Is Bond Convexity a Free Lunch?

Immunization does not imply risk-free arbitrage profits.

Joel R. Barber and Mark L. Copper

A common criticism of duration-based immunization is that it offers investors a "free lunch." Fisher and Weil [1971] show that for parallel shifts in spot rates an immunized portfolio is a convex function of the shift parameter with a minimum at the initial portfolio value. Ingersoll, Skelton, and Weil [1978] argue that this result is "too good," because:

> if it were known that the yield curve could change only by a constant noninfinitesimal shift, then it is apparent that an arbitrage profit could be earned by buying two discount bonds and shorting a third with an intermediate maturity equal to the duration of the long position [1978, pp. 635-636].

This criticism has a long history, and is evidently accepted by many researchers and practitioners. In a comment following Redington's [1952] seminal work on immunization, Rich states that:

> Immunization...[is] an outstanding example of the difference between actuarial theory and practice. How delightful it would be if the funds of a life office could be so invested that, on any change in the rate of interest...a profit would always emerge [1952, p. 319].

More recent examples of this criticism include Cox, Ingersoll, and Ross [1979], Bierwag, Kaufman,

JOEL R. BARBER is associate professor of finance and **MARK L. COPPER** is a graduate student at Florida International University in Miami (FL 33199).

and Toevs [1983], Bierwag [1987], Grantier [1988], Shiu [1990], and Kahn and Lochoff [1990].

It is our contention that the convexity property of many term structure models is not necessarily inconsistent with arbitrage-free pricing. Indeed, we show that bond portfolio convexity is a property of the arbitrage-free Ho and Lee [1986] process.

HOW CONVEXITY WORKS

Assume the initial spot rate curve $r_0(s)$ is known. Under the parallel shift model, the spot rate after an interest rate shock x is determined by $r(s) = r_0(s) + x$. Suppose we are faced with a liability of c dollars due in four years, and wish to form an immunized portfolio with two zero-coupon bonds maturing in three and five years with face values of a and b dollars. If the asset and liability values are equal, then

$$ae^{-3r_0(3)} + be^{-5r_0(5)} = ce^{-4r_0(4)}$$

For simplicity, let us choose the present value weight

$$w = \frac{ae^{-3r_0(3)}}{ce^{-4r_0(4)}}$$

as our decision variable. Duration-matching requires that $3w + 5(1 - w) = 4$ or that $w = 0.5$. By construction, this portfolio has two properties: its present value is zero, and the duration of the assets equals four years.

After an interest rate shock, the asset-to-liability ratio is

$$\frac{A(x)}{L(x)} = \frac{ae^{-3r_0(3)}e^{-3x} + be^{-5r_0(5)}e^{-5x}}{ce^{-4r_0(4)}e^{-4x}}$$

Substituting the definition of w, we find that

$$\frac{A(x)}{L(x)} = we^{x} + (1 - w)e^{-x} \tag{1}$$

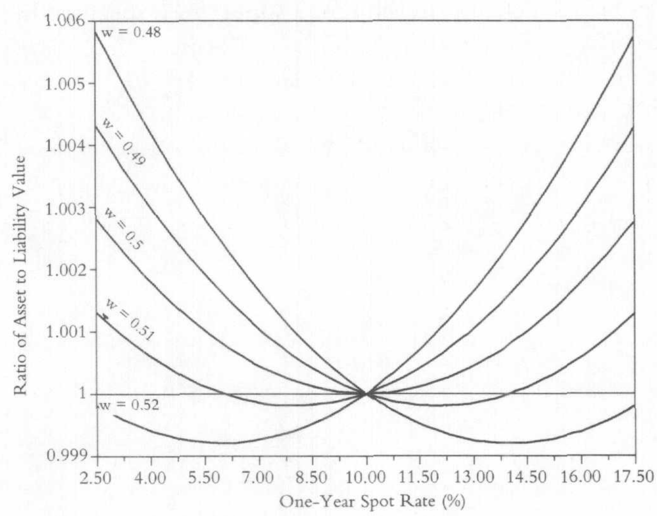
It is easy to verify that if $w = 0.5$, the asset-to-liability ratio has a minimum value of 1 at $x = 0$. Consequently, after a non-zero interest rate shock, the asset value exceeds liability value for this immunized portfolio.

