Optimal Futures Spread Positions

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Unlike the analysis of hedging behavior, theoretical analysis of spread behavior has been somewhat limited. With a few notable exceptions, study of spread trading has concentrated on empirical examination of the profitability of specific trades (e.g., Easterwood and Senchack (1986); Monroe and Cohn (1986); Rentzler (1986)). The objective here is to derive the discrete-time optimality conditions for a spread trader maximizing an expected utility function defined over the mean and variance of spread profit. The optimality conditions are found to depend on both statistical parameters and the spreader's propensity towards risk. A number of other useful corollaries are derived and the empirical implications of the optimal spread discussed. Most significantly, it is demonstrated that the optimal 'hedge ratio' is independent of the spreader's attitude towards risk.

I. INTRODUCTION

In practice, spread trading in futures markets involves a wide range of possible strategies. Generally, spread trades are either intra-commodity or inter-commodity involving either intra-exchange or inter-exchange transactions. More sophisticated spread trades (e.g., turtles) combine spread positions in one commodity with long or short positions in another commodity. Some spreads are risky while others are done as arbitrages.

The optimality conditions developed in this paper apply only to a subset of possible spread trades. Because the optimization problem assumes mean-variance expected utility on the part of the trader, the analysis is *not* applicable to arbitrage-motivated spread trades. In addition, while the results can be extended to more sophisticated trades, trades involving long or short positions in one commodity against spreads in another commodity are not directly considered. Finally, the results derived here only provide partial equilibrium information in the form of optimality conditions for spread traders. The interaction of all relevant participants in the futures market would have to be modelled to derive market equilibrium results.

The textbook approach to both hedging and spreading assumes one-to-one positions. More sophisticated approaches to hedging such as Baesel and Grant (1982); Danthine (1978); Feder, et al. (1980); Ho (1984); Holthausen (1979); McKinnon (1967); Rolfo (1980); Rolfo and Sosin (1981); Shutz (1984); and Stoll (1979) have sought to derive the appropriate sizes for the cash and futures positions using expected utility maximization procedures. The expected utility approach clarified disagreements in the earlier hedging literature which centered around whether hedgers were minimizers of price level risk or

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maximizers of expected profit. The expected utility approach has been extended to spreading by Schrock (1971) and Peterson (1977) who explain the holding of futures price spreads (fps) in a mean-variance framework. Schrock "establishes conditions leading to the holding of a particular future (asset) with a negative rate of return." He demonstrated that holding the losing leg of the spread is "rational when the holding of the particular future on which the investor expects to suffer a loss sufficiently reduces the risk associated with his overall position in the market" (p. 270). Hence, Schrock establishes that futures spreading is a risk management strategy.

Peterson extends Schrock's analysis to more accurately account for institutional features of futures markets. By 'correctly' incorporating the differential margin requirements between fps and open positions, Peterson extends Schrock's results to find: ". . . in a world of positive interest rates, the primary consideration which motivates investors to hold commodity futures straddles (spreads) is the fact that margin requirements on futures held in straddles are substantially lower than margin requirements on futures held in outright long or short positions." However, while Peterson's analysis is technically correct, the empirical importance of this result is debatable. Specifically, spreading strategies are important because they expand the *feasible* set of investment opportunities beyond the set defined by outright short and long positions. Further, the actual opportunity cost of margin is not as substantial as Peterson depicts. Eligible collateral for margin includes a wide range of interest bearing assets. In the case of clearinghouse members, margin is assessed on a net basis.

Regarding the arguments in the *EU* function, both Schrock and Peterson base their analysis on the rate of return of holding futures positions (subject to the constraint that the sum of the futures positions and cash equals one). The approach here differs by specifying profit as the relevant argument in the utility function.[1] The form of the profit function will vary depending on the activity being examined. Regarding the form of the *EU* function, invariably, if a specific functional form is selected, the underlying utility function is from the HARA (hyperbolic absolute risk aversion) class which exhibit a number of desirable properties as has been shown by Cass and Stiglitz (1975), among others. In the HARA class of functions, the mean-variance expected utility function is often selected because it is flexible enough to provide a close approximation to the results obtainable for a wide range of expected utility functions (e.g., Kroll, Levy, and Markowitz (1984)). In addition, much of the literature on optimal hedging and spreading is based on the mean-variance approach, providing benchmark results. The mean-variance approach will be used here.

