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The Dark Ages of Probability in England: The Seventeenth Century Work of Richard Cumberland and Thomas Strode

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Summary

English work on probability during the dark ages of probability (the period between Pascal and Bernoulli) is reviewed, and attention is drawn to the works of Richard Cumberland and Thomas Strode. Cumberland introduced the criterion of maximizing expected utility in a nonmathematical treatise published in 1672; Strode showed how to derive the distribution of the total thrown for an arbitrary number of dice, in the course of a 1678 treatise on combinatorics.

Key words: Combinatorics; Finite differences; History; Pascal's triangle; Utility theory.

1 Introduction

The latter half of the seventeenth century might be termed the dark ages of the theory of probability. The period began brilliantly enough, with the works of Pascal, Fermat and Huygens in the 1650's, and in the end the darkness was permanently dispelled with the great treatises by Montmort, Bernoulli and De Moivre, all published in the decade from 1708 to 1718. Between these bursts, all was apparently darkness and silence. Yet, like the dark ages of history a thousand years before, a closer look and new information can sometimes reveal faint glimmers where none had been suspected.

What is widely regarded as the first printed work in probability theory was a 15-page tract in Latin by Christian Huygens, appended to a mathematics text published by Frans van Schooten in Leiden in 1657, and reissued in Dutch two years later. Ironically, Huygens launched the subject in a most uncertain way: its title was given as *De Ratiociniis in Aleae Ludo* on the volume's title page, and as *De Ratiociniis in Ludo Aleae* on the tract's first page. Huygens had visited Paris in 1655–1656, and on that trip he had learned of unpublished work by Pascal and Fermat on the problem of points (the determination of a fair division of stakes in an interrupted dice game). Huygens's tract presented the solutions to 14 problems in probability, solutions developed with some knowledge of the French work. Pascal had written his own account in late 1654, including the successful treatment of the problem of points, but it was not published until 1665, three years after Pascal died. This posthumously published book, *Traité du triangle arithmétique,* gave a full discussion of binomial coefficients and their application to problems in probability. It is with Pascal's work that we first encounter the binomial distribution; he showed how the binomial coefficients could be calculated easily in general and how they could be used to evaluate probabilities. Edwards (1987) gives a full account of Pascal's work.

2 Probability in England

The published response to Pascal and Huygens was muted. Todhunter's (1865) encyclopedic history of probability mentions only three works touching on probability between the work of Pascal and Huygens and 1690. Huygens's tract was reprinted in 1670 by a Jesuit scholar named Caramuel; a French mathematician, Sauveur, published some formulas relating to the game of Bassette in 1679; and Jacob Bernoulli derived two infinite series for a dice game in 1685 and 1690.

In the 1690's, however, there was a flurry of activity in England. Four works appeared, including a 1692 translation of Huygens by John Arbuthnot, a 1693 note on lotteries by Francis Roberts (Roberts, 1694), and two 1699 works on the application of probability to testimony, one by John Craig (Stigler, 1986b), and one now known to be by George Hooper (Hooper, 1699). It is natural to ask if these publications were part of an older English tradition, or did they signal an entirely new interest in probability.

In fact, the English interest in chance and uncertainty has a long history. Official procedures at the Royal Mint requiring essentially random sampling and a tolerance for variability go back to the 12th century (Stigler, 1977). In 1619, the Puritan divine Thomas Gataker wrote an influential treatise, *Of the Nature and Use of Lots,* attacking the notion that the outcomes of lotteries and the throws of dice were direct signs of God's will, arguing instead that such chance events were determined by natural, not divine, law, though their causes were not known to us. Bellhouse (1988) gives a thorough discussion of Gataker's work and the reaction to it, placing it in the context of English Puritan casuistry in the sixteenth and seventeenth centuries. Against this background, we might expect that by the 1660's, English mathematicians would be receptive to the works of Huygens and Pascal.

3 Richard Cumberland and the Laws of Nature

Van Schooten's 1657 book was known in England soon after publication. For example, Isaac Newton purchased a copy in 1664 (Westfall, 1980, p. 98), and Whiteside (1967–81, vol. **1**, pp. 58–62) has reprinted three pages of notes Newton apparently made at the time, showing that he worked through Huygens's tract carefully (and even extended it in minor ways). One indication that Huygens's tract was noticed by other readers can be found in a treatise on the laws of nature by Richard Cumberland, published in London in 1672. Cumberland, who was then in Cambridge and later became Bishop of Peterborough, wrote the book primarily to refute that he took to be certain mistaken principles in Thomas Hobbes's philosophy. Cumberland was born in 1631 and died in 1718. He was a remarkable early utilitarian philosopher who argued that the proper course was the pursuit of the greatest good, and that random ('contingentis') effects of human acts could be assigned a value by following the rules for calculating chances and enumerating all the possible cases. Cumberland cited Huygens explicitly (1672, p. 183; 1750, p. 283), and presented simple calculations involving one and two dice as direct analogies to the evaluation of contingent effects in agriculture, commerce, 'and in almost all Employments, where human Industry is concern'd and Busied' (Cumberland, 1672, pp. 322–323; 1750, p. 467).