For example, suppose the current spot rate curve is a constant 10%. Exhibit 1 shows the graph of the asset-to-liability ratio versus the one-year spot rate, given by $r(1) = 10\% + x$, for $w = 0.48, 0.49, 0.5, 0.51$, and 0.52. Observe that even when the portfolio is not duration-matched, the ratio still has a minimum value, but it is less than 1 (Bierwag and Khang [1979] prove that immunization under the parallel shift model is a minimax strategy).

Further, if the duration of assets is less than 4 ($w > 0.5$), the minimum occurs at $x < 0$. In this situation, the portfolio has net reinvestment risk. If the duration is greater than 4 ($w < 0.5$), the minimum occurs at $x > 0$, and the portfolio has net liquidation or divestment risk. For large or even moderate duration mismatches, the minimum value could occur at negative or extremely large interest rates. Consequently, the asset-to-liability ratio would either be increasing or decreasing over the range of feasible interest rates.

To summarize, the minimum value of the asset-to-liability ratio equals 1 only if the portfolio is duration-matched. Otherwise, the minimum value is less than 1, and the portfolio has either net price or reinvestment risk. The risk is positively related to the magnitude of the duration mismatch, and the type of risk depends upon the direction of the mismatch.

EXHIBIT 1
ASSET-TO-LIABILITY RATIO VERSUS ONE-YEAR SPOT RATE UNDER PARALLEL SHIFT MODEL FOR DIFFERENT PORTFOLIOS



When the portfolio is duration-matched, an arbitrage opportunity apparently exists. After an interest rate shock, the asset value exceeds the liability value. An arbitrage profit can apparently be realized by shorting the four-year bond and using the proceeds to buy equal dollar amounts of the three- and five-year bonds.

IMMUNIZING OVER TIME

A limitation of this analysis, identified by Christensen and Sørensen [1994] and Reitano [1992], is that it does not consider time. The portfolio is "instantaneously" immunized. Portfolio value will change because of the passage of time, however, even if interest rates are fixed.

For example, consider a par bond that is duration-matched to a zero-coupon bond. Clearly, if one examines changes over time, the zero-coupon bond, unlike the par bond, has built-in appreciation. Also, the portfolio value will change because of expected and unexpected changes in the interest rates over time. Therefore, to examine the equilibrium aspects of a particular model, we need to 1) specify a dynamic term structure model, and 2) examine the value of the immunized portfolio at a later date.

Assume the initial term structure is known and described by the function $r_0(s)$, which assigns a spot rate to each payment date s. The current forward rate from date t to s is determined by current spot rates as follows: $F(t, s) = (r_0(s)s - r_0(t)t)/(s - t)$. In a world of certainty, the time t future spot rate from t to s equals the current forward rate.

We build uncertainty into the model as follows. A random change in the time t spot rates is specified by $b(s)x(t)$, where $x(t)$ is a stochastic process with the initial value: $x(0) = 0$, and $b(s)$ is a given function of the maturity date s. A certain change in future spot rates from the current forward rate is specified by the function $a(t, s)$ with the initial value: $a(0, s) = 0$. Consequently, the family of spot rate processes $r(t, s)$ is described by

$$r(t, s) = F(t, s) + a(t, s) + b(s)x(t) \qquad (2)$$

For example, for the parallel shift model $b(s) = 1$, and for the Khang [1979] model $b(s) = 1/(1 + cs)$.