II. THE MODEL

A. Assumptions

(A1) The term structure of futures prices is at full carry with no discontinuities. This restricts the direct scope of applicable commodities to financials, currencies, and gold. (For example, agricultural commodities are omitted because the harvest cycle introduces

[1]Theoretically, by restricting the relationship between profit and the background good or consumption bundle, this imposes homothetic separability on the utility function (Aivazian, et al. (1983)). A more formally complete specification of the investor's utility function would be with consumption as the argument, i.e., $V[C]$ where C is consumption. It is here assumed that C is a separable function of profit and other variables, denoted by z. Hence, $U[p] = U[V[C[p;z]]]$ which restricts the interaction between profit and the other determinants of consumption.

a discontinuity.) (A2) Position liquidation takes place prior to the maturity (delivery) date of the nearby contract. (A3) The profit function is linear (contra Turnovsky (1983). (A4) Transactions and variation margin costs are zero. (Elimination of these costs means that futures and forward contract results are equivalent.) (A5) The agent's maximization problem is:[2]

$$\max_{Q} EU = E[p;Q] - b \operatorname{var}[p;Q] = E[p;Q \mid I(0)] - b \operatorname{var}[p;Q \mid I(0)] \quad (1)$$

where: EU is expected utility, p is profit, b is the risk parameter, $E[p]$ is expected profit and var$[p]$ is the variance of profit, Q are the choice variables and $I(0)$ is the information set available at time 0.

The objective function defined in (1) implicitly imposes restrictions on the probability distribution of the random variables contained in the profit function (e.g., the random variables cannot be Cauchy or nonnormal Paretian). In addition, initial wealth is not directly specified. In this regard, it is implied that initial wealth is sufficient for the attainment of all finite optima.

While there is considerable debate over the most appropriate distributional hypothesis for financial variables (e.g., Poitras (1986)), in order to establish the basic results the following assumption is invoked: (A6) The stochastic processes generating both the spot and futures prices are Gaussian.

The assumption of normality is convenient because the distribution is fully defined by the first two moments (reinforcing (A5)) and Gaussian variables are linearly additive (reinforcing (A3)). Further, (A6) provides for a straight forward interpretation of the results contained in Proposition 2.

B. An Optimal Hedging Example

The procedures used in deriving the optimal spread result can be illustrated by an example adapted from the optimal hedging literature. Because hedging involves introducing considerations about the cash commodity, two additional simplifying assumptions are needed: (A7) At maturity, cash and maturing futures prices are equal:

$$F(T,T) = S(T)$$

where: $F(T,T)$ is the futures price at delivery time T, $S(T)$ is the spot price at time T. In other words, (A7) assumes that the maturity basis is zero. (A8) Cross hedging costs are zero.

In general, no restrictions need to be placed on short-selling the cash commodity, i.e., the cash commodity can be held either short or long. However, in specifying Proposition 2 the position in the cash commodity has been constrained to be positive in order to be consistent with the actual execution of cash and carry hedges. In addition, because the separation result holds for all expected utility functions obeying the usual regularity conditions, the maximization problem (A5) can be appropriately relaxed. The results for this case are summarized in the following.

Proposition 1: Optimal Cash and Carry Inventory Hedge

Assuming (A1)–(A8) and $Q_s > 0$, the cash and carry hedger's maximization problem can be stated:

$$\max_{Q_s, Q_f} E[U[(Q_s - Q_f)S(T) + F(0,T)Q_f - S(0)(1 + cc(0,T))Q_s] \mid I(0)] \quad (2)$$

[2]In defining expectations and variances, the conditioning information set is often suppressed for ease of notation, i.e., throughout, all expectations and the variances are conditional parameters where the conditioning information is the information set available at time $t = 0$.

where: Q_s is the quantity of the spot commodity, Q_f is the size of the futures position, $cc(0,T)$ is the opportunity cost of carrying the commodity from 0 to T, and $I(0)$ is the information set available at time 0.