Cumberland showed a natural tendency to reason inductively; he argued that the observed tendency of animal species to live in a state of peace rather than a state of hostility revealed a greater natural propensity for peace than for hostility in animals (and, by inference, in man). The argument was clinched by analogy:

For the Case holds here exactly the same as in the Doctrine of Chances: it is more natural to suppose, that Six will not turn up at the first cast of a Die, than that Six will; because there are five possible Chances against such a Cast, and but one Chance only for it.

(Cumberland, 1672, pp. 119–120; 1750, pp. 193–194). Again, Cumberland claimed that an individual's pursuit of the common good (rather than the selfish behavior he saw Hobbes as promoting) would more frequently benefit the individual, and hence was of greater value;

Effects, caused by all other known Acts, are naturally contingent, and, consequently, human Reason faithfully discharges its Office, in case it directs us to choose the Event, which, in the general, must frequently happen. For, a certain fix'd estimate of Value is put upon such a Contingency as most frequently happens.

Cumberland granted that in rare cases a man could benefit from the use of force and treachery, but that did not render their use wise.

The Man, for Instance, who wagers, that, at the first Cast upon a Pair of Dice, he will throw two Sixes, wins from him who lays an even Wager, that, two Sixes do not come up first: Yet, notwithstanding this lucky Cast, demonstratively true, (from the Nature, i.e. from the Cubic Figure of a Die,) it is, that the Odds, advantage against such a first Cast, are 35 to 1. And, consequently, the Chance of him, who takes up the Wager, is to the Chance of him who lays the Wager, as 35 to 1, which are call'd the Odds. This Difference between the Values of the Chances can and may, with great Propriety, be estimated and rated as the Chance, the Gain, i.e. The natural Reward of the wiser, of a more prudent Choice. And, we are to determine, in like Manner, concerning Damage or Loss, that, it is the natural Punishment of a foolish, of an imprudent Choice.

(Cumberland, 1672, pp. 322–323; 1750, pp. 466–467). Thus Cumberland argued that decisions in life should be based upon expected utility, and his analogy of a game of dice showed he had understood clearly the fundamentals of Huygens's tract.

4 Thomas Strode

Thomas Strode has been an unseen bit player on the stage of the history of mathematics. If he is mentioned at all, it is usually as the recipient of a 1676 letter from John Collins that was introduced as evidence in the dispute between Newton and Leibniz on the priority for the invention of the calculus (Whiteside, 1967–81, vol. **8**). Yet it should not surprise us that an age that could produce a Newton was a time of great mathematical curiosity, and that in some cases this curiosity resided in individuals capable of original work of their own.

Most of the little that is known about Thomas Strode's life stems from the fact that he attended Oxford University from 1642 to 1645 and thus came under the notice of Anthony à Wood, who in 1691–1692 under the title *Athenae Oxonienses* compiled 'An Exact History of all the Writers and Bishops Who have had their Education in the Most Antient and Famous University of Oxford'. Wood tells us that Strode, the son of a Somerset gentleman, matriculated at University College, Oxford, in 1642 at age 16, so Strode must have been born about 1626. Stode studied under the Roman Catholic scholar Abraham Woodhead, and in 1645 Woodhead took Strode and another student (Thomas Culpeper) on an extended tour of France. After four terms abroad, they returned to England where Strode settled at Maperston and 'followed his natural Geny to Mathematics' (Wood, 1721). Strode must have also studied law, for he was admitted to the Inner Temple as a barrister in 1657 (Foster, 1892). Strode died sometime after 1690; one source gives the year of his death as 1699.

Presumably Strode's interest in mathematics was kindled by his tutor, Woodhead, since Woodhead was widely known as an excellent instructor in mathematics. In addition to Strode and Culpeper, Woodhead was specifically engaged as mathematical tutor to the

Duke of Buckingham and to Lord Capell. Woodhead, however, appears to have published nothing on mathematics, and among his students only Strode showed a talent for original mathematical work (Culpeper did publish several tracts on interest and usury).