Define $P(t, s)$ as the time t price of a bond scheduled to pay \$1 at date $s \geq t$. The bond price function P can be expressed in terms of the spot rates as follows:

$$\begin{aligned} P(t, s) &= \exp[-(s - t)r(t, s)] \\ &= P(0, s)e^{r_0(t)t}\exp\{-(s - t)[a(t, s) + b(s)x(t)]\} \end{aligned} \qquad (3)$$

For a given portfolio of bonds, the set of payment dates is denoted by G, and the payment at date $s \in G$ is denoted by $C_s > 0$. If the first cash flow is scheduled for date s_0, then the portfolio value at time $t \leq s_0$ is given by

$$A(t) = \sum_{s \in G} C_s P(t, s) \qquad (4)$$

Substituting $P(t, s)$ from Equation (3) into Equation (4), we obtain

$$A(t) = A(0)e^{r_0(t)t}\sum_{s \in G} w_s e^{-(s-t)[a(t,s)-b(s)x(t)]} \qquad (5)$$

where $w_s = P(0, s)C_s/A(0)$ is the present value weight of the bond maturing at date s.

For simplicity, we consider the case where the liability stream consists of a single negative cash flow at some specified future date. Multiple liabilities can be handled as an extension of the single liability case by immunizing separately for each liability cash flow (see Bierwag, Kaufman, and Toevs [1983]).

Let $L(0)$ be the initial price of a liability due at date $q \geq s_0$. At some future time $t \leq q$, based upon Equation (2), the liability price is given by

$$L(t) = L(0)e^{r_0(t)t}\exp\{-(q - t)[a(t, q) + b(q)x(t)]\}$$

Let us examine the asset-to-liability ratio $\theta(x; t) = A(t)/L(t)$ at time $t \in [0, s_0]$. If asset and liability values are initially equal, then

$$\theta(x; t) = \sum_{s \in G} w_s e^{\alpha(t,s)+\beta(t,s)x(t)} \qquad (6)$$

where $\alpha(t, s) = (q - t)a(t, q) - (s - t)a(t, s)$, and $\beta(t, s) = (q - t)b(q) - (s - t)b(s)$.

It is an interesting fact that for each $t \in [0, s_0]$ the asset-to-liability ratio, $\theta(x;\ t)$, is convex. Observe that

$$\theta''(x) = \sum_{s \in G} w_s \beta(t, s)^2 \exp[\alpha(t, s) + \beta(t, s)x(t)] > 0$$

The standard definition of immunization requires choosing the present value weights (w_s) so that $\theta(x;\ 0)$ obtains a minimum value of 1 at $x = 0$. In other words, the time $t = 0$ asset value is greater than or equal to the liability value for any interest rate shock. Given the convexity of $\theta(x;\ t)$ for each t, the standard definition of immunization requires that $\theta'(x;\ 0) = 0$ at $x = 0$:

$$\sum_{s \in G} w_s \beta(0,\ s) = 0$$

For the parallel shift model, because $\beta(t,\ s) = q - s$, immunization requires:

$$\sum_{s \in G} w_s s = q \qquad (7)$$

The left-hand side of Equation (7) is the Fisher-Weil duration, and the right-hand side is the liability due date. Therefore, immunization for the general parallel shift model requires choosing the present value weights so that the duration equals the liability due date. If the duration-matching condition is satisfied, then the minimum of $\theta(x;\ 0)$ equals 1 at $x = 0$.

Having shown that θ is convex for a large class of term structure models, including the parallel shift model, we can now examine whether convexity is in fact inconsistent with equilibrium. Clearly, the absence of arbitrage opportunities requires that for any $t > 0$ the minimum of $\theta(x;\ t)$ is less than 1. Interestingly, whether a given model is consistent with arbitrage-free equilibrium depends only on the specification of $a(t,\ s)$. In other words, convexity by itself does not mean that a particular term structure model is inconsistent with equilibrium.

EXAMPLES

Time influences portfolio value through built-in price appreciation or depreciation, expected spot rate changes, and unexpected spot rate changes. Traditional immunization theory considers only unexpected changes in interest rates. The first example assumes the expected future spot rate equals the current forward rate. This assumption is often called the *unbiased expectations hypothesis.*

The second example assumes that the expected future spot rate deviates from the current forward rate so that the market price of risk is independent of maturity. In other words, the market price of risk is the same for all bonds. This is the usual no-arbitrage condition for a diffusion process, and is also the same as the Heath, Jarrow, and Morton [1992] forward drift restriction. Although in this example the asset-to-liability ratio, θ, is convex in x and has a minimum value, arbitrage opportunities are not available, because the minimum value is less than one.