It follows that the hedger's optimality condition is:

$$F(0,T) - S(0)(1 + cc(0,T)) = 0 \qquad (3)$$

Proof: See Appendix.[3]

This result involves the separation of the current hedging decision from expectations about future events (Danthine (1978); Feder, et al. (1980); Holthausen (1979)). However, (3) should not be interpreted too broadly. For example, while the hedging decision is independent of changes in the expectation of future spot prices, it is not independent of the future spot price expectations of futures speculators which affect the determination of $F(0,T)$ and, to a lesser extent, of spot speculators whose expectations affect $S(0)$. In other words, because (3) is a decidedly *partial equilibrium* result, the activities of other cash and futures market participants must be modelled to derive the market equilibrium conditions. Furthermore, if it is assumed that the hedge is 'put on' with a nonmaturing futures contract (i.e., maturity time is beyond period one) then the optimality condition no longer yields a separation result. The result also changes if financing costs for the hedge period are not certain.

III. OPTIMAL SPREADS

The analysis is somewhat different in the case of spreads. For example, the hedging related problems of a declining cash-futures basis and convenience yield on the cash commodity are of no importance in spreading decisions. (A7) and (A8) are, therefore, no longer required. In examining the spreading decision, initially, it is necessary to introduce a sign convention regarding long and short positions. While a number of different sign conventions are possible, the notation adopted here is that for both Q_T and Q_n long positions take positive values and short positions take negative values. For example, Q_T (the position in the deferred contract) would be negative and Q_n (the position in the nearby contract) would be positive for a short-long spread and vice-versa for a long-short spread. A short-long spread is defined to be short the *deferred* and long the *nearby*, i.e., the deferred position is given first. With this convention, the possibility of Q_n and Q_T of the same sign (e.g., long the nearby and long the deferred) could occur in cases such as inter-commodity spreads where the prices of the commodities are negatively correlated.

For a spread with position sizes not predetermined $E[p(1)]$ and $\text{var}[p(1)]$ can be specified:

$$E[p(1)] = Q_n(E[F(1,N)] - F(0,N)) - Q_T(E[F(1,T)] - F(0,T)) \qquad (4)$$

$$\text{var}[p(1)] = Q_T^2 \text{var}[T] + Q_n^2 \text{var}[N] - 2Q_n Q_T \text{cov}[N,T] \qquad (5)$$

where: $\text{var}[N]$ is the variance of the price of the nearby contract, $\text{var}[T]$ is the variance of the price of the deferred contract, $\text{cov}[N,T]$ is the covariance of the prices of the deferred and the nearby contracts, Q_n is the size of the position in the nearby contract, Q_T is the size of the position in the deferred contract. The optimal spread result now follows from invoking (A1)–(A6).

[3]To provide a frame of reference for this result, observe that (3) is the same result as (4) in (Danthine (1978)) with $S(0)(1 + cc) = 1$ and Danthine's marginal product = 1.

Proposition 2: Closed Form Optimal Spread Positions

Assuming (A1)–(A6) with mean and variance of the objective function defined for (4) and (5), the spreader's optimality condition can be stated:

$$-Q_n^* = (A(N) - B(N,T)A(T))/(2b \text{ var}[N](1 - r^2)) \quad (6)$$

$$Q_T^* = (A(T) - B(T,N)A(N))/(2b \text{ var}[T](1 - r^2)) \quad (7)$$

where: $A(T) = F(0,T) - E[F(1,T)]$

$A(N) = F(0,N) - E[F(0,N)]$

$B(T,N) = \text{cov}[T,N]/\text{var}(N)$

$B(N,T) = \text{cov}[T,N]/\text{var}(T)$

$r^2 = \text{cov}[N,T]^2/(\text{var}[N] \text{ var}[T])$

*represents an optimum value

Proof: See Appendix.

