Lecture 11

- Trade: Naked Position, Index Options
- Deriving the Black-Scholes model (RSD, p.438-44)
- Extending Black-Scholes to Other Cases (RSD, p.459-71)
- Currency Options: Pricing and Early Exercise (RSD, p.474-9)

The Black-Scholes Formula

THE BLACK-SCHOLES CALL OPTION FORMULA (the put solution is solved using the put-call parity condition).

$$C[S,t^*;X,r,\sigma] = C(t) = S(t)\ N[d_1] - X\ e^{-rt^*}\ N[d_2]$$

- where *N[d]* is the **cumulative** standard normal distribution evaluated at the value *d* and where:

$$d_1 = \frac{\ln[\frac{S}{X}] + (r + \frac{1}{2}\sigma^2)t^*}{\sigma\sqrt{t^*}}$$

$$d_2 = d_1 - \sigma\sqrt{t^*} = \frac{\ln[\frac{S}{X}] + (r - \frac{1}{2}\sigma^2)t^*}{\sigma\sqrt{t^*}}$$

Solving the Formula

Assume: $S(t)$ = \$36, X = \$40, τ = 3 months $\Rightarrow$ t^* = .25, r = .05, σ = .5. Both the interest rate and standard deviation are expressed in annualized form. For sigma, estimating the **historical** standard deviation over the relevant sampling frequency (e.g., weekly) requires annualizing as appropriate. Given this:

$d_1 = \{\ln[36/40] + (.05 + .5(.5)^2)\ .25\} \ / \ \{.5\ (\sqrt{.25})\} = -.25$
$d_2 = \{\ln[36/40] + (.05 - .5(.5)^2)\ .25\} \ / \ \{.5\ (\sqrt{.25})\} = -.50$

Evaluating the $N[d]$ values: $N[-.25]$ = .4013 and $N[-.50]$ = .3085 and $\exp\{(.05)(.25)\}$ = .9877, it is possible to solve for the Black-Scholes call option price:

$C = S\ [.4013] - X\ [.9877]\ [.3085] = 14.4468 - 12.1882 = \$2.26$

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(* Basics of the Black-Scholes Formula Example *)

In[32]:= ndist = NormalDistribution[0, 1]
         CDF[ndist, .25]

Out[32]= NormalDistribution[0, 1]

Out[33]= 0.598706

         (* Inputting values, r = interest rate; T = expiration date;
         S = stock price; X = Exercise price; v = volatility *)

In[34]:= r = .05
         T = .25
         S = 36
         X = 40
         v = .5

         (* Evaluation d_1 and d_2  *)

         x = ((Log[S / X] + (((r + (.5 * (v^2))) * T)))) / (v * Sqrt[T])
                     y = x - (v * Sqrt[T])

         (* The Black-Scholes Call Price *)
         Q =  (S * CDF[ndist, x]) - ((X * Exp[-.0125]) * CDF[ndist, y] )

         (*  The Black-Scholes Put price *)
             P =  (S * (CDF[ndist, x] - 1)) - ((X * Exp[-.0125]) * (CDF[ndist, y] - 1) )

         (* The inputs are listed and then the solutions for d_1,
         d_2, the call price and the put price *)

Out[34]= 0.05

Out[35]= 0.25

Out[36]= 36

Out[37]= 40

Out[38]= 0.5

Out[39]= -0.246442

Out[40]= -0.496442

Out[41]= 2.2584

Out[42]= 5.76151
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Evaluating the Cumulative Normal d.f.

- The **cumulative normal distribution function** $N[x]$ provides the area under the normal density function $n[x]$ at a given point x --- integrate the area under the density function from $-\infty$ to x to get $N[x]$ value.
- The normal density function is sometimes referred to as the bell curve.
- $N[0] = .5 \quad N[+\infty] = 1 \quad N[-\infty] = 0$
- The value of $N[x]$ is the delta for a Black-Scholes call option.

$$\frac{\partial C}{\partial S} = N[d_1] + S\,\frac{\partial N[d_1]}{\partial S} - Xe^{-rt^*}\,\frac{\partial N[d_2]}{\partial S} = N[d_1] + SN'[d_1]\frac{\partial d_1}{\partial S} - Xe^{-rt^*}N'[d_2]\,\frac{\partial d_2}{\partial S}$$

$$= N[d_1] + S\,N'[d_1]\,\frac{\partial d_1}{\partial S} - Xe^{-rt^*}\,N'[d_1]\,\frac{S}{Xe^{-rt^*}}\,\frac{\partial d_1}{\partial S} = N[d_1]$$

How to make sense of Delta and the Riskless Hedge Condition?

