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Securities Markets, Diffusion State Processes, and Arbitrage-Free Shadow Prices

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Abstract

This paper develops the parametric restrictions imposed on diffusion state processes by the requirement of arbitrage-free asset pricing. Using the equivalent martingale measure as a starting point, the diffusion property is exploited to specify the shadow pricing function, which takes conditional state variable probabilities under the reference measure into arbitrage-free contingent claim prices. The main results of the paper provide differential equations associated with the shadow price function that are used to identify restrictions on the parameters of assumed diffusion processes. The paper concludes with an application to the CIR model where the state variable, the instantaneous interest rate, is assumed to follow a square root process. Calculations are also provided for the parametric restrictions imposed on the Brownian bridge and OU state variable processes.

I. Introduction

Analysis of arbitrage-free price systems is fundamental to the study of stochastic equilibria in securities markets, e.g., Harrison and Kreps (1979), Harrison and Pliska (1981), (1983), Kreps (1981), Huang (1985), Taqqu and Willinger (1987), Duffie (1986), Back and Pliska (1991), Cheng (1991), and Flesaker (1993). Much of this work is concerned with establishing or applying the conditions required for an absence of arbitrage when security prices follow diffusions, i.e., when the commodity or state space is infinite dimensional. In this case, with "great generality," it has been shown that the existence of an equivalent martingale measure implies an absence of arbitrage opportunities, although the converse is not necessarily true (Back and Pliska (1991)). Given an equivalent martingale measure, it is possible to derive a functional relationship between assumed empirical specifications for state variables, e.g., OU interest rates (Rabinovitch (1989)) or geometric Brownian motion (Black and Scholes (1973)), and the associated shadow prices required to support the arbitrage-free equilibrium. Extending a notion originally proposed

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by Garman (1977), the primary objective of this paper is to develop the parametric restrictions this shadow price function imposes when the empirical reference measure is defined by a diffusion.

As conceived here, the practical relevance of the shadow pricing restrictions arises from the conventional practice in financial economics of constructing derivative security pricing models in a partial equilibrium setting. This approach is analytically convenient because it can exploit the riskless hedge portfolio construction to avoid the complications associated with directly specifying investor preferences. Using the equivalent martingale measure approach, absence-of-arbitrage restrictions on the coefficients of diffusion processes are imposed by the Cameron-Martin-Girsanov theorem, which gives the technical conditions the parameters must obey in order for the transformation of measure to be admissible. Cheng (1991) exploits these conditions to demonstrate that the Brownian bridge process is not an admissible process for bond prices. Unfortunately, this method of checking whether a given diffusion process is consistent with absence-of-arbitrage will, typically, be analytically complicated, e.g., due to the need to evaluate path integral conditions. A simplified method of evaluating conditions on a given diffusion process is required. By construction, such a simplification would apply only under somewhat more restrictive conditions.

Another approach, closely related to the equivalent martingale theory, is to derive the absence-of-arbitrage restrictions using general equilibrium, representative investor models as in Bick (1990) and He and Leland (1993). These papers ask the question: for a given set of diffusion asset price processes, what conditions must be imposed on the coefficients of the diffusion processes to ensure that these price processes can be supported by a representative investor? In a simplified equilibrium model, Bick shows that if the value of the market portfolio follows a specified diffusion process, the representative investor must have a certain utility function. Necessary and sufficient conditions, stated in terms of conditional moments of prices, are provided for the existence of the utility function. These conditions are used to show that, for nondividend-paying securities, the OU process is not consistent with arbitrage-free equilibrium. He and Leland show that for an equilibrium supported by a representative investor, it is necessary and sufficient that the coefficients of the market portfolio diffusion process satisfy a partial differential equation and a boundary condition. They show that in the special case where the equilibrium prices are time-homogenous diffusions, their conditions are the same as Bick's. In the more general case, their approach is different from Bick's, but is simpler to apply in that it avoids the need to compute the conditional expected utility function.

The general equilibrium approach requires explicit modeling of both aggregate wealth dynamics and the preferences of the representative investor. This approach has produced a number of interesting results regarding the relationship between a given preference assumption and arbitrage-free equilibrium (Breeden and Litzenberger (1978), Brennan (1979), and Bick (1987), (1990)). For example, it has been shown that a partial equilibrium assumption of geometric Brownian motion for stock prices is consistent with an arbitrage-free general equilibrium supported by constant proportional risk aversion. However, for purposes of evaluating restrictions on the coefficients of a given diffusion process, it is necessary to construct a general equilibrium solution. In addition to the diffusion process for the state variable of interest, this also requires modeling aggregate wealth dynamics and determining a solution for all the other endogenous variables, including aggregate wealth. In addition to being analytically complex, e.g., Cox, Ingersoll, and Ross (1985a, b), it is not immediately apparent whether the diffusion coefficient restrictions depend fundamentally on the specific preference and aggregate wealth assumptions invoked to derive the general equilibrium.

This paper develops the absence-of-arbitrage restrictions on the coefficients of the relevant diffusion process by deriving the shadow price function associated with the equivalent martingale measure. In this case, the shadow price function is defined as the ratio of the conditional density function of asset prices under the assumed diffusion process and the conditional density function of asset prices under the risk-neutral, or equivalent martingale, measure, normalized by the price of the riskless asset. In this case, for any given distribution of some asset payoff function, the product of the shadow price and the payoff, integrated over the distribution of the payoff, gives the arbitrage-free price of that asset. Using this approach requires a number of assumptions, e.g., that asset markets are dynamically complete, which permits the risk-neutral probability density to be uniquely determined. Because the absence-of-arbitrage conditions depend on differentiating the shadow price function with respect to the state variables, another key assumption is path independence. In effect, the shadow price function is assumed to be determined only by current asset prices, not the past history of prices. Finally, because the results are motivated by assuming the existence of an equivalent martingale measure, only sufficient conditions for absence-of-arbitrage are provided (Back and Pliska (1991)).

