9. Application and Extension of Option Valuation Techniques

9.1 Portfolio Management: Delta, Theta and Gamma[1]

Basic Definitions

One of the most useful applications of the Black-Scholes formula involves applying the partial derivatives of the formula to analyze and design portfolios containing derivative securities. For example, the VaR calculations in Sec. 2.2 for portfolios containing options exploited the **)** and **'** of the position to model the possible changes in position value. The presence of an option pricing formula permits partial derivatives such as **)** and **'** to be solved directly, instead of having to rely on numerical techniques. Correct evaluation of the partial derivatives permits theoretical portfolios to be precisely constructed to have desirable properties.

Delta, theta and gamma are names used to refer to the most commonly referenced partial derivatives. The partial derivatives are also referred to as ***Greeks*** after the symbols used identify the derivatives. Applied to a call option, these three Greeks are defined as:[2]

$$\Delta_C = \frac{\partial C}{\partial S} \qquad \Gamma_C = \frac{\partial \Delta}{\partial S} \qquad \theta_C = -\frac{\partial C}{\partial t^*}$$

While delta, gamma and theta are typically the most commonly referenced partial derivatives, there are a sizable number of other partial derivatives that could also be of value for certain types of situations. For example:

$$\frac{\partial C}{\partial \sigma} \qquad \frac{\partial \theta}{\partial t} \qquad \frac{\partial^2 C}{\partial \sigma \ \partial S} \qquad \frac{\partial C}{\partial r}$$

This is only a partial list. ***From put-call parity***, similar concepts can be derived for puts:

$$\Delta_P = \frac{\partial P}{\partial S} = \frac{\partial C}{\partial S} - 1 = \Delta_C - 1$$

$$\Gamma_P = \frac{\partial \Delta_C}{\partial S} = \Gamma_C \qquad \theta_P = -\frac{\partial P}{\partial t^*} = -\{\frac{\partial C}{\partial t^*} - rX\, e^{-rt^*}\}$$

The other partial derivatives for puts follow appropriately.

To employ the derivative properties of Black-Scholes to design portfolios of securities requires recognizing that the various securities that can be included in a given portfolio, such as stocks, commodities, futures and options, all possess derivative properties. From linearity, this permits the delta, theta, gamma, and other Greeks of a portfolio to be calculated. More precisely, let V be the dollar value of the portfolio, V_i represent the price of security i and n_i be the number of units of security i held where $n < 0$ indicates a short (written) position and $n > 0$ a long (purchased) position. In general, for portfolios containing a large number of securities:

$$V = n_1 V_1 + n_2 V_2 + n_3 V_3 + \ldots\ldots$$

Taking derivatives of V now permits the derivation of **)**$_V$, **'**$_V$ and the other Greeks. For example, in the ***two security case***, the delta of the portfolio can be derived:

$$\Delta_V = \frac{\partial V}{\partial S} = n_1 \frac{\partial V_1}{\partial S} + n_2 \frac{\partial V_2}{\partial S} = n_1 \Delta_1 + n_2 \Delta_2$$

This Section is primarily concerned with stating the exact analytical values for the Greeks applicable to calls and puts. Sections 9.2 and 9.3 are concerned with demonstrating how these concepts can be used in portfolio design.

The results in this Section are derived by evaluating partial derivatives of the Black-Scholes formula. As a consequence, these derivatives apply only to options on ***non-dividend paying*** securities or commodities. In cases where the assumptions are similar to the Black-Scholes case, e.g., log-normal state variables and European options, evaluating the Greeks for other types of options, such as European currency options, follow the same procedure and have similar, though not identical expressions. In cases where the assumptions are substantively different, e.g., for American options on dividend paying stocks, the Black-Scholes derivatives may not be applicable. In cases where closed form prices are not available, numerical techniques are required to evaluate the derivatives. This highlights another reason why having closed forms for option prices is important. Closed forms can be differentiated to arrive at expressions for the Greeks. Where the option price can only be solved numerically, the further evaluation of derivatives can present non-trivial numerical complications.

Delta

One application of $)_C$, the sensitivity of the call price to changes in the stock price, has already been encountered in the specification of the riskless hedge portfolio that was the basis of the derivation for the Black-Scholes formula. More precisely, after some effort and manipulation the derivative can be evaluated as (see Appendix III):

$$\Delta_C = \frac{\partial C}{\partial S} = N[d_1] > 0$$

Diagrammatically, the delta of a call measures the slope of *C[@]* given in Figures 8.1 and 8.2 of Sec. 8.1. In addition to providing a precise mathematical specification of the slope, evaluating the sensitivity of delta as different variables change also provides important information. For example, the relationship between delta and *S* is depicted in Figure 9.1. This Figure indicates that the price of deep-out-of the money calls will be unresponsive to stock price changes while deep-in-the money calls will move one-to-one with stock price. An at the money call has a delta of approximately 1/2 when *t** is small. This value for $)_C$ can be seen by evaluating d_1 when $S = X$. In this case, $ln\{S/E\} = ln\{1\} = ln\ \{exp\{0\}\} = 0$:

$$d_1 = \frac{\ln\{1\} + (r + \frac{1}{2}\sigma^2)t^*}{\sigma\sqrt{t^*}} = \frac{(r + \frac{1}{2}\sigma^2)t^*}{\sigma\sqrt{t^*}}$$

Because the remaining part of d_1 is small when *t** is small, an approximation of $N[0] = .5$ is evaluated.