Example 1

Suppose that the current spot rate curve is equal to a constant 10%, and that future spot rates are determined by the parallel shift model:

$$r(t,\ s) = F(t,\ s) + x(t)$$

For this specification, $E[r(t,\ s)] = F(t,\ s)$. In words, the expected future spot rate equals the current forward rate. This model is consistent with the unbiased expectations hypothesis.

Our only innovation is to examine the portfolio value at a future time. For the parallel shift model, the asset-to-liability ratio is given by Equation (6) with $\beta(t,\ s) = q - s$:

$$\theta(x; t) = \sum_{s \in G} w_s e^{x(t)(q-s)} \qquad (8)$$

Assume the current spot rate curve is a constant 10%. Again, suppose the liability stream consists of a single liability due in four years. The immunizing portfolio consists of two zero-coupon bonds that mature in three and five years. Based upon Equation (8), the asset-to-liability ratio is given by

$$\theta(x;\ t) = we^{x(t)} + (1 - w)e^{-x(t)} \qquad (9)$$

Because Equation (9) is the same as Equation (1) with x(t) substituted for x, the graph of θ versus the future one-year spot rate $(r(t, t+1) = 10\% + x(t))$ is the same as the graph in Exhibit 1. For the duration-matched portfolio, at some future date t the asset-to-liability ratio $\theta > 1$ for all $x \neq 0$ and $\theta = 1$ for $x = 0$. Clearly, the unbiased expectations hypothesis allows arbitrage profits. Further, even if short sales are disallowed, this situation is inconsistent with equilibrium because no investor would hold the four-year bond when a portfolio of the three- and five-year bonds has a higher return for any interest rate shock.

Example 2

Now we consider the interest rate process as given by the continuous time limit of the Ho and Lee [1986] model (see Heath, Jarrow, and Morton [1990]). This specification for interest rates does not allow arbitrage opportunities. Based upon Heath, Jarrow, and Morton's [1992] Equation (30), it is easy to show that future spot rates are determined by:

$$r(t, s) = F(t, s) + \frac{\sigma^2 ts}{2} + \sigma[W_t - \int_0^t \lambda(\tau)d\tau] \quad (10)$$

where σ is a positive constant, W_t is a Wiener process, and $\lambda(t)$ is a function of time. If we let $a(t, s) = \sigma^2 ts/2$, $b(s) = 1$, and

$$x(t) = \sigma[W_t - \int_0^t \lambda(\tau)d\tau]$$

then clearly Equation (10) is a special case of the parallel shift model.

The bond price dynamics for this spot rate process are described by a stochastic differential equation in t for each s [determined from HJM Equations (9) and (13)]:

$$dP(t, s) = P(t, s)[r(t) + \sigma(s-t)\lambda(t)]dt - P(t, s)\sigma(s-t)dW_t \quad (11)$$

where $r(t)$ is the instantaneous spot rate, $\sigma(s-t)$ is the relative volatility function, and $\lambda(t)$ is the market price of risk.

The market price of risk measures the trade-off between expected return and risk. To see this, let $\mu \times (t, s) = r(t) + \sigma(s-t)\lambda(t)$. Roughly speaking, $\mu(t, s)$ and $\sigma(s-t)$ are the expected value and standard deviation of the instantaneous return at time t on a zero-coupon bond with maturity s. Consequently, $\lambda(t) = [\mu(t, s) - r(t)]/\sigma(s-t)$ measures the excess rate of return per unit of risk for any zero-coupon bond.

The family of price processes described by Equation (11) has two desirable properties. First, the relative price volatility increases with the time to maturity $(s-t)$. In other words, long-term zero-coupon bonds display higher price volatility than short-term bonds. Second, this specification for the price process satisfies the Heath, Jarrow, and Morton [1992] equilibrium drift restriction. Therefore, the equilibrium specification for the parallel shift diffusion model is the continuous-time Ho and Lee model.

