Estimation of the Optimal Hedge Ratio, Expected Utility, and Ordinary Least Squares Regression

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In practice, commodity hedgers are faced with a fundamental question: what ratio of spot to futures positions is most appropriate to maximize the expected utility of end-of-period profit? Over time, considerable academic attention has been given to the solution of this problem (e.g., Edgerington, 1979; Herbst, Kare and Caples, 1989; Hill and Schneeweis, 1982; Johnson, 1960; Stulz, 1984; Toevs and Jacob, 1986). Much of this research focuses on estimating hedge ratios using an ordinary least squares (OLS) regression of spot prices on futures prices; the "optimal" hedge ratio being the estimated slope coefficient. Unfortunately, while it has long been recognized that OLS is optimal for only a restricted set of expected utility functions, the relationship between a general expected utility function and the OLS estimate of the hedge ratio is unavailable.

In this vein, recent work on hedge ratio estimation demonstrates that the "optimal" futures hedge ratio does depend on the objective function selected. For example, significantly different empirical estimates of the hedge ratio for T-bonds have been obtained from log and minimum variance expected utility functions (Cecchetti, Cimby, and Figlewski, 1988).[1] However, available evidence to date is restricted to comparisons of specific objective functions. With this in mind, this article formulates two different solutions to the general expected utility problem underlying the hedge ratio optimization: first, under the assumption of bivariate normality of the spot and futures returns, a general relationship between the OLS estimate and the hedge ratio implied by a general expected utility function is derived, and a number of specific expected utility functions are examined as illustrations; second, admitting riskless lending and borrowing in the wealth dynamics, the hedge ratio is shown to be independent of preferences, i.e., depending solely on parameters of the joint distribution of the return-generating processes. These results are derived for the traditional, single-period, "myopic" objective function, and then generalized to maximizing lifetime consumption admitting certain types of conditional probability distributions.

[1]However, despite the differences in the estimated hedge ratios, the returns to the hedge portfolios are not significantly different. This occurs despite the greater variability in the return to the portfolio based on the log utility hedge ratio.

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PREVIOUS RESULTS

While there are a number of roughly equivalent specifications of the hedger's optimization problem, following Cecchetti et al. (1988) the relevant problem can be expressed as maximizing the expected utility of terminal wealth for a hedged portfolio where wealth (W) is determined by

$$W_{t+1} = W_t(1 + R_s(t + 1) - h_t R_f(t + 1)) \tag{1}$$

where h_t is defined to be the ratio of the values (price times quantity) of the spot and futures positions at time t,[2] $R_s(t + 1)$ and $R_f(t + 1)$ are the t to $t + 1$ returns to holding spot and futures and $t \in [0 \ldots . T]$, i.e., $R_A(t + 1) = (A_{t+1} - A_t)/A_t$ where A is either the spot or futures price at t and $t + 1$. By construction, the selection of eq. (1) to specify the wealth dynamics, restricts the problem in order to derive implementable solutions. In particular, eq. (1) assumes a single-period decision framework with no potential for variation in the quantity of the spot commodity being held.

Given this, the conventional hedge ratio optimization problem can be generalized to admit any type of well-behaved utility function

$$\max_{h_t} \int_{R_f} \int_{R_s} U[W_t(1 + R_s(t + 1) - h_t R_f(t + 1))]\, dP_{sf}(t) = E[U[\pi_{t+1}] \,|\, X(t)] \tag{2}$$

In this form, the joint conditional probability measure associated with the expectation is P_{sf}, the profit function is $\pi_{t+1} = W_t(R_s(t + 1) - h_t R_f(t + 1))$ and $X(t)$ is the conditioning (state variable) information available at t.[3] In practice, there are restrictions on the types of commodities for which eq. (2) is the appropriate hedging problem. For example, if no allowance is taken of unexpected variation in quantity of the spot commodity, the hedged portfolio does not fully capture the wealth dynamics associated with many harvestable crops. However, eq. (2) is appropriate in the case of financial futures, such as T-bonds, money market securities, and currencies.[4]