What is d_1 ?

$$d_1 = \frac{\ln\left[\frac{S}{X}\right] + (r + \tfrac{1}{2}\,\sigma^2)\,t^*}{\sigma\sqrt{t^*}} = \frac{\ln\left[\frac{S}{X}\right]}{\sigma\sqrt{t^*}} + \frac{(r + \tfrac{1}{2}\,\sigma^2)\,\sqrt{t^*}}{\sigma}$$

Example $S = X$ –> $\ln[1] = 0$

$r = .05$ $\sigma = .25$
$t^* = .25$ (3 month) –> $d_1 = .1625$ $N[d_1] = .56$ # = 1.79
$t^* = .5$ (6 month) –> $d_1 = .2298$ $N[d_1] = .59$ # = 1.69

Deriving Black-Scholes: Basic Elements

- The Black-Scholes (1973) derivation requires some basic concepts (readings in brackets):

  The Black-Scholes Assumptions (RSD, p.439)

  The riskless hedge portfolio (RSD, p.439-40)

  Ito's Lemma (RSD, p.434-6)

  The Fundamental Partial Differential Equation

The Black-Scholes Assumptions

a) Non-dividend paying stock.

b) European option.

c) The instantaneously riskless continuous interest rate *r* is constant over time (with a flat term structure).

d) The model has only **one** source of randomness, the single state variable, the price of the stock which follows a log-normal diffusion process. This log normal process is defined only over S ε [0, ∞].

e) No transactions costs or taxes.

f) No penalties on short selling.

g) Riskless lending and borrowing at r.

h) Continuous trading.

e)-h) are conventional perfect markets assumptions.

Riskless Hedge Portfolio and Arbitrage

- Discussion of pricing for futures and forward contracts used the cash-and-carry arbitrage

- The **riskless hedge portfolio** used in Black-Scholes (1973) predates arbitrage arguments – the riskless hedge portfolio has a net investment of funds.

- The riskless hedge portfolio argument can be reconstructed as an arbitrage portfolio argument where the funds invested in the position are borrowed.

SOME BASIC CALCULUS

Derivative and Total derivative of a function of one variable

$$f[x] = x^2 \qquad \frac{df}{dx} = 2x \qquad f[x]=a\ x^3 \qquad \frac{df}{dx} = 3a\ x^2 \quad \rightarrow \quad df = (3a\ x^2)\ dx$$

Derivative/Total derivative have to be adjusted for functions of more than one variable

$$df[x,y] = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy \qquad f[x,y] = x^2\ y^3 \qquad df = 2x(y^3) + (x^2)\ 3y^2$$

Change in notation to account for the function of more than one variable.

Black Scholes option price function has more than one variable: *S* and *t*

The interest rate, exercise price, expiration date and volatility are conceived of as parameters in deriving the formula

Riskless hedge portfolio involves a partial derivative: $\dfrac{\partial V}{\partial S}$

The Riskless Hedge Portfolio

- There is both a long and a short hedge portfolio, the long portfolio combines **a long stock position with β written call options on the stock**.

- The value of required to purchase this portfolio **(*V*)** is expressed as

$$V(t) = S(t) - \beta C(t)$$

- The hedge portfolio condition can be specified as:

$$\frac{\partial V}{\partial S} = 1 - \beta \frac{\partial C}{\partial S} = 0 \quad \rightarrow \quad \beta = \frac{1}{\frac{\partial C}{\partial S}} \quad \rightarrow \quad \frac{\partial C}{\partial S} = \frac{1}{\beta}$$

Comments on the Hedge Portfolio

- The hedge portfolio condition $(\partial V / \partial S) = 0$ involves the partial derivative with respect to *S*. Do not confuse this with $dV = 0$.
- This is a riskless hedge portfolio condition because the amount of funds required to purchase *S* is less than the funds received from selling β call options, i.e., there is a net investment of funds in the position. (This violates the requirements for an arbitrage).
- The number of written calls needed to hedge the portfolio value will change as *C* and *S* change over time. The **delta** of the call $(\partial C / \partial S) = \Delta_C$ provides this information.