Section II provides the basic structure required to develop the results. Restrictions needed to derive arbitrage-free price trajectories associated with diffusion state processes are identified. In Section III, the concept of the shadow price function is derived and the connection to the more familiar Radon-Nikodym derivative discussed. Section IV develops the differential properties of the shadow price function. These properties are used to derive the fundamental restrictions on the coefficients of the assumed diffusion process. Section V provides a number of applications. Initially, both the OU and Brownian bridge processes are considered within a simplified valuation framework. An example extending the results to the general state variable case is also considered. This involves deriving the shadow price function for a state variable, the instantaneous interest rate, which follows a square root process. The one-state variable partial equilibrium case is compared with the two-state variable general equilibrium of Cox, Ingersoll, and Ross (CIR hereafter) (1985a, b). It is demonstrated that, for judicious choice of the diffusion process, the shadow price function is the indirect marginal utility for a representative investor with log utility preferences. Finally, Section VI provides a summary of the main results contained in the paper.

II. Assumptions and Structure

The initial point of reference is the equivalent martingale theory. Specifically, the structure of the model is based on the complete basic probability space (**W**,

F, **P**), which fully describes the exogenous environment. The assumed empirical reference measure **P** represents the unanimously held subjective probability assessments of the agents in the economy. The random state variable function $s(t, \omega) = s(t)$, where $\omega \in \mathbf{W}$ maps the basic probability space into the measurable *phase* space (Λ, Φ), i.e., the $n \times 1$ vector $s(t) \in \Lambda$ with $t \in [0, T]$. By definition, $d\Omega \in \Phi$ is a n-dimensional volume element $ds_1, ds_2 \ldots ds_n$ surrounding point s. Because $\Lambda = R^n$ in the present case, a particular value of $s(t)$ represents a point in n-dimensional Euclidean phase space with coordinates $(s_1, s_2, \ldots, s_n)(t)$. The behavior of the $s(t)$ over time also defines a space of *trajectories* (or sample functions) that determine the filtration (or information structure) **F**. Contained in the set **F** is the increasing family of σ-fields $F_t = \sigma\{s(u); u \leq t\}$, $t \in [0, T]$. It is further required that $F_t \in \mathbf{F}$ are right continuous with finite left limits where $F_0 = \{\mathbf{W}$, null sets of $P\}$ and $F_T = \mathbf{F}$.

In general, defined on $(\mathbf{F}, F_t)$ is a family of probability measures $\mathbf{Q}_i$, which are "equivalent" to **P** in the sense that both the **Q**s and **P** have the same null sets. Each **Q** is related to **P** through the Radon-Nikodym (RN) derivative relation,

$$(1) \qquad d\mathbf{Q}_m \quad = \quad h_m d\mathbf{P} \quad \text{where } h_m \in L^2,$$

where the index m is defined on **K**, an appropriately specified parameter space (Grenander (1981)). Associated with h_m is the conditional expectation $h_{mt} = E(h_m | F_t)$, which is a martingale under **P**. In cases where the stochastic process for the state variable is exogenously specified, it is assumed that the empirical reference measure is known and indexing by m is not relevant. However, reference to the m index will be of importance where the selection of h is treated as an estimation problem, e.g., Feigen (1976). In the case of dynamically complete markets, one of the **Q** measures takes the form of a unique "equivalent martingale measure" $\mathbf{Q}^*$ for which the appropriately specified asset price process is a martingale with respect to **F**. Jarrow and Madan (1991) and others provide further details of the conditions required for the uniqueness of the equivalent martingale measure.

In addition to the equivalent measures, adapted to the $\{F_t\}$ in the general equilibrium are $n + 1$ asset price processes $\{p_i\}$ $i = 0, 1, 2, \ldots, n$, assumed to be square integrable.[1] In the conventional case where the state variables are defined to be asset prices, the $\{p_i\}$ are taken to be functions of the filitrations generated by the state variables and, as a result, are measurable $\{F_t\}$. However, in the general state variable case where nonprice variables such as interest rates, inflation, etc., are included as state variables, the validity of assuming measurability is more complicated. In order to avoid conceptual confusion, it is expedient to take the state variables to be asset prices, even though such an assumption is not necessary to derive the results. The method of extending the results to include the general state variable case is considered in Section V. Another possible source of confusion concerns specification of the numeraire, the price of the riskless asset p_0, which can

[1]The assumption of square integrability can be weakened (e.g., Back and Pliska (1991)). In addition, the underlying assumption here is that the $n + 1$ assets are sufficient to complete the market. It is not clear in all cases that the dimension of the span is known (e.g., Kreps (1981)). However, *under appropriate conditions* (Ingersoll (1989), Jarrow and Madan (1991)), these $n + 1$ asset values are sufficient to span the state space. In particular, the diffusion state space assumption is sufficient to ensure spanning of the n states with $n + 1$ assets (e.g., Duffie (1986), p. 1173).

be specified in a number of possible ways.[2] For present purposes, the numeraire is specified as dollars. This requires the corresponding (p_0) asset to pay a dividend, r_t, the instantaneous riskless interest rate. While for analytical convenience in deriving absence-of-arbitrage conditions it is often assumed that $r = 0$, in general, this need not be the case. As it turns out, this point is of particular relevance to partial equilibrium models where the state variable of interest is not a price, e.g., the square root interest rate process in CIR (1985a, b).

In order to model the general state variable case, it is important to recognize that the shadow price function depends intimately on the state variable risk premia. In general, associated with the n state variables are risk premia for each s_i, $i = 1, 2, \ldots, n$, which reflect the market price of risk. When the state variables are assumed to be prices, then these risk premia can be postulated based on the observed dividend behavior and the coefficients of the assumed diffusion processes. In this case, asset prices and risk premia are specified to reflect expectations about both future capital gains and dividends. The general state variable case is somewhat more complicated. Because the risk premia must be specified in a nonarbitrary fashion, this requires creating notional securities whose prices mimic the behavior of the nonprice state variables by permitting *dividend payments* on the securities, where required, to adjust prices to reflect the uncertainty associated with the underlying state variables. The resulting restrictions on the risk premia of the nonprice state variables required for arbitrage-free equilibrium are determined in the process of deriving the appropriate shadow price function.