Another significant feature of eqs. (1) and (2) is the absence of portfolio theoretic considerations. In particular, by not incorporating lending and borrowing into the specification of the profit function, the size of the cash position is fixed. In effect, the resulting hedging optimization assumes away the portfolio decision by having the hedger fully invested in the spot commodity. If lending and borrowing is admitted, considerations of leveraging to buy the spot commodity, and short-selling the spot to invest in the riskless asset, enter the hedger's decision process. Theoretically, this is translated into a change in the underlying wealth dynamics to

$$W_{t+1} = W_t(1 + x_t R_s(t + 1) + (1 - x_t) r(t + 1) - H_t R_f(t + 1)) \tag{1'}$$

where x is the fraction of total wealth invested in the spot commodity, H is the value (price times quantity) of the hedge position divided by initial wealth (*not the value of the spot position*), and r is the riskless rate. In turn, (1′) produces a new

[2]Because h_t enters with a minus sign, this defines a short futures position to be a positive quantity.

[3]The transformation from terminal wealth, W_{t+1}, to terminal profit, π_{t+1}, follows because the expectation of $U[W_t]$ reduces to a constant which does not affect the optimization. In addition, it will always be assumed that the only $X(t)$ state variables of interest are R_s and R_f. However, in general, this need not be true.

[4]The appropriate trades and profit functions to be used for the currency and money market hedges are discussed in Poitras (1988, 1991).

optimization problem (2′), i.e., (2) with (1′) used instead of (1). In practice, the primary advantage of using (2) over (2′) is analytical simplification: the optimal hedge ratio requires specification of only a joint probability measure and a utility function. The addition of lending and borrowing results in the introduction of an additional choice variable.

In terms of solutions to (2), much of the early literature on hedge ratio estimation is concerned with the minimum variance solution (e.g., Edgerington, 1979) which, under some strong distributional assumptions, leads to an optimal hedge ratio that equals the slope coefficient in an OLS regression of spot on futures prices. Considerable subsequent debate focuses on whether the price variables should be expressed in levels, changes or rates of return (e.g., Myers and Thompson, 1989; Toevs and Jacob, 1986; Witt, Schroeder, and Hayenga, 1987). Specification and estimation of the "optimal" hedge is restricted to special cases. For example, assuming that the current futures price is an unbiased predictor of the distant futures price, the OLS and mean-variance optimal hedge ratio specifications are known to be identical (Benninga, Eldor, and Zilcha, 1984).

Unfortunately, there are practical problems with the OLS estimate because it depends on constant conditional variances and covariances while eq. (2) admits conditional distributions that do not, in general, have constant parameters (e.g., Poitras, 1988). In particular, the conditional expectation involves the changeable state variable information available at the time the hedging decision is made. These analytical considerations are essential to establishing a theoretical basis for the overwhelming empirical evidence available for almost all financial data: that volatility or risk measures vary intertemporally. The resulting heteroskedasticity is at least partly responsible for the high degree of kurtosis in the distributions of both the return on investment securities and the price change for derivative securities. Analytically, this creates a substantive problem for the OLS-based hedge ratio.

By construction, OLS depends fundamentally on the selection of joint probability measures which are constant over time. This assumption results in equality of conditional and unconditional parameters. When time variation in the joint probability measure is permitted, e.g., due to ARCH errors, the decision problem can be more complicated. In this case, eq. (2) typically takes on a more complicated form and has to be solved using some dynamic optimization procedure, e.g., dynamic programming, which takes account of the state variable time paths. The resulting solutions are potentially intractable and difficult to interpret. However, in the special case of *log utility* (Cecchetti et al., 1988), the dynamic solution reduces to a sequence of one-period solutions (Samuelson, 1969). This important simplification permits the introduction of certain types of temporal variation in the conditional variances and covariances without significantly complicating the solution.