4.1 Strode's treatise

Strode published only two works. One concerns probability: it appeared in London in 1678 and is entitled *A Short Treatise of the Combinations, Elections, Permutations & Composition of Quantities; Illustrated by Several Examples, with a New Speculation of the Differences of the Powers of Numbers;* see Plate 1. The book is indeed a short treatise, comprising only 55 pages, together with a two-page preface and an errata list. Strode's treatise was not unknown to his contemporaries; it is cited by John Harris in a 1710 encyclopaedia article on combinations. But the treatise has been ignored by the history of mathematics, presumably because as a work in probability it fell short of Pascal's earlier treatise, and as a work on combinations it was soon surpassed by a 1685 work of John Wallis. It was brought to my attention by a dealer in rare books. Todhunter was unaware of it, and of subsequent authors the only reference to it of which I am aware is a passing notice by Edwards (1987, p. 47). Yet Strode's treatise deserves our attention for several reasons: as an original work, as perhaps the first mathematical work on probability in English, and as a picture of the level of understanding at the time.

Strode's preface, see Plate 2, helps set the work in the proper historical setting; he explained that the treatise was originally written in ignorance of Pascal's work, and it was only slightly revised to take account of Pascal:

> Courteous Reader,
> Above a year since, on the Entreaty of a very Worthy and Publick-Spirited Friend, I gave my Consent that these Papers should come to light; afterward understanding that some *French* Authors had writ on this Subject, I put a stop to the Press; at length having obtained those Authors, and perused what they have said concerning the Argument herein handled, was willing (to prevent any abuses) that the Press should proceed. I have since out of those Books added two things; the first out of *Malbranch alias* the Author of the *Elemens des Mathematiques*; namely, to give the number of the several Compositions that may be made of 24 Letters, and that on a double account, one to correct a Mistake of the *Printer* of that Treatise; for I do suppose it to be no other, (for that the Number of Figures, as also the first and the last are true;) the other, to shew the manner of Operation which he hath omitted. The Second is a Demonstration out of Monsieur *Pascal's* Tract *du Triangle Arithmetique* of what I had before by chance found out. Which you will find in Page 33, but misplaced; for it should follow Page 42.

The reference to Malebranche was in error, though the error was a common one at the time. The book in question was in fact by Jean Prestet (Robinet, 1967), and is not particularly noteworthy (it does not discuss probability). The other statement, Strode's claim that his work was done before he saw Pascal's *Traité,* is historically more interesting. The claim seems plausible. The two works are done in markedly different mathematical styles, and where they overlap, Pascal usually goes further. Also, the portion of the argument on page 33 that Strode attributes to Pascal has the appearance of being a late addition.

While Strode had not seen Pascal's work when he wrote the greater part of the treatise, he does appear to have known of Huygens's tract. He did not cite Huygens directly, but he did refer to 'Francis Schooten in his Miscellanies' in connection with combinations, which can only be a reference to Schooten (1657). The other mathematical books referred to are John Wallis's *Arithmetica Infinitorum* (1656) and contemporary work in progress by Thomas Baker.

A SHORT

TREATISE

OF THE

COMBINATIONS, } { PERMUTATIONS
&
ELECTIONS, } { COMPOSITION

OF

QUANTITIES.

ILLUSTRATED

By ſeveral Examples, with a New Speculation of the Differences of the Powers

OF

NUMBERS.

By *THO. STRODE*, Gent.

LONDON,

Printed by *W. Godbid* for *Enoch Wyer* at the *White Hart* in S. *Paul's* Church-Yard, 1678.

Plate 1. Strode (1678): title page

TO THE READER.

Courteous Reader,

A Bove a Year since, on the Entreaty of a very Worthy and Publick-Spirited Friend, I gave my Consent that these Papers should come to light; afterward understanding that some French *Authors had writ on this Subject, I put a stop to the Press; at length having obtained those Authors, and perused what they have said concerning the Argument herein handled, was willing (to prevent any abuses) that the Press should proceed. I have since out of those Books added two things; the first out of* Malbranch, alias *the Author of the* Elemens des Mathematiques; *namely, to give the number of the several Compositions that may be made of 24 Letters, and that on a double account, one to correct a Mistake of the* Printer *of that Treatise; for I do suppose it to be no other, (for that the Number of Figures, as also the first and the last are*

A 2 *true;)*

To the Reader.

true;) the other, to shew the manner of Operation, which he hath omitted. The Second is a Demonstration out of Monsieur Pascal's *Tract* du Triangle Arithmetique *of what I had before by chance found out. Which you will find in Page 33, but misplaced; for it should follow Page 42.*

In one place I have made use of Logarithms *to calculate Numbers consisting of 30 Figures, which possibly may seem strange to some, for that the Table of* Logarithms *which I used, doth not promise to give* Logarithms *beyond 5 places: To which Doubt I answer, that exactly they cannot work beyond 5 places, but if they consist of 30 or more Figures, the first 4 Figures with the Number of Places will be true (as may easily be demonstrated) which is enough to make good my Proposition.*

Farewel.