Given this background, the probability space, the set of trading dates, the information structure, and the price processes constitute a "securities market model." Through the introduction of a convex, continuous (L^2 norm topology), and strictly increasing preference relation over "net trade vectors," the concept of an arbitrage-free price system was initially developed by Harrison and Kreps (1979) who demonstrated a fundamental connection to the "equivalent martingale measure" $\mathbf{Q}^*$ for which a zero-dividend asset price process is a martingale with respect to $\mathbf{F}$. Hence, even in the equivalent martingale approach, the process of changing from the empirical to the equivalent martingale measure involves a mapping that is directly connected to preferences, with $\mathbf{P}$ corresponding to a convex, risk-averse set of preferences, and $\mathbf{Q}^*$ a risk-neutral probability density, e.g., Back and Pliska (1991). In terms of the state variable processes, the change of measure involves transforming a stochastic process with drift, under $\mathbf{P}$, into another process without drift, under $\mathbf{Q}^*$. When the state variables are characterized as diffusions, the Cameron-Martin-Girsanov theorem gives the technical conditions that the relevant empirical processes must obey in order for the transformation of measure to be admissible, e.g., Cheng (1991).

In general, less functional structure than the diffusion assumption is required to develop the requisite theory. However, for present purposes, the diffusion assumption allows immediate application of the Kolmogorov equations to the prob-

[2]It is conventional to assume that the numeraire at time t is the price of a pure discount, zero-coupon bond that matures at time T, at which time it pays \$1. However, other approaches are possible. In the present case, simplification is provided by allowing the numeraire to be a pure floating coupon bond that pays interest continuously and instantaneously adjusts the coupon payment such that the value of the bond is \$1 at all times $t \in [0, T]$.

ability distributions of the state processes. The martingale equivalence condition can then be exploited to develop properties of the mapping from the empirical reference measure into arbitrage-free constructions. Specifically, if the state variables follow a diffusion, the conditional probability density ρ satisfies the forward equation (e.g., Arnold (1974)). The relationship between the conditional probability density and the reference measure is

$$(2) \qquad \rho(s,t;s_0,t_0) \;=\; \int\limits_{A(d\Omega,t)} \mathbf{P}\left(d\omega|s_0,t_0\right) \;\equiv\; \int\limits_{A} d\mathbf{P},$$

where $s_0 = s(t_0)$ and $0 \leq t_0 \leq t \leq T$; $A(d\Omega, t) \in \mathbf{F}$ is the set of s trajectories that begin at (s_0, t_0) and end at t in $d\Omega$ (i.e., centered at s). In words, $\rho(s,t;s_0,t_0)$ is the conditional probability that, starting from (s_0, t_0), at time t, a realization of the vector lies in the n-dimensional volume element $d\Omega$ surrounding the point $s(t)$. As with the reference measure $\mathbf{P}$, the properties of ρ depend on the nature of the filtration $\mathbf{F}$, i.e., the conditional expectation depends only on current and future values of s and t, not the past values.

III. The Shadow Price Function

Given the existence of an equivalent martingale measure, the method of obtaining ρ from $\mathbf{P}$ can now be applied to $\mathbf{Q}^*$ to specify the transition probability density γ_π associated with the arbitrage-free price processes,

$$(3) \qquad \gamma_\pi(s,t;s_0,t_0) \;=\; \int\limits_{A(d\Omega,t)} \mathbf{Q}^*(d\omega) \quad \text{where} \quad \int\limits_{A} \gamma_\pi d\Omega \;=\; 1.$$

With appropriate adjustment for the effect of the numeraire, the density, γ_π, can be used to provide an interpretation of the arbitrage-free price vector over the phase space. In particular, because γ_π is the transition probability density for a diffusion, it is possible to define π,

$$(4) \quad \pi(s,t;s_0,t_0) \;=\; \int\limits_{A} \phi(\omega)\mathbf{Q}^*(d\omega) \;=\; \int\limits_{A} \left[\exp\left\{\int\limits_{t_0}^{t} -r(\omega,u)du\right\}\right] d\mathbf{Q}^*,$$

where $\phi(\omega)$ is the appropriate discounting or normalizing factor, which, using the conventional method selected to specify the numeraire, is set equal to the ratio of pure discount bond prices at time (s,t) and at s_0, t_0.[3] The ratio of these two prices defines the interest rate discounting factor $\phi(\omega)$. Hence, the arbitrage-free

[3]To avoid confusion with the applications presented in Section V, observe that this pure discount bond is only notional, i.e., it is only used to adjust γ_π to get π; it is not the numeraire that will be used in the pricing equations to be introduced. To see this, let $g(s,t)$ be the value of a payout, *in units of the numeraire*, made in state s at time t. The $t = t_0$ value of a claim to this payout is

$$\Pi(f) \;=\; E^*(g) \;=\; \int\limits_{A}\int\limits_{A} g(s,t)\mathbf{Q}^*(d\omega)$$

contingent claims price system associated with $\mathbf{Q}^*$ has an immediate complement, $\pi(s, t; s_0, t_0)$, which is the price at time t_0 in state s_0 of a claim that pays one dollar for sure if the realization of the state vector lies in the volume $d\Omega$ about point s at time t.

The presence of two densities, ρ and γ_π, now allows the introduction of the "shadow price function" $Z(s, t; s_0, t_0)$, which relates ρ and π. The importance of Z was first recognized in the seminal paper by Garman (1977). More precisely, Z is a function that maps conditional ρ probabilities associated with the state variables under the reference measure into the arbitrage-free contingent claims prices π derived from the equivalent martingale measure,

$$(5) \qquad Z(s, t; s_0, t_0) = \frac{\pi(s, t; s_0, t_0)}{\rho(s, t; s_0, t_0)} \Rightarrow Z\rho = \pi.$$

In the phase space, Z plays a role similar to that of the Radon-Nikodym derivative h in the trajectory space of the equivalent martingale theory. More precisely, from the construction of Z,

$$(6) \qquad Z(s, t; s_0, t_0) = \int_A \phi \mathbf{Q}^*(d\omega) / \int_A \mathbf{P}(d\omega)$$
$$= E\left(\phi h_t | A(d\Omega, t)\right).$$

Hence, while related to h_t, Z does differ from h_t insofar as, in general, Z does not exhibit the martingale property.

IV. Main Results[4]

One significant implication of assuming the state variables to be Markov under the equivalent martingale measure is that the shadow price function will be path independent. This assumption permits derivation of differential equations associated with Z. These equations hold for both the case where the state variables are prices and for the nonprice state variable case where the relevant notional securities are permitted to pay dividends, $d_i(s, t)$, such that the dividend stream ensures that an asset's price is maintained at the level s_i; $i = [1, 2, \ldots, n]$. Given this, the following result applies.