In addition to complications arising from nonconstant distributional parameters, when the structure of the optimal hedging problem is altered by the introduction of riskless lending and borrowing, variation in the size of the spot position means the hedge ratio cannot be determined by choosing the relative size of the futures position. There are now two choice variables: the fraction of initial wealth invested in the riskless asset and the size of the hedged position. Again, while there is explicit recognition of riskless lending and borrowing, analysis is restricted to special cases, particularly mean-variance (Bond and Thompson, 1986; Turvey and Baker, 1989). In certain special cases (e.g., Poitras, 1989), the resulting optimal hedge ratio is shown to be independent of hedger risk preferences, i.e., depending solely on the parameters of the (un)conditional joint distribution of returns.

MYOPIC RESULTS

This section provides two propositions corresponding to the two different formulations of the "myopic" optimal hedge ratio problem, where myopia is a direct consequence of the single-period specification of the optimization problem. The first formulation is based on the conventional approach, which omits lending and borrowing from the portfolio decision, i.e., eq. (1) is the basis of the objective function. The second approach uses eq. (1') thereby admitting portfolios that allow riskless lending and borrowing. In this analysis, "myopia" dictates that future time paths of the conditioning variables are ignored; the trade is initiated at time t, and profits are taken at $t + 1$. Given this, Proposition I (presented below) extends the conventional constant distributional parameter solution to include a general expected utility function. It is shown that the optimal hedge ratio can be decomposed into the OLS-based hedge ratio (h_{OLS}) and a utility function dependent term. Proposition II incorporates riskless lending and borrowing to determine a market equilibrium hedge ratio, which is shown to be independent of the utility function selected.

Conventionally, h_{OLS} is the foundation of empirical estimation of hedge ratios. Hence, it is important to know the relationship between specific solutions to eq. (2) and the OLS estimate. More precisely, the linkage between eq. (2) and the minimum variance hedge ratio is given by the following proposition (proofs for Propositions I–III are provided in the Appendix):

Proposition I: Optimal Myopic Hedge Ratio

Under the assumption of constant parameter bivariate normality of $R_s(t)$ and $R_f(t)$, the generalized optimal hedge ratio can be specified as

$$h^* = h_{OLS} + \frac{E[R_f]}{\text{var}[R_f]} \frac{E[U'(\cdot)]}{W_t E[U''(\cdot)]} \tag{3}$$

where E[·] is the (un)conditional expectation taken with respect to the joint density, U' and U'' are the first and second derivatives of the selected utility function with respect to π, var[R_f] is the (un)conditional variance of R_f, and h_{OLS} is from a regression of spot on futures prices.

In words, Proposition I demonstrates that, for myopic agents, the optimal hedge ratio can always be decomposed into a sum of the OLS-based hedge ratio and an additional term that is fully determined by statistical parameters and the risk aversion propensity of the selected utility function.

The primary upshot of Proposition I is that, in addition to h_{OLS}, consideration must be given to the variance-deflated expected return on the futures position. When the expected return is nonzero, the properties of the particular expected utility function assumed, i.e., the inverse of the coefficient of relative risk aversion, takes on importance. Examining the effect of the statistical parameters, an important *general* corollary follows: when the current futures price is an unbiased predictor of the distant futures price ($E[R_f] = 0$), h_{OLS} is optimal. Hence, results that apply to specific utility functions (e.g., Benninga et al., 1984; Poitras, 1989) can be generalized to any type of admissible utility function. However, for many commodities, $E[R_f] = 0$ is not observed, in which case the issue of selecting an appropriate function is raised.