The delta for a put, $)_P$ follows from taking the partial derivative of the put option pricing formula:

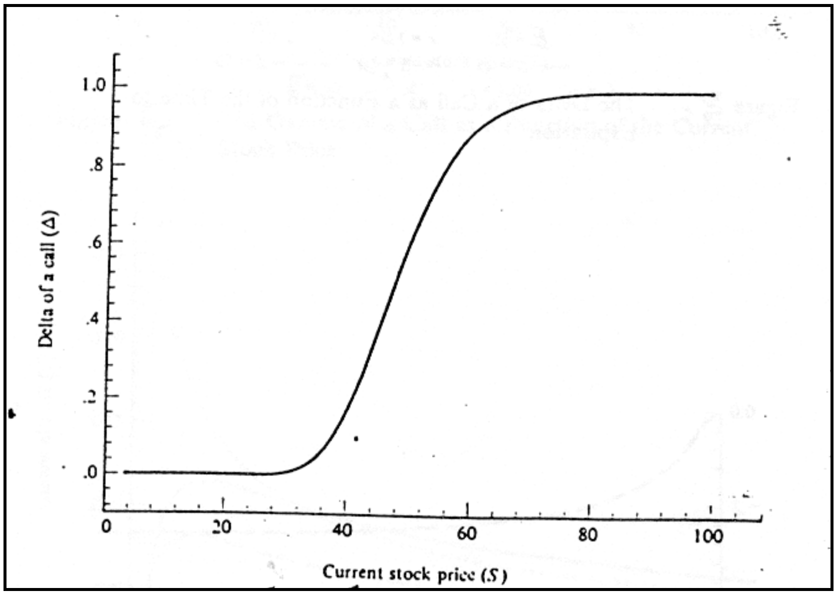
$$P = S\ (N[d_1] - 1) - X\ e^{-rt^*}\ (N[d_2] - 1)$$

$$= X\ e^{-rt^*}\ N[-d_2] - S\ N[-d_1]$$

Because all that has been done is to introduce 1, a constant, into the formula, the delta for a put follows appropriately:

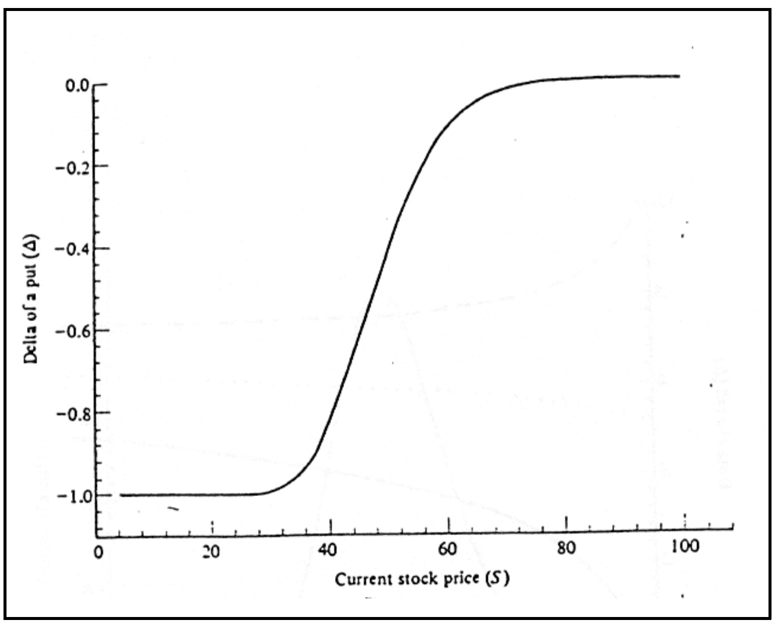
$$\Delta_P = \frac{\partial P}{\partial S} = N[d_1] - 1$$