Proposition. Arbitrage-Free Shadow Pricing Conditions
Assuming there exists an equivalent martingale measure, then arbitrage-free valuation requires the following $n + 1$ differential equations be satisfied for any priced security,

$$(7) \qquad \frac{\partial}{\partial t} Z(s, t; s_0, t_0) p_k(s, t) + Z(s, t; s_0, t_0) d_k(s, t)$$

$$= \int_A g(s, t) \int_A \mathbf{Q}^*(dw) = \int g(s, t) \frac{1}{\phi} \pi(s, t; s_0, t_0) d\Omega.$$

This presentation assumes either that the appropriate discounting function ϕ has been imbedded in π or that $r = 0$ and ϕ does not matter.

[4]Proofs are given in the Appendix.

$$+\sum_i \alpha_i(s,t)\frac{\partial}{\partial s_i}Z(s,t;s_0,t_0)p_k(s,t)$$
$$+\frac{1}{2}\sum_i\sum_j \sigma_{ij}(s,t)\frac{\partial^2}{\partial s_i \partial s_j}Z(s,t;s_0,t_0)p_k(s,t) \quad = \quad 0,$$

subject to the condition $Z(s_0, t_0; s_0, t_0) = 1$ and the state variables follow diffusion stochastic differential equations,

$$ds_i(t) \quad = \quad \alpha_i(s,t)dt + \sum_{j=1}^{n} B_{ij}(s,t)d\theta_j,$$

where $\alpha_i(s,t)$ is the expected change in s_i per unit of time,
$\sigma_{ij}(s,t)$ is the covariance per unit of time between changes in s_i and s_j, i.e.,

$$\sigma_{ij} \quad = \quad \sum_{k=1}^{n} B_{ik}B_{kj},$$

and θ_j are independent Weiner processes taking values in $\mathbf{R}^1$ under the reference measure $\mathbf{P}$.

This proposition is applicable to any priced security, including derivative securities whose values are derived from underlying state variables. In order to use this proposition to solve for a closed form Z, appropriate restrictions on the coefficients of the diffusion process are required.

To derive the restrictions on the risk premia associated with the individual asset prices, it is expedient to take the state variables to be asset prices. However, with some additional analysis, Z for the nonprice state variable case can be derived by requiring the dividend streams of the n notional asset prices to be set in such a way that the numeraire-adjusted asset prices are maintained equal to the level of the state variables s_k where $k = 1, \ldots, n$. For the case where state variables are asset prices, Corollary 1 applies.

Corollary 1. Given (7), for the n asset prices to be arbitrage free it is required that

(8)
$$[\alpha_i(s,t) + d_i(s,t) - rs_i]\,Z(s,t;s_0,t_0)$$
$$+\sum_j \sigma_{ij}(s,t)\frac{\partial Z}{\partial s_j}(s,t;s_0,t_0) \quad = \quad 0.$$

Because the s_i are taken to be security prices, the expected return on the asset is the sum of the capital gain, α_i, and the dividend, d_i, and the risk premium associated with the state variable s_i is defined as

(9)
$$\lambda_i(s,t) \quad \equiv \quad \alpha_i(s,t) + d_i(s,t) - rs_i.$$

In (8), the risk premium appears as the term associated with the shadow price function, Z. Rearranging (8) so that only λ_i is on the left side reveals the implied relationship between the risk premium for state i, λ_i, and Z. In effect, λ_i represents the appropriately weighted sensitivity of Z to changes in the state variables.

Combining (8) with a similar condition applied to the numeraire it is possible to use the λs to identify other important differential properties of Z.

Corollary 2. Given (7), $Z(s,t;s_0,t_0)$ obeys the $n+1$ first order differential equations,

$$\text{(10)} \qquad \frac{\partial}{\partial s_i} Z(s,t;s_0,t_0) = \beta_i(s,t) Z(s,t;s_0,t_0), \quad i = 1,\ldots,n,$$

$$\text{(11)} \qquad \frac{\partial Z}{\partial t}(s,t;s_0,t_0) = f(s,t) Z(s,t;s_0,t_0),$$

where β_i is the ith component of the vector $\beta(s,t) \equiv -V^{-1}\lambda$; V is the $n \times n$ positive definite covariance matrix of the state variables with i,j element $\sigma_{ij}(s,t)$; λ is the column vector of risk premiums defined by (9); and

$$\text{(12)} \qquad f(s,t) \equiv -\left\{ r(s,t) + \sum_i \alpha_i \beta_i + \frac{1}{2} \sum_i \sum_j \beta_i \sigma_{ij} \beta_j + \frac{1}{2} \sum_i \sum_j \sigma_{ij} \frac{\partial \beta_i}{\partial s_j} \right\}.$$

Corollary 2 can be used to derive a number of important results, e.g., to specify the appropriate restrictions on the state variable risk premia $\{\lambda_i\}$, which are required for Z to obey (7). Under appropriate conditions, these restrictions translate into conditions on the coefficients of the assumed diffusion processes.

Taking cross derivatives of (10) and (11) in Corollary 2 provides more direct restrictions on the diffusion coefficients, somewhat loosely referred to here as "integrability" relationships because (13) is derived from (7), which is the result of integrating the forward equation,

$$\text{(13)} \quad \frac{\partial}{\partial s_j} f(s,t) = \frac{\partial}{\partial t} \beta_j(s,t) \quad \text{and} \quad \frac{\partial \beta_i}{\partial s_j} = \frac{\partial \beta_j}{\partial s_i} \quad \text{for } j = 1,\ldots,n.$$

These conditions (13) are decidedly similar to those provided in He and Leland (1993). In particular, in the case of one state variable, which is taken to be the value of the market portfolio, these conditions reduce to those provided in He and Leland's Theorem 1.[5] It is also possible to use these integrability conditions to interpret Z as an implicit preference function. In effect, (10) and (11) can be used to derive a condition analogous to Roy's Identity (see footnote 6) in conventional utility theory. In this vein, the integrability conditions can be compared to the Slutsky conditions. These conditions are exploited in Section V to derive an explicit closed form for Z.[6]

[5]Conditions (13) are more complete than those provided in Bick (1990) and He and Leland (1993) in which only one condition $\partial f/\partial s_1 = \partial \beta_1/\partial t$ is required because the only state variable is the market portfolio. If there are two state variables, e.g., CIR, then there are three integrability conditions to be satisfied: $\partial f/\partial s_1 = \partial \beta_1/\partial t$; $\partial f/\partial s_2 = \partial \beta_2/\partial t$; and $\partial \beta_2/\partial s_1 = \partial \beta_1/\partial s_2$. In general, if there are n state variables, then $(n^2+n)/2$ integrability conditions have to be satisfied.