To better illustrate, consider some specific examples. Because of the normality assumption, if utility is taken to be a negative exponential, $U = -\exp\{-\alpha W\}$, this is equivalent to assuming mean-variance expected utility. In this case

$$h^*_{\text{MV}} = h_{\text{OLS}} - \frac{E[R_f]}{\alpha W_t \text{ var}[R_f]} \tag{4}$$

This form of solution also emerges for other methods of generating mean-variance expected utility, i.e., for quadratic utility where $U = \pi - \frac{1}{2}b(\pi - E[\pi])^2$. To contrast constant absolute and relative aversion utility functions, consider the power utility function, $U = (\pi^p/p)$, where $p < 1$. In this case

$$h^*_{\text{pow}} = h_{\text{OLS}} - \frac{E[R_f]}{\text{var}[R_f]} \frac{E[\pi^{p-1}]}{W_t(1 - p)E[\pi^{p-2}]} \tag{5}$$

For the important specific power utility case of log utility, $U = \ln(\pi)$, the solution reduces to

$$h^*_{\ln} = h_{\text{OLS}} - \frac{E[R_f]}{\text{var}[R_f]} \frac{E[\pi^{-1}]}{W_t E[\pi^{-2}]} \tag{5'}$$

From eqs. (4) and (5), it follows that a given optimal hedge ratio depends on parameters of both the conditional distribution and the expected utility function.

Significantly, Proposition I demonstrates that *when* $E[R_f] \neq 0$, it is not "optimal" to use OLS hedge ratios without making further assumptions about the return and profit generating processes and the form of expected utility. In practice, given specific distributions for the relevant processes, the h^* of interest can be approximated with numerical methods (e.g., Cecchetti et al., 1988).[5] This typically results in a substantive increase in the complexity of the estimation problem. Unfortunately, it is not possible to establish, theoretically, whether there will be corresponding increases in the value of the resulting hedge ratio estimates. While there is related work on portfolio specification that indicates there may be potential benefits (e.g., Grauer, 1986), the value added of more direct specification of the both the expected utility function and return generating processes is largely an unresolved empirical issue.

Turning to the specification of the underlying optimization problem, within the myopic model the introduction of riskless lending and borrowing alters the objective function such that the wealth dynamics are now given by eq. (1′). Solution of the resulting optimization problem, eq. (2′), leads to the following:

Proposition II: Quasi-Separation of the Hedge Ratio[6]

In the presence of riskless lending and borrowing, the myopic optimal hedge depends solely on expectations and other statistical parameters and is not affected by risk attitudes or initial wealth. Specifically the optimal hedge ratio is given by

$$h^* = \frac{\text{var}[R_s]}{\text{var}[R_f]} \frac{(\beta[F, S](E[R_s - r]) - E[R_f])}{((E[R_s - r]) - \beta[S, F]E[R_f])} \tag{6}$$

where $\beta[S, F] = (\text{cov}[R_s, R_f]/\text{var}[R_f])$; $\beta[F, S] = (\text{cov}[R_s, R_f]/\text{var}[R_s])$.

[5]This follows whenever the "other-than-OLS" term is bounded in the state space. For example, numerical algorithms can then be started at the OLS estimate and then iterated until sufficient convergence is achieved. In using numerical solution techniques, specific attention should be paid to singularities in the underlying function. Singularities rule out the pairing of certain types of measures and utility functions.

[6]The result is only quasi-separation because it is partial equilibrium.

Because eq. (6) depends only on parameters of the joint (un)conditional distribution, the optimal hedge ratio with riskless borrowing and lending is independent of both the specification of the expected utility function and initial wealth.

While not immediately apparent, the relationship between Propositions I and II can be seen by evaluating eq. (6), where $E[R_f] = 0$. In this case, $h^* = \beta[S, F] = h_{OLS}$. Significantly, as is the case without lending and borrowing, h_{OLS} is optimal when the current futures price is an unbiased predictor of the distant futures price. Hence, while it is not possible to provide a revealing closed form expression, the introduction of lending and borrowing into the hedger's optimization problem does not alter the general result that the optimal hedge ratio is decomposable into h_{OLS} and another term that depends on statistical parameters. However, admitting the ability to short-sell and leverage eliminates the need to consider the risk-aversion properties of the selected utility function. The result is that, in practice, optimization problems based on eq. (1′) may produce more implementable solutions than those based on eq. (1).

