obtain a specific solution for Z(s,t). This resulting Z will depend on maintained null hypotheses about the parameters and functional form of the underlying price processes.

It is significant that the differential equations for Z(s,t) given in Proposition 2 provide restrictions on both the risk premia and the parameters of the underlying state variable process. More precisely, Proposition 2 can be used to identify absence of arbitrage restrictions on the parameters of specified diffusion processes (Heaney and Poitras 1994). These restrictions provide necessary and sufficient conditions for Z to satisfy (4). In particular:

Corollary 2.1: **Necessary and Sufficient Coefficient Restrictions for Z(S,t)**

Given the assumptions required for Proposition 2, then the following integrability conditions:

$$\frac{\partial f}{\partial s_i} = \frac{\partial h_i}{\partial t} \qquad and \qquad \frac{\partial h_i}{\partial s_j} = \frac{\partial h_j}{\partial s_i} \qquad (5)$$

are necessary and sufficient for Z(s,t) to satisfy (4).

From this it follows that for Assumption 1 to hold, $\alpha(\cdot)$, $\sigma(\cdot)$ and $D(\cdot)$ are also required to be second order differentiable in state and first order differentiable in time. In terms of the properties of the operators {V(t)}, these restrictions are also required to motivate the derivative of {p(t)} in (1). In addition to being required for consistency of the cross derivatives of (4a) and (4b), the restrictions provided by (5) also ensure it is possible to integrate (4) to solve for Z(s,t). Under the assumptions used in Proposition 2, this integration will be **path independent** permitting the Z(s,t) solution to depend only on the initial and terminal states and not on the specific path taken by the state variables.

III. Specific Detrenders

This Section exploits (5) to derive closed form solutions for Z(s,t). Initially, it is useful to derive Z for a **non-dividend paying** asset price process. While not immediately obvious, this case is relevant to modelling the precise relationship between prices and dividends because Proposition 1 then applies to a cumulative dividend-price process. As discussed in Section II, derivation of a specific closed form for Z requires precise specification of the price process, which then becomes a nested null hypothesis under which the Z is appropriate. To this end, let the non-dividend paying price process, Y, follow the lognormal (Black-Scholes) process:

$$dY = \mu Y\, dt + \sigma Y\, dB \qquad (6)$$

In (6), Y has been interest rate detrended. The derivation of Z from the conditions associated with (4) require:

$$h = -\frac{\mu}{\sigma^2\ Y} \qquad and \qquad f = \frac{1}{2}[(\frac{\mu}{\sigma})^2 - \mu]$$

Verifying that (5) is satisfied, the Z(Y,t) can now be derived as:

$$Z(Y,t) = e^{\frac{1}{2}((\frac{\mu}{\sigma})^2 - \mu)(t-t_0)} (\frac{Y}{Y_0})^{-\frac{\mu}{\sigma^2}} \qquad (7)$$

Empirically, detrending the observed non-dividend paying asset price, firstly, by the interest rate and, secondly, by Z will produce a martingale process, under the null hypothesis (6). This detrender requires two parameters to be estimated, μ and σ.

An intuitive motivation for the use of Z is provided by contrasting (7) with results derived from representative investor models. In this context, He and Leland (1993) and others construct equilibrium which can support a given stochastic process. For example, it is demonstrated that a representative investor with constant proportional risk aversion is required to support a geometric Brownian motion in wealth. In other words, taking Y to be an aggregate wealth process it is possible to interpret Z as the indirect utility function of the representative investor (Heaney and Poitras 1994). Because the underlying price process for (7) is assumed to be geometric Brownian motion, this Z has a power utility representation, consistent with constant proportional risk aversion. This requires the investor to maintain the fraction of total wealth invested in the risky asset. Under appropriate conditions, it is possible to connect a given method of detrending prices and dividends embodied in Z with assumptions concerning the utility of the representative investor.

The representative investor framework identifies the state process Y as aggregate wealth. This suggests a method for incorporating dividends without directly specifying separate state variable processes for dividends: define Y as the cumulative dividend-price process. Taking D to be a continuous, differentiable function of S, the resulting Z(Y,t) can be applied in (3). From this, it is possible to specify an arbitrage free detrending procedure for use in empirical tests of rational stock pricing models. Dividends and prices are detrended by the interest rate and then multiplied by the appropriate Z(y,t) with the resulting series being used for the empirical tests. Observing that the functional form of the arbitrage free detrender, Z(Y,t), depends on a **null hypothesis** about the underlying cumulative dividend-price stochastic process, it follows that empirical tests of models involving security prices will also involve a joint hypothesis about the stochastic structure of the underlying state variables. Different stochastic assumptions will lead to different approaches to detrending. In other words, the method selected to detrend data implicitly imposes assumptions about the underlying equilibrium.

In order to provide another approach to specifying a detrender for dividend paying securities, it is possible to extend the analysis to the Merton (1973) case where dividends are a constant fraction (δ) of the stock price: $D = \delta S$. The asset price, S, is for simplicity assumed to follow a lognormal process: $dS = \theta S\, dt + \sigma S\, dB$. Because there is still only one state variable, there is also only one $\lambda = \theta S + \delta S$. While this approach to incorporating dividends is not fully consistent with observed dividend behaviour, it does provide the significant analytical simplification of retaining only prices as state variables. Based on Proposition 1, absence of arbitrage for a **dividend paying asset** requires the cumulative dividend-price process to be a martingale under Q. In this case:

$$h = -\frac{\theta + \delta}{\sigma^2 S} \equiv -\frac{\gamma}{S}$$

$$f = \gamma\theta - \frac{1}{2}\sigma^2 \gamma (1 + \gamma) = \frac{1}{2}\left(\frac{\theta^2 - \delta^2}{\sigma^2} - (\theta + \delta)\right)$$

Verifying that (5) are satisfied, it is possible to derive Z as:

$$Z(S,t) = e^{f(t-t_0)} \left(\frac{S}{S_0}\right)^{-\gamma} \qquad (8)$$

By construction, this Z is based on the null hypothesis of lognormal, interest rate detrended security prices and constant proportional dividends. Compared to (7), this detrender has an additional parameter to be estimated. Instead of the drift μ, there are now two values to be estimated, θ and δ.