Less formally, the spread trader maximizing expected utility of profit, with a mean-variance *EU* function, will establish a spread position only when the change in the profitable leg of the spread over the 'risk reducing' leg is expected to be greater than the *B*-predicted change. In other words, spread trading is done only to exploit abnormal deviations from typical *B*-relationships between relative price changes. The size of the respective legs of the spread are adjusted inversely to the ratio of the variances to reflect the typically larger price change associated with one leg of the spread.

The assumption of normality allows $B(N,T)$ to be interpreted as the regression coefficient in a (bivariate normal) regression of the price of the nearby contract on the price of the distant contract. If $B(N,T) < 1$ then, for a given change in the price of the distant contract, the (conditional) expectation of the change in the nearby contract will be smaller than that observed for the distant contract. The result is symmetric for $B(N,T) > 1$. Hence, for the nearby contract to be held in a short-long spread the B-adjusted expected price change for the distant or profitable contract must exceed that for the 'risk-reducing' nearby contract. This recognizes that in setting the optimal spread positions the B-relationship between the prices of the contracts in the spread is taken into account. In other words, the *EU* maximizing spread is only feasible when the difference between the legs of the contracts are expected to be larger than 'normal'.

The formal implications of Proposition 2 can be illustrated by considering two polar cases. One such case occurs when the commodities making up the legs of the spread are unrelated. In this case, the following result applies:

Corollary 1(A): Unrelated Commodities

If $r = 0$ (and hence $B = 0$), there is no benefit in spreading and the optimality conditions for the optimal spread reduce to the open position solution.

Another case applies to intra-commodity spreads.[4] By assumption (A1), intra-commodity futures prices are determined by carry costs. If these carry costs are constant through time then the (intra-commodity) term structure of futures prices is predetermined and all variances and covariances are proportionately related. This leads to the following.

[4]Theoretically, Corollaries 1(B)(i) and (ii) could be extended to any two commodities whose prices movements are either perfectly or nearly perfectly correlated, either negatively or positively.

Corollary 1(B): Degenerate Carry Cost Cases[5]

For an intra-commodity spread: (i) if *ic* is a degenerate random variable, then the optimal spread is indeterminate, i.e., the optimal spread cannot be determined because the price movements for each spread leg are perfectly correlated. (ii) if *ic* is a "nearly degenerate" random variable, then the sign of the optimal spread positions depends solely on the sign of the expected change in *ic*.

Where: *ic* is the net implied carry reflected in the term structure of futures prices.

Proof: Appendix.

Corollary 1(B)(i) is a limiting case. For practical purposes, Corollary 1(B)(ii) implies that the higher the predictability of the (intra-commodity) carry cost relationship, (i.e., the closer that r^2 approaches 1) the more sensitive will spreading behavior be to small expected deviations from normal carry relationships. This is significant because it indirectly establishes (for the set of commodities covered under (A1)) that the practical usefulness of intra-commodity spreads arises from the ability to speculate on changes in implied carry costs. While not surprising to actual spread traders, this interpretation of spread trading differs substantively from that advanced by Schrock where spreading is viewed as a risk reduction strategy. (In practice, spread trading may still be feasible as a risk reduction strategy for traders facing initial wealth constraints.)

In terms of practical implementability, the results obtained in Proposition 2 are quite general. For example, the results could be applied to intermarket spreading (eg., TBills/TBonds, Euros/TBills). In this case Q_n and Q_T would be positions in different commodities. Further, with suitable redefinition of Q_n and Q_T the analysis can be extended to include more than two contracts (eg., butterflies). Actual implementation requires information on the relative sizes of Q_n^* and Q_T^*, i.e., on the hedge ratio.[6] In this regard, Proposition 2 yields a hedge ratio which is both somewhat surprising and empirically based. In other words, the hedge ratio which falls out of Proposition 2 is independent of the spreader's attitude towards risk:

Corollary 2: Quasi-Separation of the Hedge Ratio[7]

The relative sizes of the legs of the spread depend solely on expectations and statistical parameters and are not affected by risk attitudes. Specifically, the hedge ratio is given by:

$$\frac{Q_T^*}{-Q_n^*} = \frac{\mathrm{var}[N](A(T) - B(T,N)A(N))}{\mathrm{var}[T](A(N) - B(N,T)A(T))}$$

Proof: Follows directly from Proposition 2.