INTERTEMPORAL RESULTS

This section examines the "intertemporal" case. In effect, the formulation of the underlying the optimization problem system [(2), (2′)] is being formalized. While of considerable practical interest, eqs. (3) and (6) are theoretically imprecise due to the assumption of myopia which allows expected utility of terminal wealth to be optimized without considering the entirety of lifetime consumption. This is an important theoretical development because it permits the future time paths of the conditioning variables to affect the optimization problem. As in the previous section, the results in this section feature a general expected utility function. In addition, conditional probability measures with state-dependent parameters are admitted. All the results depend on the assumption of conditional joint normality of the underlying returns. While it is possible to further generalize to include other types of joint distributions, this typically reduces the sharpness of the results.

The problem of maximizing the expected utility of lifetime consumption (not terminal wealth) can be achieved expediently by assuming additive separability of the utility function. Given this, at any time, t, the hedger's more general "intertemporal" optimization problem is

$$J(W_t, X_t) = \max_{C_i; i=[t\ldots T]} E\left\{\sum_{i=t}^{T} U[C_i] + D_T \middle| X(t)\right\} \tag{7}$$

where C_i is consumption in period i, and D_T is the bequest function for the terminal date T (e.g., Ingersoll, 1987). In eq. (7), the introduction of consumption into the intertemporal problem is dependent on using a different specification for the wealth dynamics, i.e., corresponding to eq. (1) in the myopic problem is W for the intertemporal case:

$$W_{t+1} = (W_t - C_t)\{1 + R_s(t + 1) - h_t R_f(t + 1)\} \tag{8}$$

In addition to incorporating consumption, the intertemporal problem involves conditional distributions that require the relevant state variables to be identified. While it is possible to be more general, for present purposes, the interest rate processes (R_s and R_f) are the only state variables considered. Given this, the general solution to the optimization problem must now incorporate compensation for the hedger's "nervousness" about future changes in the state variables.

Applying Bellman's dynamic programming principle (e.g., Malliaris and Brock, 1982), the generalization of Proposition I reveals the corresponding complications:

Proposition III: Optimal Intertemporal Hedge Ratio
Under the assumption of conditional bivariate normality of $R_s(t)$ and $R_f(t)$, the generalized optimal hedge ratio combining eqs. (7) and (8) can be specified as

$$h_t^* = (1 + \gamma)h_{\mathrm{OLS}}|_{x(t)=c} + \frac{E[R_f]}{\mathrm{var}[R_f]}\frac{E[J_W(\cdot)]}{E[J_{WW}(\cdot)]^*} + \frac{E[J_{Wf}]}{E[J_{WW}(\cdot)]^*}$$

where $\gamma = E[J_{Ws}]/E[J_{WW}]^*$, $E[J_{WW}]^* = (W_t - C_t)E[J_{WW}]$, c is the observed values on the state variables (X) up to and including t, and all expectations are taken conditionally on $X(t)$ with

$$E\left[\frac{\delta J}{\delta W_{t+1}}\middle| X(t)\right] \equiv J_W \qquad E\left[\frac{\delta^2 J}{\delta W_{t+1}^2}\middle| X(t)\right] \equiv J_{WW}$$

$$E\left[\frac{\delta^2 J}{\delta W_{t+1}\delta R_{s,t+1}}\middle| X(t)\right] \equiv J_{WS} \qquad E\left[\frac{\delta^2 J}{\delta W_{t+1}\delta R_{f,t+1}}\middle| X(t) \equiv J_{Wf}\right.$$

In addition to h_{OLS} now being a conditional estimate, the role of preferences in the intertemporal optimal hedge ratio is more complicated than in the myopic case. Most significantly, *h_{OLS} is no longer generally optimal when $E[R_f] = 0$.*