Figure 9.1 **The Delta of a Call as a Function of the Stock Price**



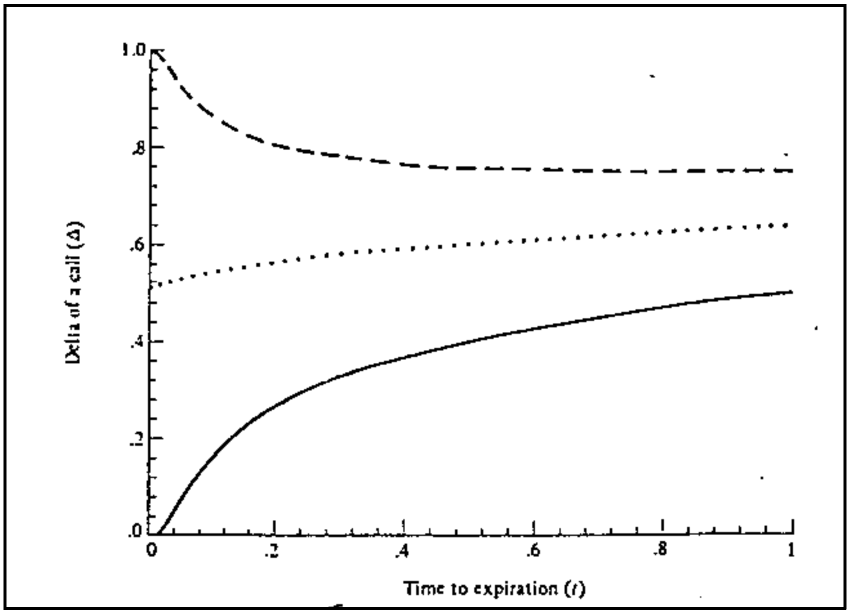
X = 50 *t* = .4* *r = .06* **F** = .3

Figure 9.2 **The Delta of a Put as a Function of the Stock Prices**



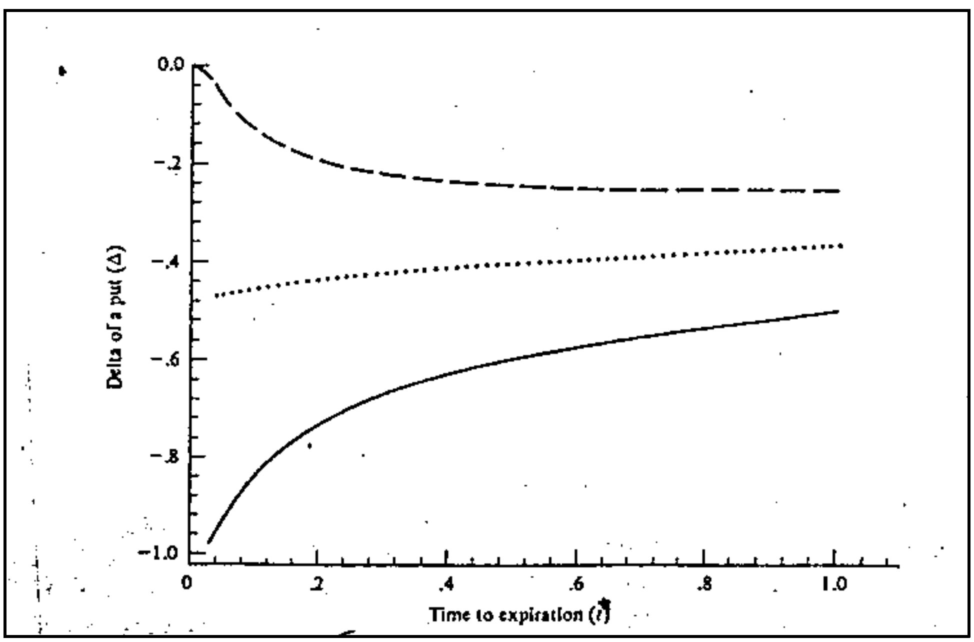
X = 50 *t* = .4* *r =.06* ***F*** = .3

Figure 9.3 **The Delta of a Call as a Function of the Time to Expiration**



X = 50 *r =* .06 **F** = .3 —— *S = 45* AAAAAAAS *= 50* – – – *S = 55*

Figure 9.4 **The Delta of a Put as a Function of the Time to Expiration**



X = 50 *r* = .06 **F** = .3 —— *S = 45* AAAAAAAS *= 50* – – – *S = 55*

Effectively, the put delta is the negative of the call delta, as illustrated in Figure 9.2. When Δ_C is 1 then Δ_P is zero and when Δ_C is zero then Δ_P is -1. Deep-in-the-money puts increase (decrease) by $1 when the stock price ***falls*** (rises) by $1. Similarly, deep-out-of-the-money puts are not sensitive to changes in the stock price.

As discussed in Sec. 8.2, one important practical implication of the delta arises in the specification of a riskless hedge portfolio: the inverse of delta gives the number of written (purchased) call options required to hedge a long (short) stock position. Recalling that the properties of both probability densities and call options require that $N[d_1] \leq 1$, the implication of Figure 9.1 is that, unless the option is deep-in-the-money, call prices will change less than $1 if the stock price changes by $1. When the stock price changes, the delta changes, therefore the riskless hedge must be ***rebalanced*** in order to maintain the hedge. A similar comment applies to changes in time to expiration. Consider the number of written ***at-the-money*** options required to form a riskless hedge for different times to expiration:

$r = .06 \quad \sigma = .3 \quad X = S$

Time to Expiration, t*	d_1	$N[d_1]$	$N[d_1]^{-1}$ = # of Options
5 years	.783	.5283	1.893
1 year	.350	.5137	1.947
6 months	.248	.5098	1.962
3 months	.175	.5069	1.973
1 month	.101	.5044	1.983

Even for at-the-money options, *ceteris parabus* rebalancing with respect to changes in time is not overly important.

Given that the delta for a put is the call delta minus 1, Figure 9.2, $\Delta_P < 0$ and a purchased put will fall (rise) in value as the stock increases (decreases). Even though the riskless hedge portfolio was derived with written call options, Δ_P could also be used to establish the number of puts that must be **purchased** in order to create a riskless hedge portfolio combining puts and stock. Because the resulting hedge ratio for a long stock position will be positive, this implies that puts will be purchased, not written as in the case of calls. As with calls, to maintain the hedge, the position will have to be rebalanced as the stock price changes in order to maintain the delta of the portfolio equal to zero. The relationship between delta and t^* is given in Figures 9.3 (call) and 9.4 (put). Observing that for $\tau = 0$ the option has expired, it is apparent that the closer the option gets to expiration, the greater the time sensitivity of delta to whether the option is in-at-or-out of the money. In addition to deltas for puts and calls, the delta for the relevant spot commodity, e.g., a stock, or for a futures contract can be seen to be 1.[3]

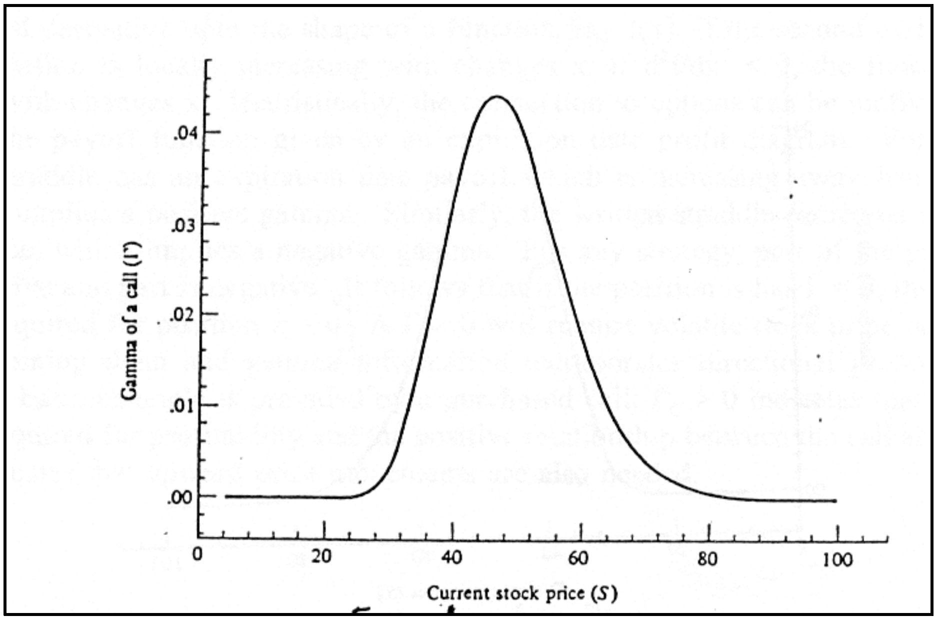
Gamma

The gamma of the position measures the sensitivity of the delta to changes in stock prices:

$$\Gamma_C = \frac{\partial \Delta_C}{\partial S} = \frac{\partial^2 C}{\partial S^2} = \frac{1}{S\ \sigma \sqrt{t^*}} N'[d_1] = \frac{1}{S\ \sigma \sqrt{t^*}}\ n[d_1]$$

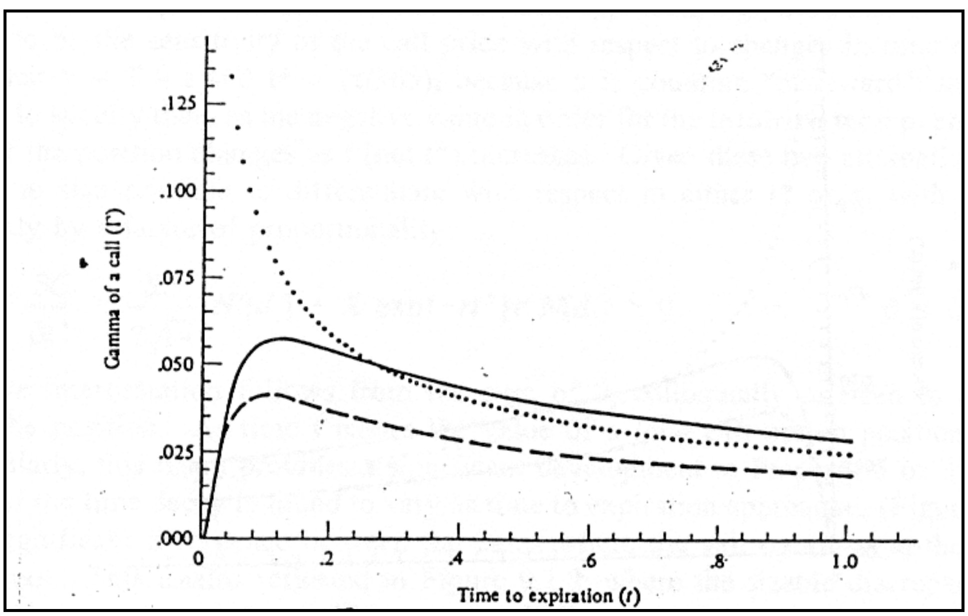
$$where: \qquad N'[d_1] \equiv \frac{1}{\sqrt{2\ \pi}}\ \exp\{\frac{-d_1^2}{2}\} \equiv n[d_1]$$