The connection between investor utility and Z provides a useful method of illustrating the equilibrium assumptions embedded in specific detrending procedures. Consider a more general form of (6), the constant elasticity of variance (CEV) process. Taking Y to be either the aggregate wealth or cumulative dividend-price process:

$$dY = \mu Y \, dt + \sigma Y^{\frac{\beta}{2}} \, dB \qquad (9)$$

For this process:

$$h = -\frac{\mu}{\sigma^2} \, Y^{1-\beta} \qquad \textit{and} \qquad f = \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 \, Y^{2-\beta} + \frac{\mu}{2}(1-\beta)$$

To satisfy the integrability conditions (5) now requires that either $\beta = 2$ or $\mu = 0$. Of the class of processes covered by the CEV, only the limiting lognormal, β=2 case is compatible with a non-zero drift, and $\lambda \neq 0$. For μ=0, because the asset involved does not pay dividends, this condition reduces to risk neutrality, $\lambda = 0$. The associated Z(t) = 1, a constant, indicating that detrending by the interest rate is all that is required for arbitrage free detrending. In the absence of a dividend process, the $\beta \in [0,2)$ CEV processes does not appear compatible with a risk averse representative investor.

Incorporating a separate diffusion process for dividends, as in Campbell and Kyle (1993), substantively complicates the derivation of the arbitrage free detrender. Two dependent state variable processes must combine to produce a cumulative dividend-price process which is a martingale under Q. In order to avoid problems associated with market incompleteness or inherent inconsistencies between the price and dividend processes, the asset price (capital gains) and dividend processes have to be, somehow, functionally related. In order to satisfy Proposition 1, specification of an asset price process imposes implicit restrictions on the dividend process. Ignoring the technical complications associated with introducing a non-price state variable, derivation of Z(t) depends on the specific diffusion processes selected to model actual empirical processes. The presence of two state variables means that h in Proposition 2 now has two elements. As a consequence, there are now three integrability conditions to satisfy, instead of one. However, there is still only one Z(t) applicable to all the state variables. The upshot is that the combined integrability conditions can usually be satisfied only in the $\lambda = 0$ or risk neutral case where the arbitrage free detrender Z(t) = 1.

IV. Empirical Results

The data is an updated version of the set used by West (1987), which was initially examined by Shiller (1981). The updated data set extends West (1987) from 1980 to 1997 and now has 127 observations starting from 1871 on four variables: p_t is an annual series of January values of the Standard and Poor's (S&P) Composite Stock Price Index; d_t is the S&P twelve-month moving total of the nominal dividends per share; r_t is the total return to investing for six months in January at the January 4-6 month prime commercial paper rate (six month starting January 1980) and for another six months at the July 4-6 month prime commercial paper rate (six month starting July 1980); and, PPI is the Producer Price Index for all finished goods (1982=100).

IV.1 Estimation of the GBM Detrender

The detrender selected for estimation is (7). Since the associated geometric Brownian motion process (6) is written on the interest-rate detrended cumulative dividend-price process Y, the first step in our detrending procedure is to detrend the observed stock price and dividend by the riskless interest rates. This is done by multiplying each element of p(t) and d(t) by:

$$\prod_{i=0}^{t} \frac{1}{(1 + r_i)} \qquad \forall\ t = (0,1,2....(T-1))$$

This interest-rate detrender equals unity at the initial date since $r_0 = 0$. The resulting series, S(t) and D(t) respectively, are given in Figures 1 and 2. A comparison can be made with the commonly used technique of deflating prices (and dividends) by the PPI, given in Figure 3. S_t and D_t are used to estimate the parameters used in the detrender Z(t) and, in turn, to calculate YD(t), the Z-detrended value of Y_t:

$$Y_t = S_t + \sum_{i=0}^{t-1} D_{t-i} \qquad YD_t = Z_t\, S_t + \sum_{i=0}^{t-1} Z_{t-i}\, D_{t-i}$$

In other words, Y_t is the interest rate detrended, cumulative dividend-price process. YD_t is the interest rate and Z(t) detrended, cumulative dividend-price process.

An important step in estimating the detrender Z(t) is the estimation of the drift μ and the volatility σ of dY/Y from (7). Following Campbell, Lo and Mackinlay (1997), the maximum likelihood estimators of the drift and volatility are respectively, the (adjusted) mean and the standard deviation of the log-differences in the state variable, Y, in each pair of adjacent time periods. To formalize:

$$\hat{\alpha} = \frac{1}{T}\sum_{t=1}^{T} \ln\left(\frac{Y_t}{Y_{t-1}}\right) \qquad \hat{\sigma} = \sqrt{\frac{1}{T}\sum_{t=1}^{T} \left[\ln\left(\frac{Y_t}{Y_{t-1}}\right) - \hat{\alpha}\right]^2} \qquad \hat{\mu} = \hat{\alpha} + \frac{\hat{\sigma}^2}{2}$$

with T being the sample size.