[6]Upon closer inspection, (10) and (11) provide a direct connection to conventional notions from utility theory. In particular, dividing (10) by (11) gives an expression that is similar in interpretation to Roy's Identity, which is based on the ratio of partial derivatives of the indirect utility function.

Using the formulations in (10) and (11), it is possible to solve the system of differential equations for Z to derive a useful decomposition property.

Corollary 3. Decomposition of Z
Given (7), it is sufficient that Z can be expressed as the product of

$$Z(s,t;s_0,t_0) = U(s;s_0,t_0)\exp\left\{\int_{t_0}^{t} f\left(s,t'\right)dt'\right\}, \quad (14)$$

where $U(s;s_0,t_0)$ is independent of t and satisfies the n equations,

$$\frac{\partial}{\partial s_i}U(s;s_0,t_0) = \beta_i(s,t_0)\,U(s;s_0,t_0) \quad (15)$$

$$\text{with} \quad U(s_0;s_0,t_0) = 1.$$

(14) provides the general result that, given s_0, t_0, Z is decomposable into a time-independent function (U) and a time-dependent function. This decomposition is a direct consequence of the assumption of path independence of Z. Given this, (15) can be used to solve for the specific functional form of U. While, in general, further restrictions are required to arrive at a readily interpretable closed form solution to U, in many practical applications, U can be solved directly and used to specify an appropriate closed form for Z. In addition to providing solutions in specific cases, interesting general results are obtainable by, for example, restricting Z to be separable in the forward and backward variables.

V. Applications

In this section, the theory developed in Section IV is adapted to the practical problem of evaluating the admissibility of a specific partial equilibrium diffusion process. In this vein, recent work has identified problems with both the Brownian bridge and the OU process in certain types of valuation problems. In particular, Cheng (1991) has demonstrated that, when used as the stochastic process for bond prices, the Brownian bridge fails the conditions for the Cameron-Martin-Girsanov theorem. Using a representative investor model, Bick (1990) has demonstrated that the OU process is inadmissible for a nondividend-paying market portfolio. Using conventional techniques, arriving at these types of results required considerable analytical effort. However, within the context of a simplified valuation model, substantially less effort is required to achieve the same results by interpreting the closed form for Z. The simplifications require that $r = 0$, which eliminates the need to consider ϕ in (5). Where possible, it will also be assumed that the asset pays no dividend or coupon. In addition, it is assumed that there is a one-good world. For most practical purposes, dropping these assumptions will introduce a number of complications that will not add substantively to understanding of specific valuation problems.

For the simplified valuation model, consider the Brownian bridge process for the relevant state variable $X(t)$, which can be specified, e.g., Cheng (1991), $dX = (-X/(T-t))dt + dW$, where $t \in [0,T)$, where dW is the standard Weiner process. The risk premium for this asset follows directly from the specification given in (9) with $d = r = 0$, i.e., $\lambda = -[X/(T-t)]$, the drift term. Taking the variance-covariance matrix for the state variables to be the identity matrix gives $\beta = [X/(T-t)]$, $f = (1/2)(X^2/(T-t)^2) - 1/(2(T-t))$. Given this, the integrability conditions (13) associated with Corollary 2 follow appropriately, $\partial\beta/\partial t = \partial f/\partial X = (X/(T-t)^2)$. Since the integrability conditions are satisfied, the closed form for Z can be derived as $Z(X,t) = \{2\pi\sqrt{(T-t)}\}\exp\{X^2/(2[T-t])\}$. In this case, as $t \to T$ the Z exhibits explosive behavior. For this reason, the Brownian bridge is not capable of avoiding arbitrage opportunities. Because the process fails only as the endpoint T is approached, the Brownian bridge will be admissible only for processes where the portions of the sample paths close to T are not considered. This method of evaluating the behavior of the Brownian bridge is substantially less complicated than the tedious approach used by Cheng (1991).

The Brownian bridge represents a stochastic process that violates the requirements for arbitrage-free valuation as the fixed endpoint is approached. However, with appropriate specification of the trajectory space, a shadow price function can be derived. In contrast, as demonstrated in Bick (1990), the OU process for a nondividend-paying security is a case where Z cannot be derived. Specifying the OU process as $dX = \theta X\,dt + dW$. For the simplified valuation model, this leads to $\lambda = \theta X$, $\beta = -\theta X$ and $f = (1/2)\theta + (1/2)\theta^2 X^2$. It follows immediately that the integrability conditions are not satisfied, i.e., $\partial\beta/\partial t \neq \partial f/\partial X$, and it is not possible to derive a Z. In effect, if the integrability conditions are not satisfied, then the specified process cannot support an arbitrage-free equilibrium. In order for the OU process to be admissible, it is required that the security pay a dividend of amount θX resulting in $\lambda = 2\theta X = -\beta$ and $f = \theta$. The closed form for Z in this case is $Z(X,t) = \exp\{\theta t\}\exp\{-\theta X^2\}$. Where the OU process refers to wealth, from $\partial Z/\partial X < 0$ for $X > 0$ this corresponds to risk-averse preferences.

While useful as a heuristic framework for practitioners seeking to evaluate the appropriateness of an assumed stochastic process (e.g., Brennan and Schwartz (1980), Rabinovitch (1989), Ritchken and Boenawan (1990)), the simple valuation approach suffers from a number of technical defects. For example, the assumption that there is only one asset in the general equilibrium obviates the possibility of pricing by arbitrage any other security. In addition, the assumption that $r = 0$ eliminates the possibility of evaluating diffusion processes for interest rates, a potential state variable that is of considerable practical importance. Evaluation of diffusion processes for interest rates requires adapting the results of Section IV to nonprice state variables. In this case, the definition of the risk premia given in (9) has to be amended to allow a given s_i to be a notional security whose price tracks the relevant nonprice state variable. The need to assume an appropriately specified risk premium for interest rates is recognized in various studies that have attempted to model the arbitrage-free equilibrium conditions for interest rate processes, e.g., Vasicek (1977).