Let us now examine a special case. Suppose the current spot rate curve is a constant 10% and the volatility coefficient (σ) is 4.49% per year. Again, suppose the liability stream consists of a single liability due in four years, and the asset portfolio consists of two zero-coupon bonds that mature in three and five years. Because the Ho and Lee model is a special case of the parallel shift model, the immunization condition is given by Equation (7), which requires that the asset duration equal the liability due date. As in the previous example, the asset duration equals four years if the present value weights for both zero-coupon bonds equal 0.5.

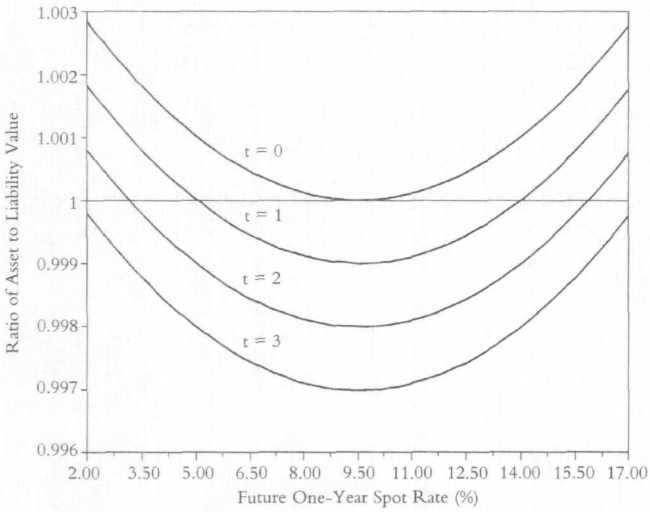
Based upon Equation (6) with $a(t, s) = \sigma^2 ts/2$ and $b = 1$, the asset-to-liability ratio for this example is given by

$$\theta(x, t) = \frac{1}{2}[A(t)e^{x} + B(t)e^{-x}] \quad (12)$$

where $A(t) = \exp[\sigma^2 t(7-t)/2]$ and $B(t) = \exp[\sigma^2 t \times (t-9)/2]$. For a given holding period t, the asset-to-liability ratio is minimum at $x^* = 1/2\log[B(t)/A(t)]$. Evaluating θ at x^*, we find that the minimum asset-to-liability ratio equals $\exp[-\sigma^2 t/2]$ for a given t. Clearly, for the Ho and Lee process, the minimum value of θ is less than 1 for $t > 0$. Because $A(t)/L(t) < 1$ at $x = x^*$, it must also be true that $A(t) - L(t) < 0$ at $x = x^*$. Obviously, the claim that immunization implies risk-free arbitrage profits is false for the parallel shift model.

Exhibit 2 shows the graph of θ versus the future one-year spot rate $r(t, t+1)$ for $t = 0, 1, 2$, and 3. Examination of this graph reveals that for $t = 0$ the minimum value of θ equals 1. For $t = 1, 2$, and 3, however, the minimum value is less than 1 and declin-

EXHIBIT 2
ASSET-TO-LIABILITY RATIO VERSUS ONE-YEAR SPOT RATE UNDER CONTINUOUS-TIME EQUILIBRIUM MODEL FOR IMMUNIZED PORTFOLIO AT t = 0, 1, 2, AND 3



ing in t. The exact values of the minimum (determined from $\exp[-\sigma^2 t/2]$) are 0.9990, 0.9980, and 0.9970 for t = 1, 2, and 3.

Exhibit 3 presents the minimum asset-to-liability ratio versus holding period for volatility coefficients of 2.5%, 5.0%, 10%, and 15%. In all four cases, the minimum decreases with the holding period, suggesting that the cost of immunizing increases with the holding period. It is also interesting to note that the minimum value for any holding period declines as the volatility increases.