Using the full 1871-1997 sample of observations raises a complication. Consistent with empirical results from previous studies, estimating the drift and volatility reveals somewhat different estimates for the pre and post-WWII periods:

Table 1

Estimated Drifts and Volatilities of Y_t

	μ	α	σ
Pre-war (1871-1944)	.00602	.0047	.0507
Post-war (1945-1997)	.00998	.00971	.02318
Combined (1871-1997)	.00761	.00674	.04173

The estimated drifts take only slightly positive values, because the prices and dividends have already been interest-rate detrended. The results show that the estimated post-war α drift is about twice the size of the pre-war α drift, even though the difference is not statistically significant (t=0.66). Further empirical examination of these observed differences in the drift and volatility estimates reveals a substantive change in dividend payout behaviour over the two subperiods, with decidedly higher dividend payout observed in the pre-WWII period. This dividend behaviour is consistent with the higher Y observed in the post-war sample.

Confronted with this empirical complication, it is sensible to proceed without adjustment, estimating the Z(s,t) detrender using the drift and volatility estimates for the combined sample. These results can then be contrasted with results where the sample is broken into two subsamples, deriving a different detrender for each period by substituting the relevant drift and volatility estimates into (7). In each case, the products of S_t and D_t with Z_t give the arbitrage-free detrended price (ZS) and dividend (ZD), respectively. In the subsample cases, the presumed structural break is directly incorporated into the detrending procedure, because the Z(t) is generated using the μ and σ estimated over that sample period. The relevant Z(t) for the full sample and the post-war sub-sample are provided in Figures 4 and 5. These Figures can be interpreted by recognizing that Z(t) is the marginal utility of Y(t). In this context, Z(t) can be used to provide a relative comparison of Y at different points in time. As expected from theory, higher values of Z are associated with bear markets and, conversely, the lowest values of Z arise in bull markets. For example, Figure 5 reveals that the market valuations circa 1997 still had not reached the levels attained by the bull market of the 1960's.

IV.2 Martingale Tests[6]

In Section II it was observed that {Z(t)} is a strictly positive martingale. This can be seen by applying Ito's lemma to (7) and using (6) to show the martingale requirement also applies to Z(t). In general, the martingale property of {Z(t)} is inherent in the theoretical framework captured in (4) and (5). Evidence that Z(s,t) generated by (7) does not follow a martingale is a rejection of the null hypothesis (6). In particular, testing empirically whether Z(t) follows a martingale is a specification test for the validity of (7). Various possible statistical methods are available for testing whether a time series is a martingale process, e.g., Campbell, et al. (1997, Sec. 2.1). Table 2 reports martingale tests derived from the result that first differences of a martingale process are orthogonal. Hence, under the null hypothesis that the Z(t) is a martingale, the slope and intercept in an AR(1) regression for ΔZ(t) are expected to be insignificant. Similar tests are also done for Y(t) and YD(t). The properties of these two series reflect the difference between using the risk averse detrender (7), for YD(t), and assuming a risk neutral detrender, Z(t) = 1, for Y(t). Results for both the full sample and the post-war samples are provided.

The results in Table 2 reveal that Z(t) generated by (7) comfortably meets the requirements of a martingale for the post-war sample, i.e., Z(t) exhibits zero autocorrelation using first differences.

However, for the full sample of 127 years, the highly significant slope coefficient indicates that Z(t) fails the martingale requirement. This rejection for the full sample, but not the post-war sample, lends empirical support to the view that there has been substantive changes in stock market valuations, in general, and firm dividend behaviour, in particular, in the post-war period. In terms of Z(t), the rejection of the martingale hypothesis for the full sample also indicates that there are possible problems with estimating the drift and volatility for such a long time period. Because these parameters are essential components in the functional form for Z(t), it may be necessary to incorporate parameter evolution in the specification of Z(t). Proposition 1 says that after detrending by Z(t), the cumulative dividend-price process YD(t) is also required to follow a martingale. The associated empirical test for the martingale property in {YD(t)} is a joint hypothesis of the martingale hypothesis and the validity of the assumed Z(t). Not surprisingly, Table 2 reveals that YD(t) also comfortably satisfies the martingale requirement for the post-war sample and fails for the full sample.

The possibility that a risk neutral Z (Z = 1) is appropriate can be tested by examining the martingale properties of Y(t). Examining Table 2 reveals the differences between the martingale tests for Y(t) and YD(t). For neither the post-war or full sample does Y(t) fully satisfy the martingale requirement. In the full sample, neither the intercept or slope is significant at the 5% level, but the F test that both coefficients are zero is significant. For the post-war sample, the intercept is significant while the slope is insignificant. This is consistent with {Y(t)} being a martingale with drift. Yet, drift is indicative of something more than contained in Proposition 1 with Z(t) = 1. This begs the question: is the failure of Y(t) to comfortably satisfy the martingale hypothesis due to using a risk neutral detrender for Z(t)? In any event, the martingale tests suggest that full sample results associated with the Z(t) derived from (7) will be unreliable. Being the only sample that comfortably satisfies the martingale requirement, the post-war results for the Z(t) derived from (7), and the associated YD(t), are of particular interest In contrast, the martingale tests indicate that the full sample results using the risk neutral detrender may be more reliable.

IV.3 The Rational Bubble Specification Tests

The basic empirical problem in rational stock pricing models is to test the null hypothesis of $p_t = p_t^f$ versus $p_t = p_t^f + B_t$. A number of previous studies, e.g., West (1987), construct hypothesis tests by applying the dynamic linear rational expectations model (DLREM), e.g., Hansen and Sargent (1981). More precisely, if dividends are allowed to be endogenous and H_t is a subset of F_t, which consists of current and lagged dividends, in discrete time the null hypothesis implies that:

$$p_t = \sum_{i=1}^{\infty} [b^i E(d_{t+i} \mid H_t)] + e_t \qquad (10)$$

where:

$$e_t = \sum_{i=1}^{\infty} b^i [E(d_{t+i} \mid F_t) - E(d_{t+i} \mid H_t)] \qquad (11)$$

and e_t is not correlated with H_t, such that $E[H_t e_t] = 0$ and b is the discrete discount factor corresponding to the continuous discount factors in the V(t). $E(d_{t+i} \mid H_t)$ is by definition the expectation of future dividends given the current and lagged dividends. This implies that $E(d_{t+i} \mid H_t)$ can be calculated as an ARIMA forecast for d_{t+i}. In other words, as Hansen and Sargent (1981) suggest, if d_t is stationary, then there is a closed-form expression for $E(d_{t+i} \mid H_t)$ in terms of a distributed lag of the current and past dividends. In the conventional DLREM, the coefficients of the distributed lag are functions of the discount factor and the parameters associated with the univariate ARIMA process for d_t.