Plate 2. Strode (1678): preface

4.2 Strode on combinatorics

The first third of the treatise is concerned with counting. Strode discussed combinations, showing how the arithmetic triangle can be used to determine their number. He referred to this as the Table of Figurate Numbers. He showed how to compute the entries by multiplication without constructing the table. Of course, all of this and more was in Pascal, but Strode was working at this point without the benefit of reading Pascal, and we can at a minimum take this as testimony that Strode was a capable mathematician. He gave a nonspecific reference to Tacquet, but he seems to have come upon the equivalent to the formula

$$\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{1 \times 2 \times 3 \times \ldots \times k} \tag{1}$$

by himself. This is a significant achievement, although earlier mathematicians had known of it (Edwards, 1987).

Strode next discussed elections. The number of elections of n quantities is the number of nonempty subsets that may be formed; it is given by

$$\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1, \tag{2}$$

a result Strode attributed to Schooten. He then gave the rule for counting 'variations' (what we would now call permutations). He even treated the case where not all of the objects were distinguishable. Strode did not use abstract mathematical notation, but if we let P_k^n be the number of permutations of n distinguishable objects taken k at a time, then Strode described how to compute

$$P_k^k = k!, \quad P_k^n = \binom{n}{k} \times P_k^k. \tag{3}$$

He even explained that, if the n objects included a subset of r that were indistinguishable,

$$P_k^n = \binom{n}{k} P_k^k / P_r^r; \tag{4}$$

if there were R indistinguishable subsets of sizes $r_1, r_2, \ldots, r_R$, then

$$P_k^n = \left\{\binom{n}{k} P_k^k\right\} \Big/ \sum_{i=1}^{R} P_{r_i}^{r_i}. \tag{5}$$

In describing this last rule, he went a step beyond Pascal.

These discussions of methods of counting were accomplished through and enlarged upon by a series of examples. Some were of a routine nature: the number of 'words' consisting of 3 letters from the Latin alphabet of 24 letters is

$$\binom{24}{3} P_3^3 = 2024 \times 6 = 12144; \tag{6}$$

the number of conjunctions of the (then) seven planets (presumably including the earth and moon) is

$$\binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7} = 2^7 - 7 - 1 = 120.$$

But some were a bit more involved. After asking what the value of 100 sheep offered in exchange for

$$\binom{100}{50}$$

barley corns would be, Strode labored for three pages, using logarithms and the latest data on the size of the earth to conclude

That if the Terrestrial Globe should be covered with Guineys ten foot thick, there would not be enough to pay for the Sheep: Neither if the Terrestrial Globe should be converted to Barley, would there be Barley enough to satisfie for the Sheep.

Strode was imaginative in describing other large numbers as well. To give the reader a sense of how large was the number of different possible hands of 12 cards that could be extracted from a 52 card pack, he wrote

So that if 1000 men shall constantly deal 12 Cards for 12 hours each day, excepting *Sundays,* they cannot deal all the several Games that are on 12 Cards in a Pack, in 54 Years, accounting that each man may deal a Thousand Games in an hour.

The number Strode described,

$$1000 \times 12 \times 1000 \times 54 \times 6 \times 52 = 2{\cdot}02 \times 10^{11}$$