Summary

The two examples we have considered are special cases of the parallel shift model. Consequently, the

EXHIBIT 3
MINIMUM ASSET-LIABILITY RATIO VERSUS HOLDING PERIOD BY VOLATILITY

Volatility	t = 0	t = 1	t = 2	t = 3
2.5	1.0000	0.9997	0.9994	0.9991
5.0	1.0000	0.9988	0.9975	0.9963
10.0	1.0000	0.9950	0.9900	0.9851
15.0	1.0000	0.9888	0.9778	0.9668

immunization condition is the same for both examples. It requires matching the Fisher-Weil duration of the assets to the liability due date. Further, in both examples the asset-to-liability ratio for an immunized portfolio is convex with a minimum value of one when the holding period is zero. Hence, both models provide *apparent* arbitrage profits for *instantaneous* interest rate shocks.

Yet the second model, the continuous Ho and Lee model, does not allow arbitrage opportunities, so the claim that immunization necessarily provides arbitrage profits is false for the parallel shift model. We suspect this claim is also false for other convex term structure models, but proof requires further research.

CONCLUSION

By explicitly including time in the term structure model and examining the immunized portfolio at future dates, we have shown that bond convexity can be consistent with arbitrage-free pricing. Under the no-arbitrage equilibrium condition, convexity has a price. Our result provides theoretical support to empirical work (such as that of Kahn and Lochoff [1990]) that suggests that the objective of pursuing convexity does not result in superior portfolio performance.

REFERENCES

Bierwag, G.O. "Bond Returns, Discrete Stochastic Processes, and Duration." *Journal of Financial Research*, Vol. 10 (1987), pp. 191-209.

Bierwag, G.O., G.G. Kaufman, and A. Toevs. "Immunization Strategies for Funding Multiple Liabilities." *Journal of Financial and Quantitative Analysis*, Vol. 18 (1983), pp. 113-123.

Bierwag, G.O., and C. Khang. "An Immunization Strategy is a Minimax Strategy." *Journal of Finance*, Vol. 24 (1979), pp. 389-399.

Christensen, P.O., and B.G. Sørensen. "Duration, Convexity, and Time Value." *Journal of Portfolio Management*, Winter 1994, pp. 51-60.

Cox, J.C., J.E. Ingersoll, and S.A. Ross. "Duration and the Measurement of Basis Risk." *Journal of Business*, Vol. 52 (1979), pp. 51-61.

Fisher, Lawrence, and Roman L. Weil. "Coping with the Risk of Interest-Rate Fluctuations: Returns to Bondholders from Naive and Optimal Strategies." *Journal of Business*, Vol. 44 (1971), pp. 408-431.

Grantier, B.J. "Convexity and Bond Performance: The Benter the Better." *Financial Analysts Journal*, Vol. 44 (1988), pp. 79-81.

Heath, David, Robert Jarrow, and Andrew Morton. "Bond Pricing and the Term Structure of Interest Rates." *Econometrica*, Vol. 60 (1992), pp. 77-105.

——. "Bond Pricing and the Term Structure of Interest Rates: A Discrete Time Approximation." *Journal of Financial and Quantitative Analysis*, Vol. 25 (1990), pp. 419-440.

Ho, Thomas S., and Sang-Bin Lee. "Term Structure Movements and Pricing Interest Rate Contingent Claims." *Journal of Finance*, Vol. 41 (1986), pp. 1011-1030.

Ingersoll, J.E., J. Skelton, and R.L. Weil. "Duration Forty Years Later." *Journal of Financial and Quantitative Analysis*, Vol. 13 (1978), pp. 627-650.

Kahn, R.N., and R. Lochoff. "Convexity and Exceptional Return." *Journal of Portfolio Management*, Winter 1990, pp. 43-47.

Khang, C. "Bond Immunization When Short Rates Fluctuate More than Long-Term Rates." *Journal of Financial and Quantitative Analysis*, Vol. 14 (1979), pp. 1085-1090.

Redington, F.M. "Review of the Principle of Life-Office Valuations." *Journal of the Institute of Actuaries*, Vol. 78 (1952), pp. 286-340.

Reitano, R.R. "Nonparallel Yield Curve Shifts and Convexity." *Transactions of the Society of Actuaries*, Vol. XLIV (1992), pp. 479-507.

Shiu, E.S.W. "On Redington's Theory of Immunization." *Insurance, Mathematics and Economics*, Vol. 9 (1990), pp. 171-175.