Conventional tests based on the DLREM are altered somewhat when arbitrage free detrended prices (ZS) and dividends (ZD) are used. As a consequence of the detrending procedures, these series imply much stronger coefficient restrictions than those for the conventional DLREM. More precisely, the equations to be estimated are:

$$ZD_{t+1} = \phi_0 + \phi_1 \, ZD_t + \upsilon_{t+1} \qquad (12)$$

$$ZS_t = \delta_0 + \delta_1 \, ZD_t + w_t \qquad (14)$$

$$ZS_t = \beta_0 + \beta_1 \, (ZS_{t+1} + ZD_{t+1}) + u_t \qquad (15)$$

The time series model selected for {ZD(t)} in (12) is AR(1). This specification was chosen empirically, using Box-Jenkins identification procedures. Equation (14) is derived directly from (10), (11) and (12); and, (15) is the absence of arbitrage condition. Under the null hypothesis of no rational bubbles, the parameter estimates for (12)-(15) are unbiased and therefore will be statistically indifferent. But if the null is false, one relevant variable (B_t) is omitted in (14). As long as the omitted variable is somehow correlated with ZD_t, coefficient estimates in (14) will be biased. However, the coefficients for (12) and (15) are unaffected. Therefore an empirical test of the null hypothesis of no rational bubbles can be realized by testing the significance of the associated cross equation restrictions.

Solving (12) with (14) permits the derivation of one set of restrictions for δ_0 and δ_1, while solving with (15) with (12) and (14) permits an additional set of restrictions. Under the null hypothesis of no rational bubbles, both restrictions will be satisfied. However, if the null is incorrect, then the restrictions from (12) and (14) will be unaffected, while the restrictions derived by combining (12) and (14) with (15) will be rejected. The relevant cross equation restrictions are (see Appendix):

$$\delta_0 = \left[\frac{T}{1 - \phi_1} - \frac{\phi_1 (1 - \phi_1^{T})}{(1 - \phi_1)^2}\right] \phi_0 = \frac{\beta_0 + \beta_1 \phi_0 + \beta_1 \delta_1 \phi_0}{(1 - \beta_1)} \tag{16}$$

$$\delta_1 = \frac{\phi_1 (1 - \phi_1^{T})}{(1 - \phi_1)} = \frac{\beta_1 \phi_1}{1 - \beta_1 \phi_1} \tag{17}$$

Interpreting (17), the first equality indicates that ϕ_1 reflects the T period annuity associated with the AR(1) dividend process. For $T < \infty$, (17) also requires indicates that $\beta_1 < 1$ is expected to prevail. In the limit, as $T \rightarrow \infty$, $\beta_1 \rightarrow 1$ is required. The cross equation restrictions in (16) are more difficult to interpret. Arbitrage free detrending is expected to eliminate trends in prices and dividends which implies intercepts which are equal to zero. In addition, as (17) requires the condition that $\beta_1 \rightarrow 1$, this condition also implies that the intercepts in the numerator of (16) be equal to zero in order to avoid the rhs term exploding at the singularity point $\beta_1 = 1$.

IV.4 Estimation Results

Estimation of system (12)-(15) and testing the associated cross equation restrictions (16)-(17) proceeds by estimating (12) with (14) and then estimating (12) with (15). The estimation method is maximum likelihood, with estimates obtained by concentrating out the variance parameters and then maximizing the negative of the log determinant of the residual covariance matrix.[7] This will result in two different sets of coefficient estimates for (12). The cross equation restrictions are then evaluated. This results in four different equalities being tested. Regression equation results are tabulated in Table 3. A number of these coefficient estimates are of interest. For example, comparison of the dividend yield coefficient, δ_1, for the interest rate detrended data indicates a substantial change in dividend payout behaviour in the post-war period. Yet, this change is not captured in the δ_1 estimates for the Z detrended samples. Another interesting estimation result arises with the β_1 value which decreases for the post-war Z detrended sample, compared to the full sample. This indicates that convergence of β_1 is not monotonic as T increases. In addition, the β_1 for the Y(t) post-war sample is substantially lower than the other β_1 estimates, with a higher standard error. Finally, differences in the estimates of ϕ_0 and ϕ_1 are not found to be significant.

Results for the tests of the cross equation restrictions are given in Table 4. Restrictions derived from (12) and (14) are initially tested. Then restrictions from (12) and (15) are tested, using estimated parameter values from (14) to represent parameters from that equation. The Table 4 results provide a number of sharp inferences about questions of interest. In particular, on the question of whether (7) or a risk neutral detrender is more appropriate, for the post-war sample the martingale evidence favoured (7) because of the drift in Y(t). Table (4) reveals that detrending by Z(t) successfully smooths the drift. For both samples, the cross equation restrictions associated with δ_0 could not be rejected. In contrast, for the interest rate detrended data, all cross equation restrictions, for both δ_0 and δ_1, for the post-war sample are rejected. The full sample gives conflicting results, with one of the δ_0 restrictions being rejected while the other is accepted.