is not a bad approximation to

$$\binom{52}{12} = 2{\cdot}06 \times 10^{11},$$

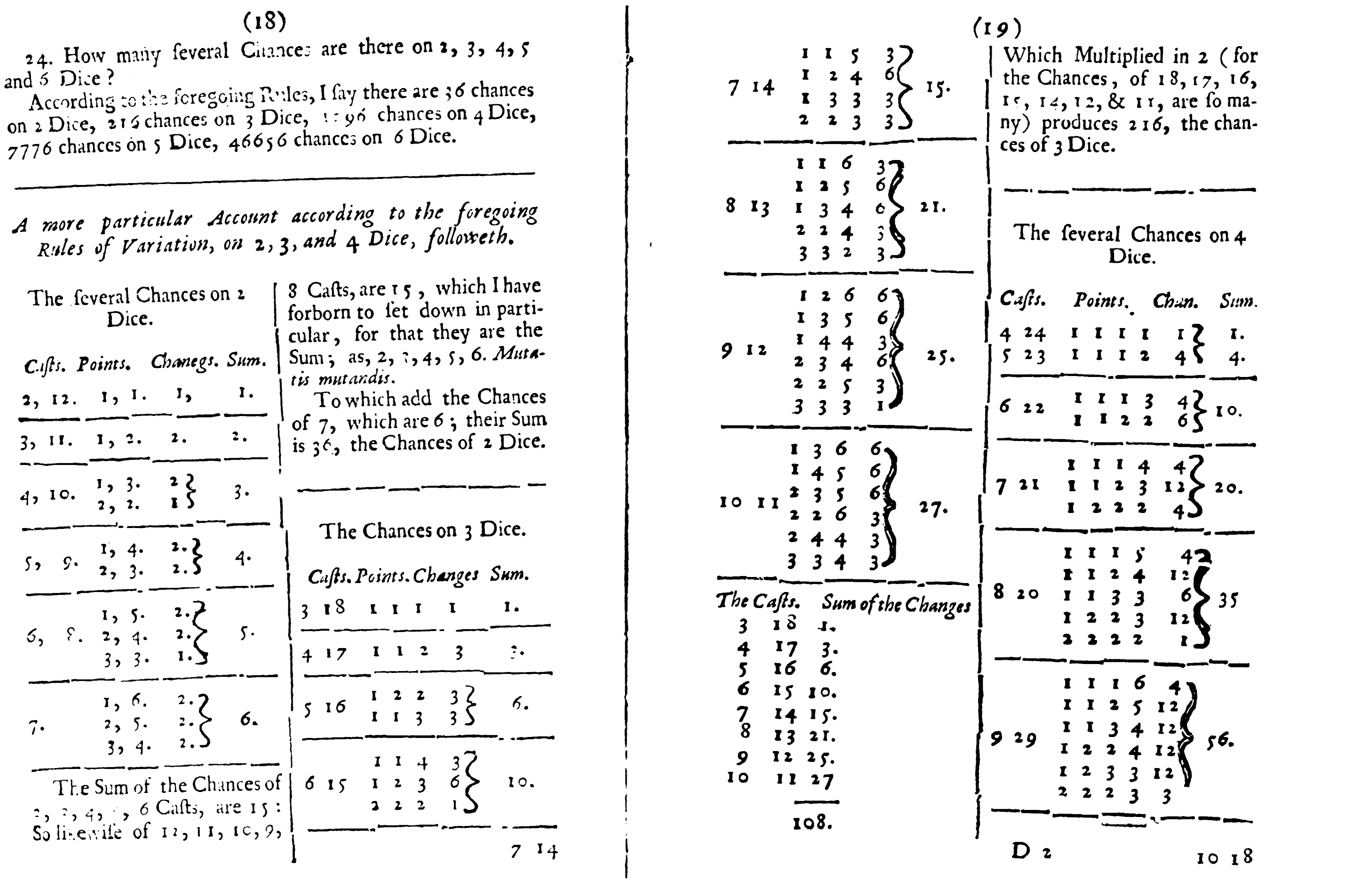
though it is smaller, as Strode stated.

4.3 *Strode on the chances at dice*

A portion of the treatise is concerned with dice, and some of the applications to the enumeration of chances Strode gave are more complicated than can be found in earlier literature. The first mention of dice occurs on page 15, where the rules we described by formulas (3)–(5) for permutations are illustrated. For example, for 3 dice the number of ways the outcome $(1, 1, 2)$ can be rearranged is 3; for 4 dice the outcomes $(1, 1, 2, 2)$ and $(1, 2, 3, 4)$ have 6 and 24 possible 'changes' or rearrangements; for 6 dice the outcomes $(1, 1, 1, 1, 2, 2)$, $(1, 1, 2, 2, 3, 4)$ and $(1, 1, 2, 3, 4, 5)$ have 15, 180 and 360 different 'changes'. After noting (p. 18; Plate 3) that the total number of outcomes for a set of n dice is 6^n, a result he gives for $n = 2$, 3, 4, 5 and 6, Strode then proceeded to compute and table the probability distribution of the total thrown with 2, 3 or 4 dice. He did this by enumeration, using his counting rules. He listed all possible totals, and for each possible total he listed the ways the total could be obtained, up to permutations, and summed those which corresponded to the given total to find the number of chances out of 6^n that give that total. Thus with 3 dice, there are 3 ways to get a total of 6: $(1, 1, 4)$, $(1, 2, 3)$ and $(2, 2, 2)$. There are, respectively, 3, 6 and 1 distinguishable ways of obtaining these results, and so the chance of a total of 6 is $3 + 6 + 1 = 10$ chances out of $6^3 = 216$. The procedure was orderly, and it was carried out correctly.