Figure 9.5 **The Gamma of a Call as a Function of the Stock Price**



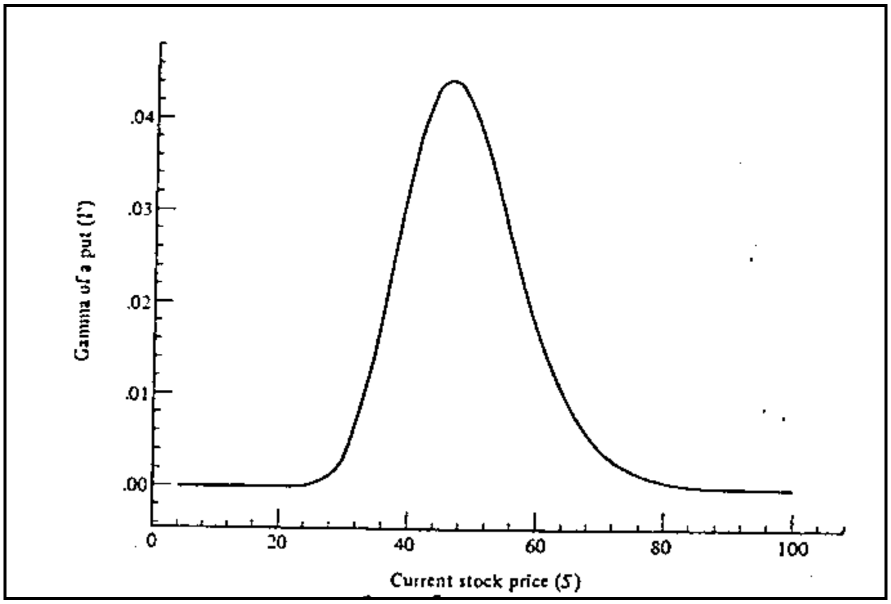
X = 50 *t* = .4* *r = .06* **F** = .3

Figure 9.6 **The Gamma of a Call as a Function of the Time to Expiration**



X = 50 *r* = .06 **F** = .3 —— *S = 45* *AAAAAAAS = 50* – – – *S = 55*

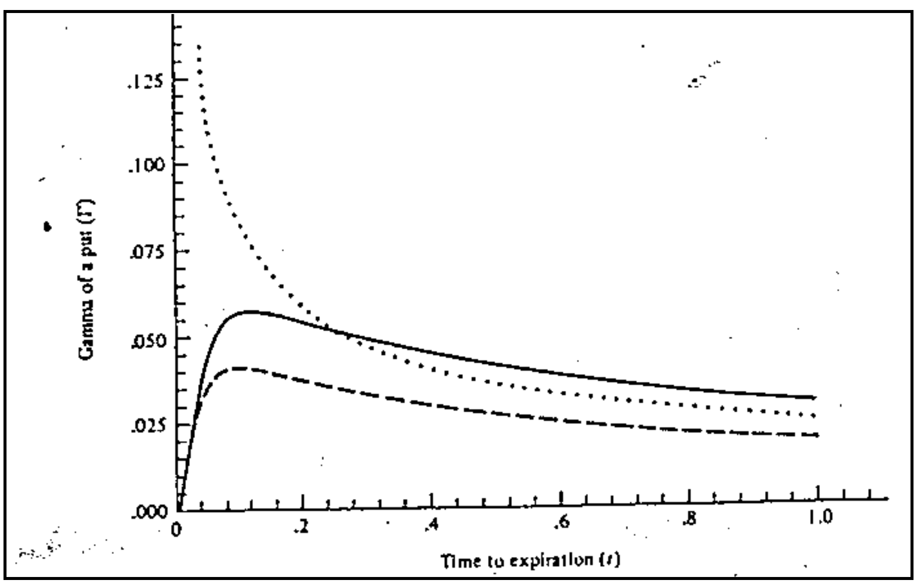
Figure 9.5A **The Gamma of a Put as a Function of the Stock Price**



X = 50 *t**

= .4 *r* = .06 *F* = .3

Figure 9.6A **The Gamma of a Put as a Function of the Time to Expiration**



X = 50 *r* = .06 **F** = .3 —— *S* = 45 ﾠ*S* = 50 – – – *S* = 55

This value is the same for both puts and calls; the stock (futures contract) has a gamma of zero. In effect, gamma reexpresses the information in Figures 9.1 and 9.2. Much as with delta, gamma can be plotted as a function of the stock price (Figure 9.5) and time to expiration (Figure 9.6). Gamma has two important practical features: size and sign. ***For cases such as the riskless hedge portfolio that require rebalancing in order to achieve* a *delta target***, the size of gamma determines how frequently the position has to be adjusted to maintain the hedge portfolio feature. "High" values indicate frequent adjustments are required, "low" values mean the position delta is relatively immune to stock price changes and rebalancing can be done infrequently. A ***gamma neutral*** (Γ = 0) position is one that the delta is 'locally' protected from changes in the stock price. For example, from Figure 9.5, long positions for deep in- and out-of-the-money options are found to be gamma neutral.

The size of gamma also has significance for positions that do not have to be rebalanced. Examples would be portfolios insured with purchased puts or option strategy positions such as straddles and vertical spreads. These positions are established with a specific delta and gamma that will provide information about the speed that delta changes as the stock price changes. To see this, consider a stock position insured with an at-the-money put. This position has a positive delta and, given that the put is at-the-money, a sizable gamma. As the stock price increases the delta on the stock-plus-put position will increase, because the delta of the put will become less negative, to the point where the put delta is zero and the stock-plus-put position delta approaches one and the gamma is zero. What gamma indicates is that the delta of the position will change the most when the option is at-the-money. As discussed in Section 8.2, positions with equal delta and gamma will experience the same local movement in delta but may have substantively different expiration date payoffs. In these cases, once the stock price has changed, the delta and gamma of the positions will not typically be equal.

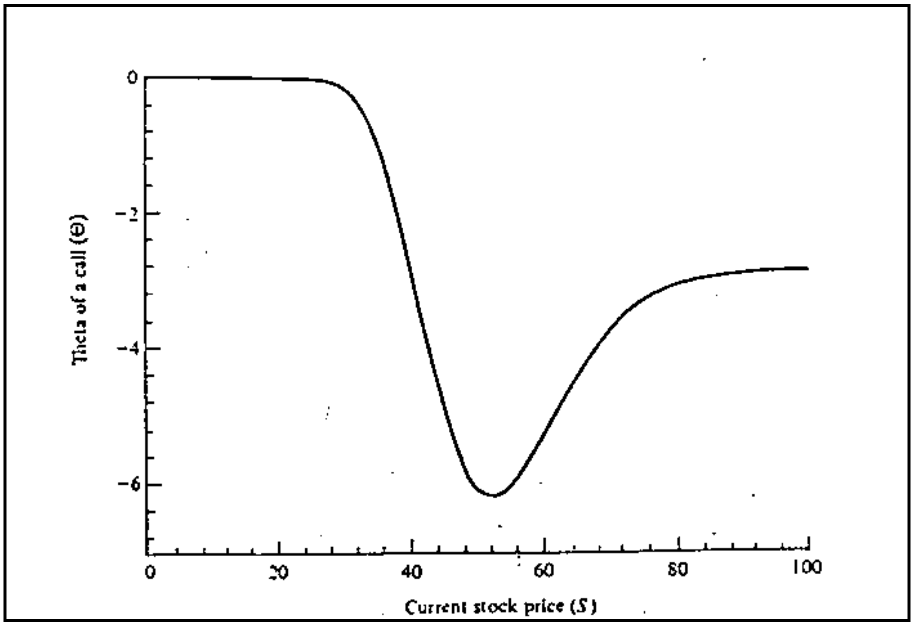
The other important practical feature of gamma is the sign. Basic calculus identifies the sign of the second derivative with the shape of a function, say $f(x)$. If the second derivative, $d^2 f / dx^2 > 0$, the function is locally increasing with changes in x; if $d^2 f / dx^2 < 0$, the function is locally decreasing with changes in x. Heuristically, the connection to options can be motivated by taking $f(x)$ to be the payoff function given by an expiration date profit diagram. For example, the purchased straddle has an expiration date payoff that is increasing away from the exercise price. This implies a positive gamma. Similarly, the written straddle decreases away from the exercise price, which implies a negative gamma. For any strategy, part of the payoff function will be positive and part is negative. It follows that ***if the position has $\Gamma < 0$, then stable stock prices are required for position $\pi > 0$***. A $\Gamma > 0$ will require volatile stock price behavior for $\pi > 0$. Combining delta and gamma information incorporates directional preference into the analysis. A basic example is provided by a purchased call: $\Gamma_C > 0$ indicates that volatile stock prices are required for profitability and the positive relationship between the call and stock prices ($\Delta > 0$) indicates that upward price movements are also needed.