Turning to the central question of interest, whether there is evidence of rational bubbles in stock prices, the tests of the restrictions on δ_1 using the (7) detrended data provide strong, if somewhat ambiguous, evidence against the null hypothesis of no rational bubbles. Examining the δ_1 restriction derived using the absence of arbitrage condition (15), the δ_1 restrictions are rejected for all samples and both detrenders. Yet, the δ_1 derived from (12) and (14) is marginally accepted for the post-war sample for the Z(t) from (7). The (12) and (14) δ_1 restriction is also accepted for full sample using the risk neutral detrender. Observing that (12) and (14) are empirical relationships, while (15) is the absence of arbitrage valuation equation, the immediate implication is that the annuity represented by the future dividend payments is insufficient to account for the value of current stock prices. There is an additional component, the rational bubble which is also being reflected in stock prices. This conclusion is based on the overwhelming failure of the δ_1 restriction derived using (15), at the same time the restriction based on (12) and (14) is marginally accepted. These results are strongest in the samples which the martingale tests indicated would be most reliable.

V. Conclusions

The essence of a rational bubble is that there is a component of the observed price which is not explained by the discounted value of expected future dividends. If there are rational bubbles, then discounted dividend stock pricing models, such as the Gordon model, are misspecified. In turn, this brings into question the widespread use of such models in applied valuation studies, e.g., Damodaran (1994). Rational bubbles can be ruled out if the transversality condition derived from (1) is satisfied. In the present context, it is required that:

$$\lim_{T \to \infty} ZS(T) \to 0$$

Unfortunately, direct econometric tests of this transversality condition are not available. The absence of a direct test is due to the difficulty of determining whether the observed level of ZS(T) is insignificantly different from zero. All this dictates that another approach is needed to determine whether there is evidence of a rational bubble in stock prices.

To this end, much of this paper is concerned with developing properties of the R-N derivative associated with the equivalent martingale measure. These properties are shown to have a practical application to the problem of deriving detrending methods for empirical tests of asset pricing models. The derivation of specific closed form solutions for the detrender depend on the null hypothesis that the assumed price process is the actual empirical process. More precisely, closed forms were derived under the assumption that the price processes were Markov, particularly diffusions. The connection to absence of arbitrage is embedded in the R-N derivative connecting the assumed empirical measure and the associated equivalent martingale measure. Differential properties of the conditional martingale density associated with the R-N derivative provide restrictions on the coefficients of the price processes (Heaney and Poitras 1994) which can be used to provide an arbitrage free transformation method for detrending observed price processes.

Several issues are raised in this paper. Of particular interest to empirical researchers, it is demonstrated that detrending procedures can impose significant equilibrium restrictions on empirical studies of rational stock pricing models. Intuitively, this is due to the implications that absence of arbitrage has for the empirical behaviour of security prices. Implementation of the detrending procedure proposed here involves making specific stochastic process assumptions about prices in order to generate a Z(s,t). Because the specific closed form detrender selected embeds an assumption about the risk aversion propensity of the representative investor, the empirical tests involve a joint hypothesis concerning the detrending process and the rational stock pricing model. Given this, the arbitrage free detrended data provided substantial evidence against the null hypothesis of no rational bubbles in stock prices. This result is consistent with the numerous anecdotal observations from market practitioners claiming that there is much more to stock pricing than dividend behaviour, alone, can explain.

References

K. Back and S. Pliska (1991), "On the Fundamental Theorem of Asset Pricing with an Infinite State Space", Journal of Mathematical Economics 20: 1-18.

Blanchard, O. and M. Watson (1982), "Bubbles, Rational Expectations and Financial Markets", chp. 11 in P. Wachtel, Crises in the Economic and Financial Structure, Toronto: Lexington Books.

Campbell, J. and A. Kyle (1993), "Smart Money, Noise Trading and Stock Price Behaviour", Review of Economic Studies, January.

Campbell, J. and R. Shiller, (1987), "Cointegration and Tests of Present Value Models", Journal of Political Economy 95: 1062-88.

Campbell, J., A. Lo and A. McKinley (1997), Econometrics of Financial Markets, Princeton, NJ: Princeton U. Press.

Craine, R. (1993), "Rational Bubbles: A Test", Journal of Economic Dynamics and Control 17: 829-46.

Damodaran, A. (1994), Damodaran on Valuation, New York, Wiley.

Diba, B. and H. Grossman (1988a), "The Theory of Rational Bubbles in Stock Prices", Economic Journal 98: 746-54.

_________________________ (1988b), "Explosive Rational Bubbles in Stock Prices?", American Economic Review 78: 520-30.

Dothan, M. (1990), Prices in Financial Markets, New York: Oxford U. Press.

Duffie, D. (1992), Dynamic Asset Pricing Theory, Princeton: Princeton U.P., 1992.

Evans, G. (1991), "Pitfalls in Testing for Explosive Bubbles in Asset Prices", American Economic Review 81: 922-30.

Gregory, A. and G. Smith (1996), "Measuring Business Cycles with Business-Cycle Models", Journal of Economic Dynamics and Control 20: 1007-25.

Grossman, S. and R. Shiller (1981), "The Determinants of the Variability of Stock Market Prices", American Economic Review 71: 222-227.

Hansen, L. and T. Sargent (1981), "Formulating And Estimating Dynamic Linear Rational Expectations Models", in R. Lucas, and T. Sargent (ed.), Rational Expectations And Econometric

Practice, Minneapolis, MN: The University of Minnesota Press, 91-126.

Harrison, M. and D. Kreps (1979), "Martingales and Arbitrage in Multiperiod Securities Markets", Journal of Economic Theory 20: 318-408.

He, H. and H. Leland (1993), "On Equilibrium Asset Price Processes", Review of Financial Studies 6: 593-617.

Heaney, J. and G. Poitras (1994), "Security Prices, Diffusion State Processes and Arbitrage Free Shadow Prices", Journal of Financial and Quantitative Analysis: (June) 29: 223-239.