Strode used his lists of possible outcomes to answer one further question that was of interest to gamblers. Of the 1296 ways that 4 dice could fall, he found that 216 gave double pairs of faces agreeing (or *In and In* as he referred to the event). He found there were 360 chances of no pairs, and thus 720 chances of a single pair.

In a two-page appendix (pp. 52–53; Plate 4), Strode discussed how the approach and results generalized to dice with other than 6 faces, and how for some of the possible totals



(18)

24. How many several Chances are there on 2, 3, 4, 5 and 6 Dice?

According to the foregoing Rules, I say there are 36 chances on 2 Dice, 216 chances on 3 Dice, 1296 chances on 4 Dice, 7776 chances on 5 Dice, 46656 chances on 6 Dice.

A more particular Account according to the foregoing Rules of Variation, on 2, 3, *and* 4 *Dice, followeth.*

The several Chances on 2 Dice.

Casts.	*Points.*	*Changes.*	*Sum.*
2, 12.	1, 1.	1,	1.
3, 11.	1, 2.	2.	2.
4, 10.	1, 3. 2, 2.	2 1	3.
5, 9.	1, 4. 2, 3.	2. 2.	4.
6, 8.	1, 5. 2, 4. 3, 3.	2. 2. 1.	5.
7.	1, 6. 2, 5. 3, 4.	2. 2. 2.	6.

The Sum of the Chances of 2, 3, 4, 5, 6 Casts, are 15: So likewise of 12, 11, 10, 9, 8 Casts, are 15, which I have forborn to set down in particular, for that they are the Sum; as, 2, 3, 4, 5, 6. *Mutatis mutandis.*

To which add the Chances of 7, which are 6; their Sum is 36, the Chances of 2 Dice.

The Chances on 3 Dice.

Casts.	*Points.*	*Changes*	*Sum.*
3 18	1 1 1	1	1.
4 17	1 1 2	3	3.
5 16	1 2 2 1 1 3	3 3	6.
6 15	1 1 4 1 2 3 2 2 2	3 6 1	10.

7 14

(19)

7 14	1 1 5 1 2 4 1 3 3 2 2 3	3 6 3 3	15.
8 13	1 1 6 1 2 5 1 3 4 2 2 4 3 3 2	3 6 6 3 3	21.
9 12	1 2 6 1 3 5 1 4 4 2 3 4 2 2 5 3 3 3	6 6 3 6 3 1	25.
10 11	1 3 6 1 4 5 2 3 5 2 2 6 2 4 4 3 3 4	6 6 6 3 3 3	27.

The Casts.		*Sum of the Changes*
3	18	1.
4	17	3.
5	16	6.
6	15	10.
7	14	15.
8	13	21.
9	12	25.
10	11	27
		108.

Which Multiplied in 2 (for the Chances, of 18, 17, 16, 15, 14, 12, & 11, are so many) produces 216, the chances of 3 Dice.

The several Chances on 4 Dice.

Casts.	*Points.*	*Chan.*	*Sum.*
4 24	1 1 1 1	1	1.
5 23	1 1 1 2	4	4.
6 22	1 1 1 3 1 1 2 2	4 6	10.
7 21	1 1 1 4 1 1 2 3 1 2 2 2	4 12 4	20.
8 20	1 1 1 5 1 1 2 4 1 1 3 3 1 2 2 3 2 2 2 2	4 12 6 12 1	35
9 29	1 1 1 6 1 1 2 5 1 1 3 4 1 2 2 4 1 2 3 3 2 2 3 3	4 12 12 12 12 3	56.

D 2 10 18

Plate 3. Strode (1678): pages 18 and 19, chances at dice

(52)

APPENDIX.

To the End of Pag. 21, *may be further added,*

THat the Sum of all the Chances on 2, 3, or more Dice, and the Chances on the Regular Bodies, *viz.* the *Tetrahedron*, *Octahedron*, *Dodecahedron*, and *Icosahedron*, being marked on their sides as Dice, not exceeding the number of their sides, are Figurate Numbers, and if the Sum of their Chances do exceed their sides, then they are not Figurates; they beginning at an *Unit*, do proceed on until they do come to their middle Chance, then decreasing in the same manner, they do end in an *Unit*, whether it be on 2, 3, 4, or more Dice, or such Bodies: The Chances on 2 Dice, or 2 such Bodies, are natural Numbers; on 3 Dice, or 3 such Bodies, are triangular Numbers; on 4, are Pyramidal, and so forward.