This quasi-separation result is significant if risk attitudes do not affect the relative sizes of the positions. This allows the hedge ratio to be solely defined by potentially observable statistical parameters (the variances and B's) and the spreader's expectations (the A's). (In turn, given the B's and variances then r will be determined.) The resulting set of variance, B, A, r and hedge ratio combinations is considerable. Table I illustrates a subset of the possible spread combinations for the nearly degenerate carry cost case. While not immediately obvious, the hedge ratios defined in the last two columns of Table I have a direct connection to the accepted methods of determining intra-commodity

[5]The nearly degenerate intra-commodity spread could be more practically described as a spread where the variability of carry costs is small relative to spot price variability.

[6]The term hedge ratio is carried over from the optimal hedging literature. Futures traders often refer to the hedge ratio as the spread tail.

[7]This result is a quasi-separation result because it is only a partial equilibrium condition.

Table I
OPTIMAL SPREAD POSITIONS: NEARLY DEGENERATE CASE*
ASSUMING: $r \to 1$ $B(T,N) \to (1 + ic)$ $B(N,T) \to 1/(1 + ic)$
$\text{var}[T] \to (1 + ic)^2 \text{ var}[N]$ $A(T), A(N) > 0$

	$ic = 0$	$ic > 0$	$ic < 0$
$A(T) - (1 + ic)A(N) > 0$	$Q_T = Q_n$ short-long	$Q_T < Q_n$ short-long	$Q_T > Q_n$ short-long
$A(T) - (1 + ic)A(N) < 0$	$Q_T = Q_n$ long-short	$Q_T < Q_n$ long-short	$Q_T > Q_n$ long-short

*The $ic < 0$ case would apply to cases where the carry return exceeded the carry cost, e.g., for TBill futures where the TBill rate exceeded the repo rate.

hedge ratios (e.g., Monroe and Cohn (1986), Rentzler (1986), Jones (1981)). To see this heuristically, consider the nearly degenerate carry cost case. In this case, $\text{var}[T] \to (1 + ic)^2 \text{ var}[N]$ and $\text{cov}[N,T] \to (1 + ic) \text{ var}[N]$. Substituting these results into the hedge ratio formula and noting that $A(T)$ differs from $(1 + ic)A(N)$ by the assumed small change in carry costs, gives the conventional result that the $Q_n^* = (1 + ic)Q_T^*$. In other words, the optimal intra-commodity hedge ratio is determined by the cost of carry.

To illustrate this convention and to provide a benchmark for the optimal spread result, consider the case of a one-to-one spread, i.e., where the size of spread legs have been constrained to be equal.[8] In this case, the following result emerges.

Proposition 3: Constrained One-to-One Optimal Spread

Assuming (A1)–(A6), the maximization problem for a spreader constrained to take one-to-one positions is:

$$\max_Q EU[p;Q] = Q((F(0,T) - E[F(1,T)]) - (F(0,N) - E[F(1,N)])) - Q^2 b(\text{var}[N] + \text{var}[T] - 2\text{ cov}[N,T])$$

It follows that:

$$Q = Q_n^* = -Q_T^* = \frac{F(0,T) - F(0,N) - (E[F(0,T)] - E[F(0,N)])}{2b(\text{var}[N] + \text{var}[T] - 2\text{ cov}[N,T])} \tag{8}$$

Proof: Appendix.