Jarrow, R., X. Jin and D. Madan (1999), "The Second Fundamental Theorem of Asset Pricing", Mathematical Finance 9: 255-273.

Marsh, T. and R. Merton (1986), "Dividend Variability and Variance Bounds Tests for the Rationality of Stock Market Prices", American Economic Review 76: 483-98.

Olivier, J. (2000), "Growth Enhancing Bubbles", International Economic Review 41: 133-51.

Osborn, D. (1995), "Moving Average Detrending and the Analysis of Business Cycles", Oxford Bulletin of Economics and Statistics 57: 547-58.

Pantula, S., G. Gonzalez-Farias, W. Fuller (1994), "A Comparison of Unit-Root Test Criteria", Journal of Business and Economic Statistics, April: 167-76.

Sarno and M. Taylor (1999), "Moral hazard, asset price bubbles, capital flows and the East Asian crisis: the first tests", Journal of International Money and Finance 18: 637-657.

Scott, L. (1989), "Estimating the Marginal Rate of Substitution in Intertemporal Capital Asset Pricing Models", Review of Economics and Statistics 71: 365-75.

Shiller, R. (1981), "Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends", American Economic Review 71: 421-35.

West, Kenneth D. (1987), "A Specification Test For Speculative Bubbles", Quarterly Journal of Economics (August): 554-579.

Wu, Y. (1997), "Rational Bubbles in the Stock Market: Accounting for the US Stock-Price Volatility", Economic Inquiry 35: 309-19.

APPENDIX

Proof of Proposition 1:

Since the cumulative dividend-price process is a martingale under Q:

$$S(t_0) = E^Q(S(t) + \int_{t_0}^{t} D(u)\ du\ |F_{t_0}) \qquad (A.1)$$

Since S(t) is F_t measurable, then for $t > t_0$:

$$E^Q(S(t)|F_{t_0}) = \frac{E^P(\frac{dQ}{dP}S(t)|F_{t_0})}{E^P(\frac{dQ}{dP}|F_{t_0})} = \frac{E^P(E^P(\frac{dQ}{dP}S(t)|F_t)|F_{t_0})}{Z(t_0)}$$

$$= \frac{E^P(E^P(\frac{dQ}{dP}|F_t)S(t)|F_{t_0})}{Z(t_0)} = \frac{E^P(Z(t)S(t)|F_{t_0})}{Z(t_0)} \qquad (A.2)$$

where $Z(t) \equiv E^P[\ dQ/dP \mid F_t] = E^P[\ dQ/dP \mid S_t]$ due to the Markov property. From its definition Z(t) is a martingale under P. Also since D(u) is F_u measurable, then:

$$E^Q(\int_{t_0}^{t} D(u)\ du\ |F_{t_0}) = \frac{E^P(\frac{dQ}{dP}\int_{t_0}^{t} D(u)\ du\ |F_{t_0})}{E^P(\frac{dQ}{dP}\ |F_{t_0})} = \frac{E^P(E^P(\int_{t_0}^{t} \frac{dQ}{dP}\ D(u)\ du|F_u)|F_{t_0})}{Z(t_0)}$$

$$= \frac{E^P(\int_{t_0}^{t} E^P(\frac{dQ}{dP}|F_u)D(u)\ du|F_{t_0})}{Z(t_0)} = \frac{E^P(\int_{t_0}^{t} Z(u)D(u)\ du|F_{t_0})}{Z(t_0)} \qquad (A.3)$$

where in the third step above the order of integration over ω and t is interchanged, by Fubini's Theorem. For the purpose of using Fubini's theorem it is assumed that:

$$E^Q(\int_{t_0}^{t} |D(u)|\ du) < \infty$$

Substituting (A.2) and (A.3) into (A.1) gives equation (3) of the text.

Proof of Proposition 2:

These equations follow from both {Z(t)} and transformed cumulative dividend-price processes being martingales under P. From Proposition 1:

$$E^P(d\{\int_t^{t+dt} Z(u)\ D(u)\ du + Z(t+dt)\ S(t+dt)\} \mid F_t) = 0 \qquad (A4)$$

It follows on using Ito's lemma that Z(s,t) satisfies the following partial differential equation:

$$\frac{\partial}{\partial t}Z(s,t)\ s + Z(s,t)\ D(s) + \sum_{i=1}^{K} \alpha_i(s,t)\frac{\partial}{\partial s_i}Z(s,t)\ s$$
$$+ \frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K} \Sigma_{ij}\frac{\partial}{\partial s_i \partial s_j}Z(s,t)\ s = 0 \qquad (A.5)$$

where s is the vector of specific values of S(t) and $\sigma\sigma' = \Sigma$. Equation (A.5) holds for the cumulative dividend-price process. In particular for the k^{th} such asset:

$$s_k\frac{\partial Z}{\partial t} + ZD_k + s_k\sum_{i=1}^{K}\alpha_i\frac{\partial Z}{\partial s_i} + \alpha_k Z + \sum_{j=1}^{K}\Sigma_{kj}\frac{\partial Z}{\partial s_j}$$

$$+ s_k\ \frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K}\Sigma_{ij}\frac{\partial Z}{\partial s_i \partial s_j} = 0 \qquad (A6)$$

Since {Z(t)} is a martingale, $E(dZ(t) \mid F_t)=0$ so that, using Ito's lemma once again:

$$\frac{\partial Z}{\partial t} + \sum_{i=1}^{K}\alpha_i\frac{\partial Z}{\partial s_i} + \frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{k}\Sigma_{ij}\frac{\partial Z}{\partial s_i \partial s_j} = 0 \qquad (A7)$$

Equations (A6) and (A7) together imply that Z satisfies the K equations:

$$[\alpha_i(s,t) + D(s)]\ Z(s,t) = -\sum_{i=1}^{k}\Sigma_{ij}(s,t)\frac{\partial}{\partial s_i}Z(s,t) \qquad (A8)$$

Substituting back into (A7) and inverting yields equations (4a) of the text. Substituting these back into (A8) gives (4b) of the text.