Basic Calculus and Stochastic Calculus

- Conventional calculus deals with deterministic functions, *f[x,y]*, where *x* and *y* variables are non-random.
- The total derivative rule:

$$df[x,y] = (\partial f / \partial x)\, dx + (\partial f / \partial y)\, dy$$

- The call price function ***C[S,t*;X,r,σ]*** contains the random variable *S* (Semantics: random = stochastic).

- Stochastic Calculus is for functions with stochastic variables. This is more complicated because the process of taking the slope at a point is not possible because the function is not well defined at a point if there is a random component.

Ito's Lemma and Stochastic Calculus for Diffusion Processes

- Ito's Lemma provides the rule for totally differentiating functions of random variables where the randomness is associated with a **diffusion process**.

Ito's Lemma: Let $y(x,t)$ be a continuous random function with continuous partial derivatives y_t, y_x, and y_{xx}. If $x(t)$ is a random process obeying a diffusion of the form:

$$dx(t) = a(t)\,dt + v(t)\,dW(t)$$

where $W(t)$ is a standard Wiener process and $a(t)$ and $v(t)$ are the drift and volatility of the diffusion, then the function $y(t) = y(x(t),t)$ also has a differential on $[0,T]$ given by:

$$dy(t) = \{y_t + y_x\,a(t) + 1/2\,y_{xx}\,v(t)^2\}\,dt + y_x\,v(t)\,dW(t)$$

Ito's Lemma for the Call Price Function

The non-dividend paying stock price is assumed to follow a log-normal diffusion:

$$dS = \alpha\, S\, dt + \sigma\, S\, dW$$

In this case, ***a(t)* = *αS* and *v(t)* = *σS*** where α and σ are constants.

- The functional relationship between the call option price *(C)* and the stock price takes the form: $C = C[S,t; X]$. Application of Ito's lemma gives:

$$dC = \{C_t + C_S\, \alpha\, S + 1/2\, C_{SS}\, \sigma^2\, S^2\}\, dt + \{C_S\, \sigma\, S\}\, dW$$

Solving for the fundamental Black-Scholes PDE

$$\{S - \frac{C}{C_S}\}\ r\ dt = dV \qquad \rightarrow \qquad dV = dS - \frac{dC}{C_S}$$

At this point, Ito's lemma can be applied to solve for dC:

$$dC = \{C_t + C_S\ \alpha S + \frac{1}{2}\ C_{SS}\ \sigma^2\ S^2\}\ dt + C_S\ \sigma\ S\ dW$$

The solution for dV can be derived as:

$$dV = dS - \frac{dC}{C_S}$$

$$= \alpha S\ dt + \sigma S\ dW - \frac{1}{C_S}\{[C_t + C_S\ \alpha S + \frac{1}{2}\ C_{SS}\sigma^2\ S^2]\ dt + C_S\sigma S\ dW\}$$

$$= -\frac{1}{C_S}\{C_t + \frac{1}{2}\ C_{SS}\sigma^2 S^2\}dt = (S - \frac{C}{C_S})r\ dt$$

Solving for the Fundamental PDE

- The Black-Scholes derivation uses two conditions:

$$dV = dS - \beta\, dC \quad \text{(Total Derivative)}$$

$$dV = (S - \beta\, C)\, r\, dt \quad \text{(Riskless Return Condition)}$$

The first condition can be solved by substituting in: the diffusion equation for dS, $\beta = C_S$ (from the riskless hedge solution) and dC from Ito's Lemma.

The two conditions are equated and solved to get the **fundamental partial differential equation** (PDE) for a European call on a non-dividend paying stock (**subject to the boundary and terminal conditions of** $C[T] = \max[0, S-X]$ **and** $C[S=0] = 0$)

$$C_t = rC - rS\, C_S - \frac{1}{2}\sigma^2 S^2 C_{SS}$$

Solving the PDE

- The Black-Scholes formula is the function that solves the fundamental PDE problem.
- There are a number of possible methods that can be used to derive this solution – solving PDE's is an active research area in mathematics.
- In this class the solution will be verified when the Greeks are being discussed (see RSD, p.497-8).

(Notice that the fundamental PDE provides a relationship among delta, gamma and theta).