So that in a *Tetrahedron*, the Sums of the 4 first Chances are Figurate Numbers; in a Cubick Die, the Sums of the six first Chances; in an *Octahedron*, the Sums of the 8 first; in a *Dodecahedron*, of the 12 first; and in an *Icosahedron*, the 20 first Chances are Figurates.

As in 3 *Tetrahedrons*, the Chances are 3,4,5,6,7,8,9,10,11,12.

Their Sums are 1,3,6,10,12,12,10,6,3, 1.

The first and last 4 Figures of their Sums are Triangular Numbers; 12, which is the Sum of the Chances, 7, is not Figurate; for that 1+1+5=7. and the *Tetrahedron* having but 4 sides, and consequently no 5, the Changes of 1+1+5=7 are 3; which being substracted from the next Figurate in the same Row, *viz.* 15, rest 12; as before.

So

(53)

So on the 3 Cubick Dice,

The Chances are 3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18.

Their Sum 1,3,6,10,15,21,25,27,27,25,21,15,10,6,3,1

The first and last six are Triangulars; 25 and 27 are not Figurates, for that 1+1+7=9 (there being not any 7 on any one Die) cannot be on the Dice, and consequently its Changes 3, must be deducted from the next Figurate 28, remains 25, as before. Again, 1+1+8=10, and 1+2+7=10, neither of which can be on the Dice, there being neither 7 nor 8, their Changes 3 and 6=9, must be deducted from 36, the 8*th* Number in that Row; then you have 27, as before.

So on 4 Dice; the first and last six Chances are Pyramidal Numbers, *viz.* 1, 4, 10, 20, 35, 56; the other, 80, 104, 125, 140, and 146 are not on the same reason.

If they had been all Figurates, their Sum would have been 715+715+286=1716, exceeding 1296, the Chances on 4 Dice by 420.

Upon a Re-examination of the *Table*, these Errors were found, which the Reader is desired to Correct thus.

Range 7, *Latus* 21, 230230. *R.* 9. *L.* 22. 4292145. *L.* 25. 10518300. *L.* 30. 38608020. *R.* 12. *L.* 12. 705432.

In the Continuation.

Range 4. *Lat.* 77. 79079. *L.* 80. 88560. *R.* 5. *L.* 38. 101270. *L.* 89. 2794155. *L.* 90. 2919735. *R.* 6. *L.* 41. 1221759. *L.* 48. 2598960. *L.* 80. 308720016. *R.* 7. *L.* 67. 1562389008. *L.* 78. 377447148 *R.* 8. *L.* 63. 1078897248. *L.* 68. 1799579064. *L.* 95. 1719961320. *L.* 97. 1981350785.

Plate 4. Strode (1678): Appendix

the chances could be found easily from his Table of Figurate Numbers. In modern notation, Strode observed that if a number n of s sided dice were tossed, then of the total possible number of outcomes s^n, the number of chances that the total equals k is

$$\binom{k-1}{n-1}$$

as long as $k < n + s$. That is,

$$P(\text{Total} = k) = s^{-n}\binom{k-1}{n-1} \quad \text{for } k = n, \ldots, s + n - 1.$$

For $s + n \leqslant k \leqslant (n-1)s$ he realized that more complicated enumeration would be needed, but, for $k > (n-1)s$,

$$P(\text{Total} = k) = P(\text{Total} = n(s+1) - k).$$

Strode's work on dice went beyond any that was published at the time. Cardano, who died in 1576, left a short work that enumerated the chances for 2 and 3 dice; it was published in 1663 (Todhunter, 1865, p. 3). Galileo had likewise treated the case of 3 dice but, though Galileo died in 1642, his work was only published in 1718 (Todhunter, 1865, p. 5). Neither Cardano nor Galileo had generalized the rules for enumeration as had Strode. Pascal and Fermat had gone well beyond Strode in some respects (as in their treatment of the problem of points), but neither had considered the general problem of the distribution of the total for an arbitrary number of arbitrary dice, and their correspondence, while dating from 1654, was not published until 1679. The tracts of Huygens and Pascal showed a level of mathematical attainment capable of dealing with Strode's problem, but the interests of those authors lay elsewhere: they wanted to know how to divide the stakes in interrupted games, and how many throws of two dice would be expected to be required before a 12 would occur. It could be argued that their problems were more difficult than Strode's, but Strode's problem of determining the distribution of a sum has turned out to be historically more important.