Two distinct uses of gamma have been identified. For portfolios designed to achieve a delta target, such as delta neutrality, gamma provides analytical information on the rebalancing frequency. For portfolios aiming to achieve trading profits, such as straddles and vertical spreads, gamma indicates the degree of movement in stock prices required to achieve profitability. Given this, it is possible to combine these two uses for gamma to analyze portfolios where discrete rebalancing is being used to achieve a delta target. In practice, it is not possible to trade continuously. In general, discrete rebalancing will not be able to exactly achieve the delta target. Recognizing there will be some slippage, gamma can provide information about whether the slippage will tend to generate positive or negative profits.

Theta

The theta of the position measures the sensitivity of the option price to changes in time. Because "time" in options counts backwards, two different definitions of theta are encountered, depending how the impact of time is evaluated. One approach, e.g., Stoll and Whalley (p.228), takes theta to be the sensitivity of the call price with respect to changes in time to expiration. Recalling that $\tau = T - t$ and $t^* = (\tau/365)$, because τ is counting "backward", the alternative approach is to specify theta as the negative value in order for the ***intuitive interpretation*** of how the value of the position changes as t (not t^*) increases. Given these two alternative definitions of theta, it is also possible to differentiate with respect to either t^* or τ, with these results differing only by a factor of proportionality:

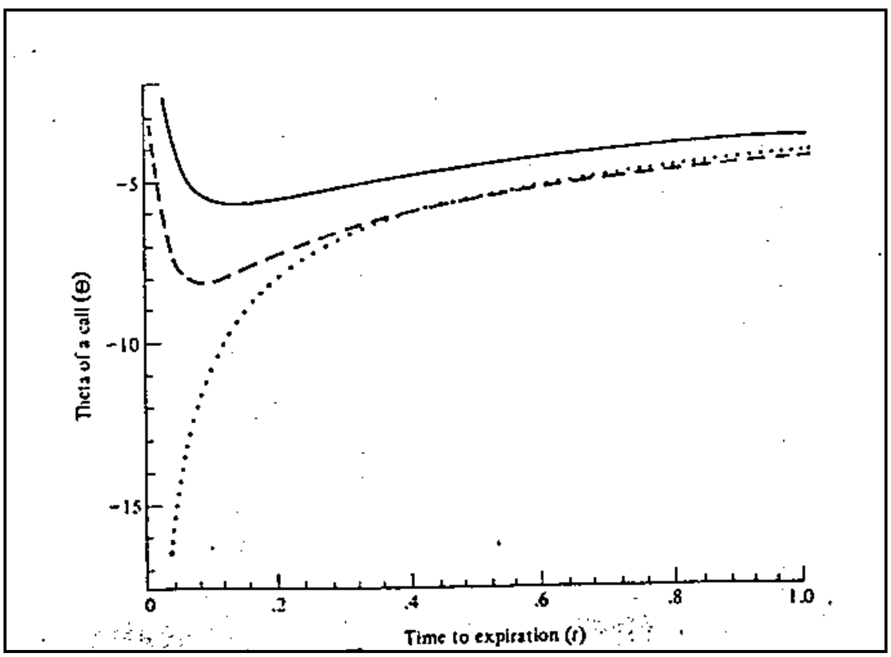
Figure 9.7 **The Theta of a Call as a Function of the Stock Price**