To provide a more developed frame of reference, the square root interest rate model of CIR (1985b) is considered. The closed form for Z is first solved in a partial equilibrium setting and, subsequently, is contrasted to Z derived by exploiting CIR's general equilibrium solution procedure. It was demonstrated in Section IV that, to do this correctly, restrictions on the risk premia compatible with arbitrage-free equilibrium are required. General equilibrium approaches such as CIR (1985a, b), Bick (1990), and He and Leland (1993) develop relevant valuation results by explicitly modeling aggregate wealth dynamics. In this case, the risk premium for a given state variable is determined simultaneously with the risk premia for all other endogenous variables, including aggregate wealth. Hence, a general equilibrium solution is required to get the risk premium for a given state variable. The approach proposed in Section IV permits the evaluation of "partial equilibrium" prices and risk premia that are compatible with arbitrage-free valuation without solving for the general equilibrium.

For the partial equilibrium example under consideration, it is assumed that there is one state variable, the instantaneous interest rate r, which follows a mean-reverting square root process,

$$dr = \kappa(\Theta - r)dt + \sigma\sqrt{r}\, d\eta, \tag{16}$$

where $d\eta$ is the standard Weiner process, and κ, Θ, and σ are constants. Given this, the objective of the exercise is to derive a closed form expression for Z based on the results of Section IV. Taking the price of the notional security for interest rates to be q_r, with appropriate scaling, the derivative $\partial q_r/\partial r = 1$. This permits Corollary 2 to be applied as follows,

$$\frac{\partial Z}{\partial r} = -\frac{\lambda}{\sigma^2 r}Z \quad \text{and} \tag{17}$$

$$\frac{\partial Z}{\partial t} = fZ, \tag{18}$$

$$\text{where } f = -\left\{r - \kappa(\Theta - r)\frac{\lambda}{\sigma^2 r} + \frac{1}{2}\frac{\lambda^2}{\sigma^2 r} + \frac{1}{2}\sigma^2 r\frac{\partial \beta}{\partial r}\right\}, \tag{19}$$

$$\beta = -\lambda/\left(\sigma^2 r\right). \tag{20}$$

Observing that the final solution will depend intimately on λ, in order to compare the resulting solution directly to CIR, choose a general form for the λ given in Section IV,

$$\lambda = \lambda_1 r + \lambda_0, \tag{21}$$

where λ_1 and λ_0 are constants with $d(s,t)$ and $\alpha(s,t)$ captured in λ_0.

Given this, a more precise expression for λ can be obtained by manipulating the second derivative conditions associated with Corollary 2,

$$\frac{\partial f}{\partial r} = (\partial\beta/\partial t) = 0. \tag{22}$$

Substituting (21) into (19) and (20) and using (22) gives the relevant restrictions on λ[7]

$$\lambda_1^2 + 2\kappa\lambda_1 + 2\sigma^2 = 0 \quad \text{and} \quad \lambda_0 = 0. \tag{23}$$

[7]The cross derivative or integrability conditions on λ associated with Corollary 2 imply that λ_0 will be equal to 0, i.e., λ_0 must be zero to avoid arbitrage opportunities. In turn, analysis of the solution for λ_0 reveals that there are actually two solutions to the discount bond pricing equations. This nonuniqueness problem was not considered in CIR.

Equation (23) can now be used to specify both β and Z. In this case, $\beta = -\lambda_1/\sigma^2$ so that using Corollary 3,

$$
\begin{aligned}
(24) \qquad Z(r,t;r_0,t_0) &= e^{-q(t-t_0)} e^{-\frac{\lambda_1}{\sigma^2}(r-r_0)}, \\
\text{where} \quad q &= \frac{-\kappa\Theta\lambda_1}{\sigma^2} \quad \text{and} \quad \lambda_1 = -\kappa \pm \sqrt{\kappa^2 - 2\sigma^2}.
\end{aligned}
$$

The implications of using this approach can be illustrated by considering the general equilibrium form for Z derived using the CIR approach.

To model the general equilibrium, CIR require that there are two state variables, W (per capita wealth) and r. These variables follow the joint processes,

$$
\begin{aligned}
(25) \qquad dW &= (\alpha(r) - c)W\,dt + \left(\left(\lambda_1\sqrt{r}\right)/\sigma\mu\right))W\,d\Gamma, \\
(26) \qquad dr &= \kappa(\Theta - r)dt + \sigma\sqrt{r}\,d\eta,
\end{aligned}
$$

where $\mu = \mathrm{Cov}(d\eta, d\Gamma)$ and μ, σ, λ_1, c, κ, Θ are constants. Deriving Z requires choosing the appropriate risk premia that satisfy the three conditions associated with (13) when there are two state variables,

$$
(27) \qquad \lambda_w = \left(\left(\lambda_1^2 rW\right)/\sigma^2\mu^2\right) \quad \text{and } \lambda_r = \lambda_1 r.
$$

Then, using Corollaries 2 and 3, it follows that Z is now

$$
(28) \qquad Z = \exp\{-c(t-t_0)\}\left(W_0/W_t\right).
$$

In effect, in the general equilibrium CIR model, Z has the more conventional utility theoretic interpretation as the ratio of the marginal utilities of wealth at $t = 0$ and $t = T$ for an investor with log utility.

Comparing the partial and general equilibrium approaches, the derivation of Z appropriate for the valuation of the price of claims contingent on r are the same except that, in partial equilibrium, an explicit restriction is imposed on λ_1 arising from the cross-derivative restrictions on λ given by Corollary 2. Given the risk premia have been correctly specified, the partial equilibrium solution is compatible with the arbitrage-free general equilibrium. Hence, satisfaction of the integrability conditions ensures that the risk premium of interest can, for practical purposes, be modeled in a partial equilibrium setting. For the practitioner, evaluation of the integrability conditions can provide an effective method for checking whether the coefficients on assumed partial equilibrium diffusion processes are capable of avoiding arbitrage opportunities without building a general equilibrium model. Analytically, previous approaches to specifying the coefficient restrictions have involved specifying second order partial differential equations arising from a general equilibrium model. Exploiting the properties of Z, the second order PDE can be replaced with two first order differential equations. In certain situations, this will provide a significant reduction in the complexity of arriving at a valuation formula.