Proof of Cross Equation Contraints:

Let ZD follow an AR(1) process: $ZD_{t+1} = \phi_0 + \phi_1\ ZD_t + v_{t+1}$. It follows that $E[ZD_{t+1} \mid H_t] = \phi_0 + \phi_1\ ZD_t$. Given this, the following iterative process applies:

$$E[ZD_{t+2} \mid H_t] = \phi_0 + \phi_1\ E[ZD_{t+1} \mid H_t] = \phi_0 + \phi_1\ (\phi_0 + \phi_1\ ZD_t) = \phi_0(1 + \phi_1) + \phi_1^2\ ZD_t$$

$$E[ZD_{t+3}] = \phi_0(1 + \phi_1 + \phi_1^2) + \phi_1^3\ ZD_t \quad .....$$

$$E[ZD_{t+n} \mid H_t] = \phi_0(1 + \phi_1 + \phi_1^2 + ... + \phi_1^{n-1}) + \phi_1^T\ ZD_t$$

Under the null hypothesis that the detrending process is correct, then the joint null of no rational bubbles implies that:

$$ZS_t = ZS_t^f = \sum_{t=1}^{T} E[ZD_{t+1} \mid H_t]$$

It follows that (12) and (14) can be solved as:

$$ZS_t = (1 + (1 + \phi_1) + (1 + \phi_1 + \phi_1^2) + ... + (1 + \phi_1 + \phi_1^2 + ... + \phi - 1^{T-1})) \phi_0$$
$$+ (\phi_1 + \phi_1^2 + ... + \phi_1^T) ZD_t = \delta_0 + \delta_1 ZD_t$$

$$where: \quad \delta_0 = [1 + (1 + \phi_1) + ... + (1 + \phi_1 + ... + \phi_1^{T-1})] \phi_0$$
$$= [T + (T-1)\phi_1 + (T-2)\phi - 1^2 + ... + 2\phi_1^{T-2} + \phi_1^{T-1}] \phi_0$$

$$(1 - \phi_1)\delta_0 = \{T - [\phi_1 + \phi_1^2 + \phi_1^3 + ... + \phi_1^{T-1} + \phi_1^T]\} \phi_0$$
$$\delta_0 = \left[\frac{T}{(1 - \phi_1)} - \frac{\phi_1(1 - \phi_1^T)}{(1 - \phi_1)^2}\right] \phi_0$$

$$Similarly: \quad \delta_1 = \frac{\phi_1(1 - \phi_1^T)}{(1 - \phi_1)}$$

The derivations make repeated use of the geometric progression.

A further set of restrictions can now be derived from substitution of (12) and (14) into (15):

$$ZS_t = \beta_0 + \beta_1 (E[ZS_{t+1}] + E[ZD_{t+1}]) + v_t$$

$$= \beta_0 + \beta_1 [\delta_0 + \delta_1 (\phi_0 + \phi_1 ZD_t) + \phi_0 + \phi_1 ZD_t] + v_t$$

$$ZS_{t+1} = \delta_0 + \delta_1 ZD_{t+1} + w_{t+1} \quad \rightarrow \quad E[ZS_{t+1}] = \delta_0 + \delta_1 E[ZD_{t+1}]$$

$$ZD_{t+1} = \phi_0 + \phi_1 ZD_t + v_{t+1} \quad \rightarrow \quad \therefore E[ZS_{t+1}] = \delta_0 + \delta_1 (\phi_0 + \phi_1 ZD_t)$$

$$ZS_t = (\beta_0 + \beta_1 \delta_0 + \beta_1 \phi_0 + \beta_1 \delta_1 \phi_0) + ((\beta_1 \delta_1 \phi_1 + \beta_1 \phi_1) ZD_t + v_t$$

$$= \delta_0 + \delta_1 ZD_t + w_t$$

Equating coefficients provides the cross equation restrictions stated in (16)-(17).

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Absence of Arbitrage, Detrending and Rational Stock Prices

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ABSTRACT

This paper examines the connection between absence of arbitrage in security prices and detrending procedures used in empirical tests of discounted dividend models of stock prices. The convention in these empirical studies is to invoke some method to detrend price-dividend data in order to achieve desirable statistical properties, such as covariance stationarity. Other studies have shown that the detrending process selected involves utility theoretic assumptions about the underlying pricing equilibrium. Under appropriate assumptions on the state variable probability distributions sufficient to ensure absence of arbitrage, this paper provides a method for determining an arbitrage free detrending procedure. Some examples of closed form detrenders are also provided. The paper concludes with an empirical application of the detrending procedure to testing for rational bubbles.

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Table 2*

Martingale Tests: Estimation Results for

$$\Delta \mathbf{X}_t = \theta_0 + \theta_1 \, \Delta \mathbf{X}_{t-1} + \mathbf{u}_t$$

ΔX	Sample	θ_0	θ_1
ΔZ	1871-1997	-.0027 (.0157)	.2919 (.0853)
ΔZ	1945-1997	-.0103 (.0073)	-.0523 (.1054)
ΔYD	1871-1997	.0151 (.0235)	.3368 (.0798)
ΔYD	1945-1997	-.0027 (.0036)	.0055 (.128)
ΔY	1871-1997	.0409 (.0220)	.1297 (.0891)
ΔY	1945-1997	.0874 (.0310)	-.0512 (.1420)

* Standard errors are in brackets below estimated coefficient.