X = 50

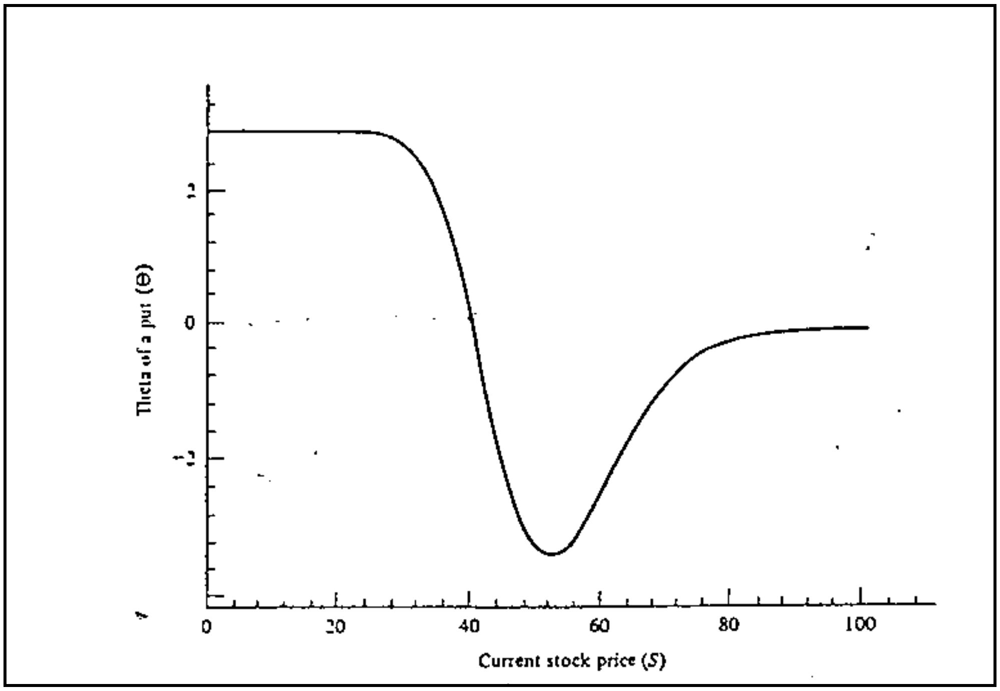
t = .4* *r = .06* ***F*** *= .3*

Figure 9.8 **The Theta of a Call as a Function of the Time to Expiration**



X = 50 *r* = .06 **F** = .3 —— *S = 45* AAAAAAAA*S = 50* – – – *S = 55*

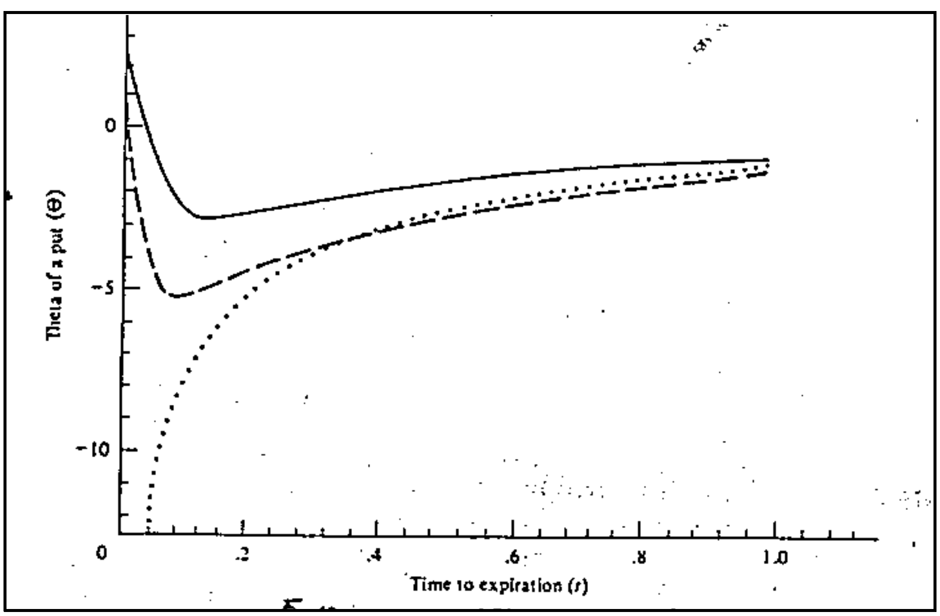
Figure 9.9 **The Theta of a Put as a Function of the Stock Price**



X = 50
t = .4*

r =.06 ***F**= .3*

Figure 9.10 **The Theta of a Put as a Function of the Time to Expiration**



X = 50 *r* = .06 **F** = .3 —— *S = 45* ········*S = 50* – – – *S = 55*

$$\frac{\partial C}{\partial t^*} = \frac{S\sigma}{2\sqrt{t*}} N'[d_1] + X \exp\{-rt^*\}r\ N[d_2] > 0 \qquad \Rightarrow \qquad \theta < 0$$

The intuitive interpretation follows from the sign of **2**, colloquially referred to as the "time decay" of the position. As time *t* passes the value of a long ***call*** option position will fall in value. Similarly, this result provides a significant development to Property 5 of Sec. 8.1. The sensitivity of the time decay is found to vary as time to expiration approaches (Figure 9.7). Again, there is a significant divergence between the cases where the call option is at-the-money and where it is not. This is also reflected in Figure 9.8, where the sizable discrepancy between deep-out-of the money and deep-in-the-money call options is apparent: the relationship is not symmetric.

A seemingly counterintuitive results applies to the theta of a European put. Recalling that the put values are determined from put-call parity:

$$\frac{\partial P}{\partial t^*} = \frac{\partial C}{\partial t^*} - X \exp\{-rt^*\}r$$

The value of this derivative can be either positive or negative. (This result only holds for European puts, American puts will be always positive.) This result is illustrated in Figures 9.9 and 9.10 that provide the relationship between the put theta and both stock prices and time to expiration. From Figure 9.9, the theta of the put being positive tends to occur when the stock price is low, implying the put is in-the-money. From Figure 9.10, in-the-money puts are required, though this condition is not restrictive in the case where the stock price approaches zero.

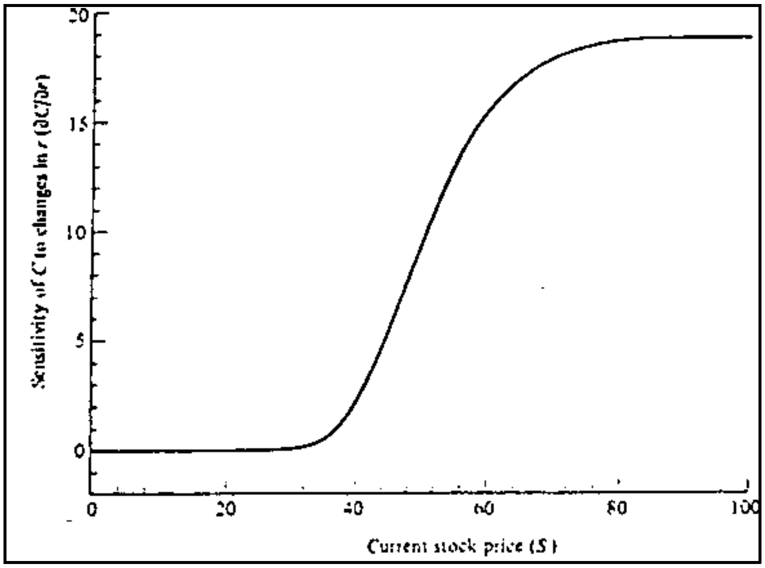
To see why this ambiguous result occurs, reformulate the put-call parity condition in the form: *S(t) + P[S,t*,X] = C[S,t*,X] + X exp{-rt*}*. Differentiating both sides by t* produces the condition stated for the European put theta. As *S* approaches zero, the theta of a call gets close to zero leaving the put theta to be determined by the last term on the rhs associated with the discounted value of the exercise price. In effect, when the stock price approaches zero, a longer time to maturity will have a negative impact on put price by lowering the cost of acquiring the bond investment needed to replicate the insured stock portfolio that, when the stock price is close to zero, is almost completely associated with the intrinsic value of the put. However, on balance, while positive ***theta*** values are possible, the theta of a put should typically be negative, just as with the call.