4.4 *Strode on the integration of difference equations*

Strode presented one more 'Mathematical Observation on Dice' (pp. 22–24). Noting that, for two dice, the number of chances of obtaining both dice below a number is the square of the number, he generalized the result to more than 2 dice. In modern notation he had, for two dice,

$$P(\text{both faces} \leqslant k) = k^2/6^2,$$

and, for m dice,

$$P(\text{all faces} \leqslant k) = \frac{k^m}{6^m}.$$

This led him to an interesting question, and to what may be among the earliest results on the integration of finite differences. For Strode explained in some detail a rule he had discovered inductively that enabled him to use his Table of Figurate Numbers to solve problems of the type: given any power and its differences, find another power and its differences. For example, given $3^3 = 27$ and its (backwards) differences,

$$\nabla 3^3 = 3^3 - 2^3 = 19, \quad \nabla^2 3^3 = \nabla 3^3 - \nabla 2^3 = 19 - 7 = 12, \quad \nabla^3 3^3 = 6,$$

find 10^3. Strode's solution was

$$10^3 = 3^3 + \binom{7}{1}\nabla 3^3 + \binom{8}{2}\nabla^2 3^3 + \binom{9}{3}\nabla^3 3^3.$$

In general, we could write his rule as, for $N > n$,

$$N^k = \sum_{l=0}^{k} \frac{(N-n)^{[l]}}{l!} \nabla^l n^k,$$

where $h^{[l]} = h(h+1)\ldots(h+l-1)$ is the ascending factorial. It is clear from Strode's discussion that he came upon this relationship essentially by integrating the difference equation $\nabla^k n^k \equiv k!$. The formula is of course a special case of the interpolation formula

$$f(x+h) = \sum_{l=0}^{\infty} (h^{[l]}/l!)\nabla^l f(x),$$

which is one of a class of formulas often referred to as Newton's difference interpolation formulas. They are analogues for finite differences of Taylor's theorem. Strode did not generalize his result beyond monomials and their differences, but in that respect he clearly anticipated later published work by Newton. Other versions of this relation had been found earlier, by Harriot, Gregory and Newton, but none of this work was published by 1678. In published earlier work, Briggs and Mercator had shown themselves familiar with predecessors of the formula; see Whiteside (1961, pp. 232–252; 1967–81, vol. **4**, pp. 14–73) and Goldstine (1977, pp. 68–84) for those and other references. It would be interesting to discover if Newton was aware of Strode's work.

Most of the remainder of Strode's *Treatise* consisted of discussion added after he had read Pascal. The promised addition starting on p. 33 presented a rule Strode credited to Pascal; we would write it

$$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}.$$

Strode next gave a Table of Figurate Numbers, a table of

$$\binom{n+m-1}{m}$$

for $0 \leqslant n \leqslant 30$, $1 \leqslant m \leqslant 12$; later in an appendix he added a table of this for $31 \leqslant n \leqslant 100$, $1 \leqslant m \leqslant 7$. He then returned to a discussion of Pascal. Strode reported that Pascal had given 19 consequences of the rule for constructing the table, and he repeated the 5 'choicest'. In modern notation,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \ldots + \binom{k-1}{k-1},$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-2}{k-1} + \ldots + \binom{n-k-1}{0},$$

$$\binom{n}{k} = \binom{n}{n-k}, \quad \sum_{k=0}^{n}\binom{n}{k} = 2\sum_{k=0}^{n-1}\binom{n-1}{k}, \quad \binom{n}{k} = \left(\frac{n-k+1}{k}\right)\binom{n}{k-1}.$$

Strode published one more book, a short treatise on 'dialling', explaining the use of certain geodetic instruments (Strode, 1688). That work was reprinted in 1697, and referred to by Taylor (1954, pp. 224–225, 405).

5 Conclusion

While Richard Cumberland's treatise played a role in the development of early utilitarianism, Thomas Strode seems to have had no impact at all upon the development of mathematical probability. Those small advances he made beyond Pascal were soon swept away by the full emergence of probability as an area for mathematical research in the eighteenth century (Hacking, 1975; Stigler, 1986a). If De Moivre knew of the existence of Strode's work, he did not think it sufficiently important to require citation. Rather, the importance of Strode's (and Cumberland's) work for historians of probability is that they reveal that, even in the 'dark ages' of the end of the seventeenth century, ideas of applying mathematics to chance were circulating widely, and were accepted without apparent resistance by mathematically educated people.

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Résumé

Dans la dernière moitié du dix-septième siècle, deux oeuvres remarquables ont été publiées dans l'Angleterre qui sont presque inconnues aujourd'hui. Richard Cumberland a discuté la théorie d 'utilité dans un traité de 1672, et Thomas Strode a etudié des problèmes des dés et du triangle arithmétique dans un traité de 1678.

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