Solving for the Put Pricing Formula

$$P = C + X e^{-rt^*} - S = S\,N[d_1] - X e^{-rt^*} N[d_2] + X e^{-rt^*} - S$$

$$= S\,\{N[d_1] - 1\} - X e^{-rt^*} \{N[d_2] - 1\} = X e^{-rt^*} N[-d_2] - S\,N[-d_1]$$

- Solving for the Black-Scholes Put Price involves using the Put-Call parity condition and using the result that (see Appendix 3 in RSD for derivation)

$$N[x] - 1 = N[-x]$$

Extending Black-Scholes: Dividends

- Extensions to the basic Black-Scholes formula follow from dropping specific assumptions.
- Dropping the no dividends assumption can involve either ***continuous (D = δS)*** or ***discrete*** payment frequencies.
- In both cases, the riskless hedge condition is adjusted. For the continuous dividend case, the call option formula becomes:

$$C = S \exp\{-\delta t^*\} N[d_1] - X \exp\{-rt^*\} N[d_2]$$

where:

$$d_1 = \frac{\ln\{\frac{S}{X}\} + \{(r - \delta) + \frac{1}{2}\sigma^2\}t^*}{\sigma\sqrt{t^*}}$$

$$d_2 = d_1 - \sigma\sqrt{t^*}$$

Extending Black-Scholes: Forward Contracts

- When the underlying security is a forward contract, instead of a stock, there are no funds required to purchase the underlying security. All that remains is the cash inflow from selling the call. The call option formula in this case becomes:

$$C = \exp\{-rt^*\}\{F(t,T)\, N[d_1] - X\, N[d_2]\}$$

where:

$$d_1 = \frac{\ln\{\frac{F(t,T)}{X}\} + (\frac{\sigma^2}{2})t^*}{\sigma\sqrt{t^*}} \qquad d_2 = d_1 - \sigma\sqrt{t^*}$$

Extending Black-Scholes: American Options

- Almost all exchange-traded options are American, not European. Except in specials cases (perpetuals, calls on non-dividend paying stocks), there are no formulas for American options.
- Solution procedures for Americans involve binomial methods.
- For the implications of using Black-Scholes to value the American, recall from distribution free properties:

$$C_A = C + EEP_C$$

Extending Black-Scholes: Other Diffusions

- Geometric Brownian motion, the log-normal diffusion, has the desirable property that prices have to be non-zero.
- Arithmetic Brownian motion permits the process to assume negative values. For a non-dividend paying security, this process produces the solution (see RSD, p.469 for definition of *g).*

$$C_G(S, t^*; r, \sigma, X) = e^{-rt^*}\{(S_t\, e^{rt^*} - X)\, N[g] + \sigma\sqrt{t^*}\, n[g]\}$$

$$g = \frac{S_t\, e^{rt^*} - X}{\sigma\sqrt{t^*}}$$

Extending Black-Scholes: Currencies

- The security underlying the currency option is a foreign money market security that pays interest continuously at rate *r**.
- The solution is known as the Garman-Kohlhagen currency option formula:

$$C = S \exp\{-r^* t^*\}\, N[d_1] - X \exp\{-r t^*\}\, N[d_2]$$

where:

$$d_1 = \frac{\ln\{\frac{S}{X}\} + (r - r^* + \frac{1}{2}\sigma^2)\, t^*}{\sigma \sqrt{t^*}}$$

$$d_2 = d_1 - \sigma \sqrt{t^*}$$

Currency Options and Early Exercise

- Early exercise of currency options is a common event.
- In practice, early exercise for stocks is associated with a discrete payment. Currency options have continuous-pay underlying securities.

- Key results for early exercise:

An American call on a non-dividend paying stock will not be exercised early (see Property 10, RSD, p.378-80)

Only in-the-money options are exercised.

Early exercise involves the loss of time value.

Currency Option Early Exercise (cont'd)

- The early exercise decision does **not** involve an arbitrage.

- Arbitrage support for the American call option is provided by Property 10:

  $C_A[S,\tau,X] \geq C[S,\tau,X] \geq Max[0, S(t)PV[r^*,\tau] - X\ PV[r,\tau]]$

However, as the currency option goes deep-in-the money
$C \rightarrow Se\text{-}r^*t^* - Xe\text{-}rt^*$ (because $N[d] \rightarrow 1$ and $EEP \rightarrow 0$)
which is $< S - X$ when $r^* > r$.