This result is a straight-forward extension of the mean-variance optimality condition for a speculator using open positions. The size of the spread position depends on the (risk adjusted) expected change in the futures basis. When the futures basis is expected to narrow, $Q > 0$ and the speculator goes short the deferred position and long the nearby. When the basis is expected to widen, $Q < 0$, the speculator takes a negative spread position, i.e., a long deferred and short nearby position. The (conditional) variance of spread profit acts to increase or decrease the size of the position, i.e., $dQ^*/d \text{ var}[p(1)] < 0$. The variance does not affect the type of spread position (i.e., short or long). The spreader's risk attitudes enter through b. As risk neutrality is ap-

[8]In practice, if the spread legs are not one-to-one, this will alter the margin requirements and the transactions costs on those positions which are not matched by an offsetting position. Following Peterson, this could be an important reason to engage in (one-to-one) spreads.

proached, reflected in $b \to 0$, the supply of speculative funds becomes 'infinitely' elastic at the expected future basis.

Corollary 2 can also be used to establish the relationship between Propositions 2 and 3. Proposition 2 clearly demonstrates that one-to-one spreads are not, in general, optimal.[9] However, examining the correspondence between a one-to-one spread and the optimal spread is still revealing.

Corollary 3: Optimal One-to-one Spread

The optimal spread is both nontrivially one-to-one and independent of expectations if and only if $B(N,T) = B(T,N) = 1$. Proof: Appendix.

This is clearly a special case. In practice, for a carrying charge market, the more distant contract has a higher price and will exhibit a greater change in price for a given change in price for the nearby contract. For example, for gold and other precious metals, $B(T,N) > B(N,T)$. In addition, it is incorrect to conclude that as a special case of Proposition (2) that Proposition (3) can be used to approximate the results of Proposition (2). For example, there are instances where a short-long spread would be indicated based on an evaluation of $A(T)-A(N)$ and a long-short spread is indicated based on $A(T)-B(T,N)A(N)$. In other words, the two Propositions at times will indicate substantively different types of spread positions.

III. SUMMARY

In a discrete-time model, the comparative static conditions for a futures spread trader maximizing a mean-variance expected utility function were derived. The optimality conditions were found to depend on (subjective) statistical parameters and the spreader's attitude towards risk. Based on these conditions a number of results were derived. It was demonstrated that, for small variations in carry costs, intra-commodity spread strategies depend solely on the expected change in carry costs and not on the need to reduce risk as was previously thought. More significantly, it was also demonstrated that the optimal hedge ratio is independent of the spreader's attitude towards risk. Under certain conditions, the optimal hedge ratio is shown to be consistent with prevailing practices of spread traders. It was demonstrated also that the optimal hedge ratio will not usually be one-to-one.

Appendix

Proof of Proposition 1: The cash-and-carry hedger's maximization problem is defined as:

$$\max_{Q_s, Q_f} E[U[(Q_s - Q_f)S(1) + F(0,1)Q_f - S(0)(1 + cc(0,1))Q_s]\,|\,I(0)]$$

where U is differentiable on the finite range of Q_s and Q_f and the conditional expectation converges.

Given this, the EU function is differentiable and the first order conditions give (primes indicating derivatives):

$$E[U[p]'(F(0,1) - S(1)] = 0$$

[9]In the notation of Proposition 3, Proposition 2 can be expressed: $Q = Q_n(-Q_d) = (A(T) - A(N))/2b(\text{var}[p(1)])$. In addition, because margin and transactions costs are not directly incorporated into the decision problem, the result stated in Corollary 3 does not *directly* address the issue raised by Peterson.

$$E[U[\mathrm{p}]'(S(1) - S(0)(1 + cc(0,1))] = 0$$

Rearranging:

$$E[U[\mathrm{p}]'S(1)] = E[U[\mathrm{p}]']S(0)(1 + cc(0,1)) = E[U[\mathrm{p}]']F(0,1)$$

or:

$$F(0,1) - S(0)(1 + cc(0,1)) = 0 \qquad \text{QED}$$

Proof of Proposition 2: The optimal spread problem is defined as:

$$\max_{Q_n, Q_T} EU = Q_n(E[F(1,N)] - F(0,N)) + Q_T(F(0,T) - E[F(1,T)]) - b(Q_T^2 \operatorname{var}[T] + Q_n^2 \operatorname{var}[N] - 2Q_n Q_T \operatorname{cov}[N,T])$$