Table 3

Estimation Results for (12)-(15)*

	Y	Y	YD	YD
Date	1871-1997	1945-1997	1945-1997	1871-1997
δ_0	.427 (.0558)	-0.3196 (0.260)	.01798 (.0051)	0.1104 (0.0202)
δ_1	14.95 (0.6612)	31.18 (4.52)	14.708 (0.520)	16.796 (0.282)
ϕ_0	0.00407 (0.0017)	0.00464 (0.00236)	.00013 (.00050)	0.00210 (0.00176)
ϕ_1	0.918 (0.0196	0.921 (0.041)	0.8907 (.0475)	0.9095 (0.0234)
β_0	.0000 (.0459)	0.0676 (0.0907)	.0057 (.0043)	0.0256 (0.0322)
β_1	.969 (0.0268)	.9027 (.0567)	.9672 (0.0275)	.9493 (0.0244)
ϕ_0	0.0039 (0.0017)	0.0048 (0.0024)	.00022 (.00050)	0.00128 (0.00176)
ϕ_1	0.921 (0.0196)	0.919 (0.0411)	0.8756 (.0476)	0.9273 (0.0233)

* Standard errors, given in brackets below the coefficient estimates, are computed from the quadratic form of the analytic first derivatives.

Table 4

Tests of Cross Equation Restrictions, (16) and (17)*

	Y	Y	YD	YD
Restriction	1871-1997	1945-1997	1945-1997	1871-1997
$\delta_0 = \{(T/(1-\phi_1)) - (\phi_1(1-\phi_1^T)/(1-\phi_1)^2\} \phi_0$	-5.770 (1.640) $X^2 =$ 12.38 (P= 0.000)	-3.302 (0.413) $X^2 =$ 63.83 (P= 0.000)	-.0453 (0.218) $X^2 =$ 0.433 (P= 0.835)	-2.780 (2.064) $X^2 =$ 1.818 (P= 0.178)
$\delta_0 =$ $\{\beta_0 + \beta_1\phi_0 + \beta_1\delta_1\phi_0)/(1-\beta_1)$	-1.502 (1.167) $X^2 =$ 1.656 (P= 0.198)	-2.441 (0.875) $X^2 =$ 7.780 (P= 0.005)	-.2574 (0.300) $X^2 =$ 0.734 (P= 0.392)	-0.821 (0.676) $X^2 =$ 1.477 (P= 0.224)
$\delta_1 = (\phi_1(1-\phi_1^T))/(1-\phi_1)$	3.728 (2.931) $X^2 =$ 1.618 (P= 0.203)	19.67 (8.328) $X^2 =$ 5.577 (P= 0.018)	6.582 (4.042) $X^2 =$ 2.651 (P= 0.103)	6.741 (2.932) $X^2 =$ 5.283 (P= 0.022)
$\delta_1 = \beta_1\phi_1/(1 - \beta_1\phi_1)$	6.677 (2.639) $X^2 =$ 6.402 (P= 0.011)	26.313 (2.227) $X^2 =$ 139.6 (P= 0.000)	9.179 (2.310) $X^2 =$ 15.78 (P= 0.000)	9.443 (1.904) $X^2 =$ 24.59 (P= 0.000)

*The first value given in each cell is the calculated parameter value defined by the difference between the rhs and lhs of the restriction evaluated using the estimated coefficients from Table 3. The value below the parameter estimate is the standard error. The X^2 value is for the Wald test that the set of parameter values in the restriction are (jointly) zero. The value below the X^2 is the P-value of the statistic.

NOTES

1. A potential confusion arises with the sometimes conflicting usage of "arbitrage". The convention in rational stock pricing models is that absence of arbitrage requires "assets are voluntarily held and that no agent can, given his private information and the information revealed by prices, increase his expected utility by reallocating his portfolio". This definition of absence of arbitrage is much weaker than that encountered in the pricing of derivative securities where an arbitrage is a riskless trading strategy which generates a positive profit with no net investment of funds.

2. One type of simple security valuation problem which has the form of (1) is the deterministic, continuous time bond pricing problem, where p(t) is the current bond price, T is the maturity date, p(T) is the principal, c(t) is the continuous coupon paid at t and $\{V_b(t)\}$ is the associated discounting operator. Taking the valuation date to be t=0:

$$V_b(0,T) = \exp\{\int_0^T -r_u \, du\} = \exp\{\int_0^t -r_v \, dv\} \, \exp\{\int_t^{t+n} -r_w \, dw\} = V_b(0,t) \, V_b(t,t+n)$$

where r_i is the deterministic continuous interest rate for period i (exp $\equiv$ e).

3. There are some models of rational bubbles which permit the bubble to collapse to zero prior to the limit being reached, e.g., Evans (1991). This subclass of bubbles is not being considered here. It is also possible that the B(t) will increase at such a rate that the stock price will be dominated by the bubble. However, this case is neither necessary or sufficient for there to be a rational bubble. Because stock prices cannot go negative, the possibility of the bubble dominating the price is sometimes used to rule out the possibility of a negative bubble.

4. The cumulative dividend-price process could also be defined as the total return process. The change in notation from p and d to S and D is intended to recognize the difference between these variables: S and D are interest rate detrended while p and d are observed prices and dividends.

5. The substitution of Z(s,t) for Z(S,t), where s represents a specific point realization of S, is deliberate. This follows because, much like a Taylor series, Ito's lemma involves expanding a function about a specific point.

6. Creating subsamples is not straight forward. Another practical problem which arises in constructing and interpreting the various Z-detrended series is associated with the presence of initial transients in the various data series: it takes a number of observations from the start of the detrended series for the process to damp down to an equilibrium state. This initial transient behaviour is expected, given that the Z(t) are solved by integrating (5) which induces an initial starting value into the solution. The transient is generic to the pricing problem being examined and is independent of the starting value selected.

7. More precisely, the LSQ routine in TSP was used to estimate the regression equations with the ANALYZ routine being used to test the cross equation restrictions.