VI. Summary

This paper develops the shadow pricing function (Z) associated with the equivalent martingale theory of arbitrage-free security price processes. Under the assumptions of diffusion state variable processes and dynamically complete markets, the existence of an equivalent martingale measure (risk-neutral probability density) permits derivation of the Z that maps conditional probabilities associated with the state variables into arbitrage-free contingent claims prices. It was demonstrated that differential equations associated with Z can be used to derive restrictions on the coefficients of assumed diffusion processes such that the absence-of-arbitrage is ensured. A heuristic, "simple valuation" framework was proposed to facilitate practitioners concerned with evaluating the appropriateness of assuming a given diffusion process for the relevant state variables. Both the Brownian bridge and OU processes were used as illustrations of the simplifications provided by shadow price function restrictions in evaluating the admissibility of a specific diffusion assumption. Finally, a specific closed form for Z was derived for the case where the nonprice state variable of interest followed a square root process. This Z is contrasted with the Z derived from the general equilibrium model of Cox, Ingersoll, and Ross (1985a, b).

Appendix

Proof of Proposition 1. A number of possible approaches can be used to derive (7). One method exploits the change of measure property, i.e., by the martingale property,

$$E^{\mathbf{Q}}\left\{d\left[\int_0^t d_k(s)ds + p_k(t)|s,t\right]\right\} = 0. \tag{A-1}$$

Under the **P** measure this becomes

$$E^{\mathbf{P}}\left\{d\left[\int_0^t Z(s)d_k(s)ds + Z(t)p_k(t)|s,t\right]\right\} = 0. \tag{A-2}$$

Using Ito's lemma, it follows that $Z(s,t)$ satisfies Equation (7) of the proposition. Another, more tedious but potentially more revealing derivation involves integrating the relevant forward equation. Using this approach involves observing that the following pricing relationship holds for all t, $t_0 \leq t \leq T$,

$$\begin{aligned} p_k(s_0,t_0) &= \int_\Lambda \int_{t_0}^t \pi\left(s,t';s_0,t_0\right) d_k\left(s,t'\right) d\Omega\, dt' \\ &\quad + \int_\Lambda \pi\left(s,t;s_0,t_0\right) p_k(s,t) d\Omega, \end{aligned} \tag{A-3}$$

where $0 \leq t_0 < t \leq T$; $p_k(s,t)$ is the price of asset k in state s at time t during the time interval 0 to T; $d_k(s,t)$ is the dividend paid per unit of time in state s at time

t; and $\int_\Lambda d\Omega$ represents the integral over the volume of phase space. Substitute $\pi = \rho Z$ in (A-3), differentiate Equation (A-3) with respect to t, and use the forward equation for the diffusion state variables to obtain,

$$\text{(A-4)} \quad 0 = \int \rho(s,t;s_0,t_0)\left[Z(s,t;s_0,t_0)\,d_k(s,t) + \frac{\partial}{\partial t}Z(s,t;s_0,t_0)\,p_k(s,t)\right]d\Omega - \int Z(s,t;s_0,t_0)\,p_k(s,t)\sum_{i=1}^{k}\frac{\partial}{\partial s_i}\left\{\alpha_i(s,t)\rho(s,t;s_0,t_0) - \frac{1}{2}\sum_{j=1}^{k}\frac{\partial}{\partial s_j}\sigma_{ij}\rho(s,t;s_0,t_0)\right\}d\Omega.$$

Integrate the last two terms in Equation (A-4) by parts. By assumption, the probability density on the boundary of integration vanishes so that

$$\text{(A-5)} \quad 0 = \int \rho(s,t;s_0,t_0)\left[Z(s,t;s_0,t_0)\,d_k(s,t) + \frac{\partial}{\partial t}Z(s,t;s_0,t_0)\,p_k(s,t) + \sum_i \alpha_i(s,t)\frac{\partial}{\partial s_i}Z(s,t;s_0,t_0)\,p_k(s,t)\right]d\Omega - \frac{1}{2}\int\sum_i\sum_j\frac{\partial}{\partial s_j}\{\sigma_{ij}(s,t)\rho(s,t;s_0,t_0)\}\frac{\partial}{\partial s_i}Z(s,t;s_0,t_0)\,p_k(s,t)d\Omega.$$

Integrate the last term in Equation (A-5) by parts. Because $\sigma_{ij}\rho$ vanishes on the boundary, Equation (A-6) is obtained,

$$\text{(A-6)} \quad \int_\Lambda \rho(s,t;s_0,t_0)\left[\frac{\partial}{\partial t}Z(s,t;s_0,t_0)\,p_k(s,t) + Z(s,t;s_0,t_0)\,d_k(s,t) + \sum_i \alpha_i(s,t)\frac{\partial}{\partial s_i}Z(s,t;s_0,t_0)\,p_k(s,t) + \frac{1}{2}\sum_i\sum_j\sigma_{ij}(s,t)\frac{\partial^2}{\partial s_i\partial s_j}Z(s,t;s_0,t_0)\,p_k(s,t)\right]d\Omega = 0,$$

$$\text{for } t_0 \leq t \leq T,$$

with the initial condition $Z(s_0,t_0;s_0,t_0) = 1$. (Observe that to derive (A-6) requires that $(\partial p_k/\partial t) = 0$ in (A-3).) Equation (7) follows immediately from (A-6) since (7) insures that Equation (A-6) holds. A special case of interest occurs in (A-6) where only ρ depends on s_0 and t_0 . If s_0 and t_0 are varied freely, the value of the integral will vary unless the expression in the square brackets is identically zero. This is the case where Z is separable in the forward and backward variables. □

Proof of Corollary 1. This result requires combining (7), evaluated with $p_k = s_k$, which is the expression given in the proposition, with (7) evaluated for the special case of the numeraire where $p_0(s,t) = 1$, which pays $d_0 = r$ continuously. For the numeraire, this produces

$$\text{(A-7)} \qquad \left[\frac{\partial}{\partial t}Z + r(s,t)Z + \sum_i \alpha_i(s,t)\frac{\partial}{\partial s_i}Z + \frac{1}{2}\sum_i\sum_j \sigma_{ij}(s,t)\frac{\partial^2}{\partial s_i \partial s_j}Z\right] = 0.$$

Multiply (A-7) by s_k, equating this with (7) and solving provides the result in the corollary.