Solving the first order conditions gives:

$$Q_n^* = (-A(N)/(2b \operatorname{var}[N])) + Q_T(\operatorname{cov}[T,N]/\operatorname{var}[N])$$

$$Q_T^* = A(T)/(2b \operatorname{var}[T]) + Q_n(\operatorname{cov}[T,N]/\operatorname{var}[T])$$

Substituting Q_n^* and Q_T^* on the right hand side where appropriate and solving gives (6) and (7). QED

Proof of Corollary 1B(ii): To define the "nearly degenerate" case let:

$$F(1,T) = (1 + ic^* + e)F(1,N)$$

where: ic^* is the constant component of ic, e is an allowable 'small' amount of change in ic where e is normally distributed with:

$$E[e] \text{ not equal to zero}$$

For ease of exposition it will also be assumed that:

$$\operatorname{cov}[e, F(1,N)] = \operatorname{cov}[eF(1,N), F(1,N)] = 0$$

Given this the following results apply:

$$A(T) = (1 + ic^*)A(N) + E(e)E[F(1,N)] \qquad (\text{using } \operatorname{cov}[e, F(1,N)] = 0)$$

$$\operatorname{cov}[T,N] = (1 + ic^*) \operatorname{var}[N] + \operatorname{cov}[eF(1,N), F(1,N)]$$

$$\operatorname{var}[T] = (1 + ic^*)^2 \operatorname{var}[N] + \operatorname{var}[eF(1,N)] + 2(1 + ic^*) \operatorname{cov}[eF(1,N), F(1,N)]$$

$$B(N,T), B(T,N) \text{ and } r \text{ follow appropriately.}$$

Observing that position sign depends only on the numerator of (6) and (7), imposing the zero restrictions gives immediately that Q_T is proportional with $E[e]E[F(1,N)]$. Because prices are nonnegative, the sign of Q_T will depend solely on the expected change in carry costs. Q_n is somewhat more complicated. In this case, recall that "nearly degenerate" refers to small changes. Applying this condition as $\operatorname{var}[eF(1,N)] \to 0$, then it follows that Q_n is also proportional to $E[e]E[F(1,N)]$. QED

Proof of Proposition 3: The one-to-one spread problem is defined:

$$\max_{Q} EU[p;Q] = Q((F(0,T) - E[F(1,T)]) - (F(0,N) - E[F(1,N)])) - Q^2 b(\operatorname{var}[N] + \operatorname{var}[T] - 2\operatorname{cov}[N,T])$$

Assumptions (A5) and (A6) are sufficient for the problem to be regular. The first order condition gives:

$$\frac{dEU}{dQ} = (F(0,T) - E[F(1,T)]) - (F(0,N) - E[F(1,N)]))$$
$$- 2bQ^*(\text{var}[N] + \text{var}[T] - 2\,\text{cov}[N,T]) = 0$$

Solving for Q^* gives:

$$Q^* = \frac{F(0,T) - F(0,N) - (E[F(0,T)] - E[F(0,N)])}{2b(\text{var}[N] + \text{var}[T] - 2\,\text{cov}[N,T])}$$

Proof of Corollary 2: From Corollary 2 and the one-to-one assumption:

$$1 = \frac{\text{var}[N]A(T) - \text{cov}[T,N]A(N)}{\text{cov}[T,N]A(T) - \text{var}[T]A(N)}$$

Manipulating gives:

$$(A(T)/A(N))(\text{var}[N] - \text{cov}[T,N]) + (\text{var}[T] - \text{cov}[T,N]) = 0$$

To be independent of $A(T)/A(N)$ and nontrivial, the equality holds if and only if $\text{var}[N] = \text{var}[T] = \text{cov}[T,N]$. This is identical to $B(N,T) = B(T,N) = 1$. QED

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