Specifically, the additional preference-dependent terms arise from expected changes in the state variables affecting the marginal utility of wealth. In this situation, utility function selection takes on added importance. For example, log utility possesses the important simplifying property that $J_{Ws} = J_{Wf} = 0$, which allows the intertemporal solution to correspond directly to the myopic case, with the caveat of potential inequality of conditional and unconditional statistical parameters. Empirical estimation proceeds by assuming a specific form of conditional distribution, e.g., ARCH (Cecchetti et al., 1988). Significantly, other important types of utility functions such as quadratic and power are not so well behaved. Estimation of intertemporal optimal hedge ratios for these types of functions may be problematic.[7]

Similar complications arise when riskless lending and borrowing is admitted. In this case, the wealth specification is:

$$W_{t+1} = (W_t - C_t)\{1 + (1 - x_t)r(t + 1) + x_tR_s(t + 1) - H_tR_f(t + 1)\} \tag{9}$$

In an intertemporal context this leads to the introduction of an additional conditioning (state) variable, r, which has to be taken into account. As in Proposition III, the risk associated with the potential changes in the state variables must be compensated. This leads to an equilibrium condition for the generalized hedge ratio that does not involve quasi-separation. In particular, there are terms that involve J_{Ws}, J_{Wf}, and J_{Wr}, i.e., indirect utility function (J) terms appear.[8] The exact expression for the optimal hedge ratio is quite complicated and not revealing and, as a result, is not given here. However, as before, log utility provides an important simplification: because the J_{Ws}, J_{Wf}, and J_{Wr} terms are zero, the myopic result (Proposition II) applies.

[7]Advanced estimation techniques, e.g., Scott (1989), would likely have to be applied.

[8]However, J_{WW} terms do not appear and, as a result, the hedge ratio does not depend on risk attitudes of the hedger.

SUMMARY

This article provides a number of useful generalizations and clarifications of prevailing theoretical analyses on optimal hedge ratio estimation. In turn, these results are of direct relevance for accepted methods of empirical estimation. For example, in the context of a myopic model, it is established that h_{OLS} is a robust estimate of the optimal hedge ratio whenever $E[R_f] = 0$. In addition, if the optimization problem is appropriately specified [eq. (2′)], deviations from robustness of h_{OLS} depend solely on distributional parameters. Finally, it is demonstrated that the formally correct method of specifying the underlying optimization problem, i.e., maximization of the expected utility of lifetime consumption, leads to the result that h_{OLS} is a potentially biased estimate of the optimal hedge ratio.

Appendix

Proof of Proposition I: At a given point in time, the problem is to maximize the expected utility of terminal wealth. Observing that the level of initial wealth matters, this involves solving for the choice variable, h, in the problem

$$\max_h E[U[W_t(1 + R_s - hR_f)]]$$

The first order condition (foc) gives

$$E[U'[\cdot]\, R_f] = 0$$

Observing U' and R_f can be treated as random variables, expanding the definition of covariance and substituting the foc gives

$$\text{cov}[U', R_f] + E[U'[\cdot]]E[R_f] = 0 \qquad \text{(A1)}$$

Bivariate normality of R_s and R_f permits the introduction of the Stein–Rubinstein lemma (Rubinstein, 1976)

$$\text{cov}[U', R_f] = E[U''[\cdot]]\, \text{cov}[\cdot, R_f]$$

Substituting this result into eq. (A1) produces

$$W_t E[U''[\cdot]]\, \text{cov}[1 + R_s - hR_f, R_f] + E[U'[\cdot]]E[R_f] = 0$$

Evaluating the covariance term gives

$$W_t E[U''[\cdot]]\, [\text{cov}[R_s, R_f] - h\, \text{var}[R_f] = -E[U'[\cdot]]E[R_f]$$

It follows immediately that the optimal hedge ratio is specified as

$$h = \frac{\text{cov}[R_s, R_f]}{\text{var}[R_f]} + \frac{E[R_f]}{\text{var}[R_f]} \frac{E[U'[\cdot]]}{W_t E[U''[\cdot]]} = h_{\text{OLS}} + \frac{E[R_f]}{\text{var}[R_f]} \frac{E[U'[\cdot]]}{W_t E[U''[\cdot]]}$$