Proof of Corollary 2. In vector notation, (8) can be written

$$\text{(A-8)} \qquad -\lambda Z = V \text{ grad } Z \quad \text{where grad } Z \text{ is the appropriate gradient}$$

$$\text{or} \quad \text{Grad } Z = \beta Z,$$

where $\beta = -V^{-1}\lambda$. Equation (10) is (A-8) in component form. Substituting (10) into (A-7) gives

$$\text{(A-9)} \qquad \frac{\partial Z}{\partial t} + rZ + \sum_i \alpha_i(s,t)\beta_i Z + \frac{1}{2}\sum_i\sum_j \sigma_{ij}(s,t)\frac{\partial}{\partial s_i}\beta_j Z = 0.$$

Using (10) once again, the last term in (A-9) can be written as

$$\text{(A-10)} \qquad \frac{1}{2}\sum_i\sum_j \sigma_{ij}(s,t)Z\frac{\partial \beta_j}{\partial s_i} + \frac{1}{2}\sum_i\sum_j \sigma_{ij}(s,t)\beta_j\beta_i Z.$$

(A-9) and (A-10) yield (11) in the text. Taking cross partial derivatives of (10) and (11) and manipulating gives

$$\text{(A-11)} \qquad \frac{\partial}{\partial s_j}f(s,t) = \frac{\partial}{\partial t}\beta_j(s,t), \quad j = 1, \ldots, k, \quad \text{and}$$

$$\text{(A-12)} \qquad \frac{\partial \beta_i}{\partial s_j}(s,t) = \frac{\partial \beta_j}{\partial s_i}(s,t), \quad \text{all } i \text{ and } j,$$

Proof of Corollary 3.

Equation (14) is obtained by integration from Equation (11). To obtain Equation (15), substitute (14) into (10) and use Condition (A-11). □

References

Arnold, L. *Stochastic Differential Equations*. New York, NY: John Wiley (1974).

Back, K., and S. Pliska. "On the Fundamental Theorem of Asset Pricing with an Infinite State Space." *Journal of Mathematical Economics*, 20 (1991), 1–18.

Bick, A. "On the Consistency of the Black-Scholes Model with a General Equilibrium Framework." *Journal of Financial and Quantitative Analysis*, 22 (1987), 259–275.

________________. "On Viable Diffusion Price Processes of the Market Portfolio." *Journal of Finance*, 45 (1990), 673–689.

Black, F., and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, 81 (1973), 637–659.

Breeden, D., and R. Litzenberger. "Prices of State-Contingent Claims Implicit in Option Prices." *Journal of Business*, 51 (1978), 621–651.

Brennan, M. "The Pricing of Contingent Claims in Discrete Time Models." *Journal of Finance*, 34 (1979), 53–68.

Brennan, M., and E. Schwartz. "Conditional Predictions of Bond Prices and Returns." *Journal of Finance*, 35 (1980), 405–417.

Cheng, S. "On the Feasibility of Arbitrage-Based Option Pricing when Stochastic Bond Price Processes are Involved." *Journal of Economic Theory*, 53 (1991), 185–198.

Cox, J. C.; J. E. Ingersoll; and S. A. Ross. "An Intertemporal General Equilibrium Model of Asset Prices." *Econometrica*, 53 (March 1985a), 363–384.

________________. "A Theory of the Term Structure of Interest Rates." *Econometrica*, 53 (March 1985b), 385–407.

Duffie, D. "Stochastic Equilibria: Existence, Spanning Number and the 'No Expected Financial Gains from Trade' Hypothesis." *Econometrica*, 54 (1986), 1161–1184.

________________. *Dynamic Asset Pricing Theory*. Princeton, NJ, Princeton Univ. Press (1992).

Feigen, P. "Maximum Likelihood Estimation for Continuous-Time Stochastic Processes." *Advances in Applied Probability*, 8 (1976), 712–736.

Flesaker, B. "Arbitrage Free Pricing of Interest Rate Futures and Forward Contracts." *Journal of Futures Markets*, 13 (1993), 77–92.

Garman, M. "A General Theory of Asset Valuation under Diffusion State Processes." Center for Research in Management Science Working Paper #50, Univ. of California at Berkeley (1977).

Grenander, U. *Abstract Inference*. New York, NY: Wiley (1981).

Harrison, J., and D. Kreps. "Martingales and Arbitrage in Multiperiod Securities Markets." *Journal of Economic Theory*, 20 (1979), 381–408.

Harrison, J., and S. Pliska. "Martingales and Stochastic Integrals in the Theory of Continuous Trading." *Stochastic Processes and Applications*, 11 (1981), 215–260.

________________. "A Stochastic Calculus Model of Continuous Trading." *Stochastic Processes and Applications*, 15 (1983), 313–316.

He, H., and H. Leland. "On Equilibrium Asset Price Processes." *Review of Financial Studies*, 6 (1993), 593–617.

Huang, C. "Information Structures and Viable Price Systems." *Journal of Mathematical Economics*, 14 (1985), 215–240.

________________. "An Intertemporal General Equilibrium Asset Pricing Model." *Econometrica*, 55 (1987), 117–142.

Ingersoll, J. *Theory of Financial Decision Making*. Totawa, NJ: Rowan Littlefield (1989).

Jarrow, R., and D. Madan. "A Characterization of Complete Securities Markets on a Brownian Filtration." *Mathematical Finance*, 1 (1991), 31–43

Kreps, D. "Arbitrage and Equilibrium in Economies with Infinitely Many Commodities." *Journal of Mathematical Economics*, 8 (1981), 15–35.

Rabinovitch, R. "Pricing Stock and Bond Options when the Default-Free Rate is Stochastic." *Journal of Financial and Quantitative Analysis*, 24 (1989), 447–457.

Ritchken, P., and K. Boenawan. "On Arbitrage-Free Pricing of Interest Rate Contingent Claims." *Journal of Finance*, 45 (1990), 259–264.

Rogers, L., and D. Williams. *Diffusions, Markov Processes and Martingales, V.2 Ito Calculus*. New York, NY: Wiley (1987).

Taqqu, M., and W. Willinger. "The Analysis of Finite Security Markets Using Martingales." *Advances in Applied Probability*, 19 (1987), 1–25.

Vasicek, O. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics*, 5 (1977), 177–188.