Proof of Proposition II: In admitting riskless lending and borrowing, observe that initial wealth can be ignored because it will cancel out, it follows that the appropriate optimization problem is

$$\max_{x,h} E[U[1 + (1 - x)r + xR_s - HR_f]]$$

where $x = Q_sP_s/(Q_sP_s + (Q_b/(1 + r)))$ and $H = Q_fP_f/(Q_sP_s + (Q_b/(1 + r)))$ with Q and P being the appropriate quantities and prices. The first order conditions give

$$H\colon E[U'[\cdot]\, R_f] = 0$$

$$x\colon E[U'[\cdot]\,(R_s - r)] = 0$$

Bivariate normality of R_s and R_f allows the application of the Stein–Rubinstein lemma

$$E[U']E[R_f] + E[U''[\cdot]]\operatorname{cov}[xR_s - HR_f, R_f] = 0$$

$$(E[R_s - r])E[U'] + E[U''[\cdot]]\operatorname{cov}[xR_s - HR_f, R_f] = 0$$

Evaluating the covariances and solving gives the linear system

$$[x, H]^T = B\sum{}^{-1}[(E[R_s - r]), E[R_f]]^T$$

where Σ is the variance covariance matrix for the R's and $B = -[E[U']/E[U'']]$ and T indicates transposition. Observing that $h = x/H = Q_sP_s/Q_fP_f$ gives the solution

$$h^* = \frac{\operatorname{var}[R_s]}{\operatorname{var}[R_f]}\frac{(\beta[F,S](E[R_s - r]) - E[R_f])}{((E[R_s - r]) - \beta[S,F]E[R_f])}$$

This result is independent of investor preferences because it does not depend on parameters from the expected utility function.

Proof of Proposition III: Applying Bellman's dynamic programming principle (e.g., Malliaris and Brock, 1982) to eq. (7) gives:

$$J[W_t, X(t)] = \max_{C_t, h_t}\{U[C_t] + E[J[W_{t+1}, X(t + 1)] \mid X(t)]\}$$

From eq. (8), there is a direct relationship between first derivatives

$$\frac{\partial}{\partial C_t}J[W_{t+1}, X(t + 1)] = -\frac{\partial J}{\partial W_{t+1}}(1 + R_s - hR_f)$$

$$\frac{\partial}{\partial h_t}J[W_{t+1}, X(t + 1)] = -\frac{\partial J}{\partial W_{t+1}}(W_t - C_t)R_f$$

Suppressing the conditioning notation, the first order conditions give

$$C_t\colon \frac{\partial U}{\partial C_t} - E[J_W(1 + R_s - hR_f)] = 0$$

$$h_t\colon E[J_WR_f] = 0 \qquad \text{(A2)}$$

Recalling eq. (A1), from eq. (A2) gives

$$\operatorname{cov}[J_W, R_f] + E[J_W]E[R_f] = 0$$

Using the intertemporal form of the Stein–Rubinstein lemma produces

$$\operatorname{cov}[J_W, R_f] = (W_t - C_t)E[J_{WW}]\operatorname{cov}[1 + R_s - hR_f, R_f] + (E[J_{Ws}]\operatorname{cov}[R_s, R_f]) + (E[J_{Wf}]\operatorname{var}[R_f])$$

Evaluating the covariances, substituting and manipulating gives

$$h_t^* = (1+\gamma)\frac{\text{cov}[R_s, R_f]}{\text{var}[R_f]} + \frac{E[J_W]}{E[J_{WW}]^*}\frac{E[R_f]}{\text{var}[R_f]} + \frac{E[J_{Wf}]}{E[J_{WW}]^*} \tag{A3}$$

where $\gamma = E[J_{Ws}]/E[J_{WW}]^*$ and $E[J_{WW}]^* = (W_t - C_t)E[J_{WW}]$. Eq. (A3) is the result given in Proposition III.

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