This ambiguous result for the European put can be further motivated by considering the theta for an American put. The possibility of positive theta for European puts is due to the inability to exercise the deep-in-the-money put early. Recalling the discussion of Properties 10 and 11 of Sec. 8.1, when the European put goes deep-in-the-money, the value of the put is bounded below by *Max[0, X exp{-rt*} - S(t)]*. However, in the case of the American put, the lower bound will be the exercise boundary *X - S(t)*. Hence, as *S* approaches zero and the put goes deep-in-the-money, the associated call is deep-out-of-the-money indicating that the call theta goes to zero. For the American put, the deep-in-the-money put price will fall to the exercise boundary *X - S(t)* and, combined with the result that the call theta is zero in this case, the American put theta is bounded below by zero.

Rho and Vega

While delta, gamma and theta are the most commonly considered features, the partial derivatives with respect to both sigma and r also receive attention. The relationship between option prices and volatility was initially referred to as the ***lambda*** but the convention is now to refer to the ***vega***. The is more uniformity with the partial derivative between option prices and interest rates that is referred to as the ***rho*** of the position. Evaluating the relevant values for calls and puts gives:

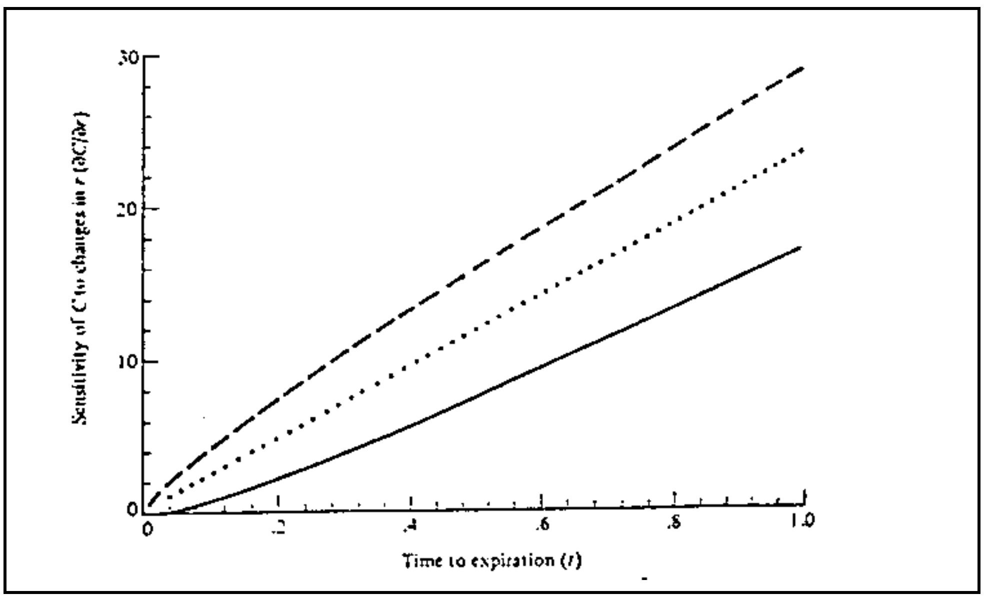
Figure 9.11 **The Sensitivity of *C* to Changes in *r* as a Function of the Stock Price**



X = 50 *t* = .4*

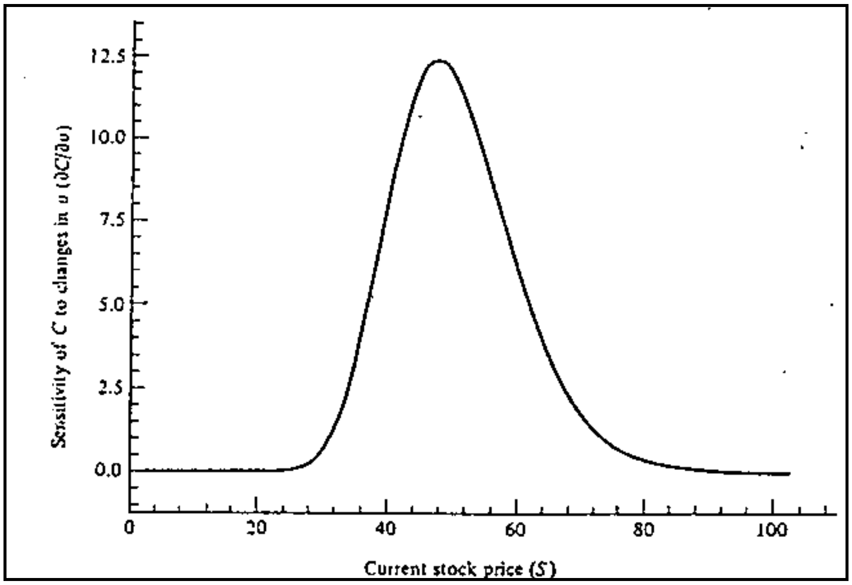
r =.06 ***F*** *= .3*

Figure 9.12 **The Sensitivity of *C* to Changes in *r* as a Function of the Time to Expiration**



X = 50 *r* = .06 **F** = .3 —— *S = 45* AAAAAAAA*S = 50* – – – *S = 55*

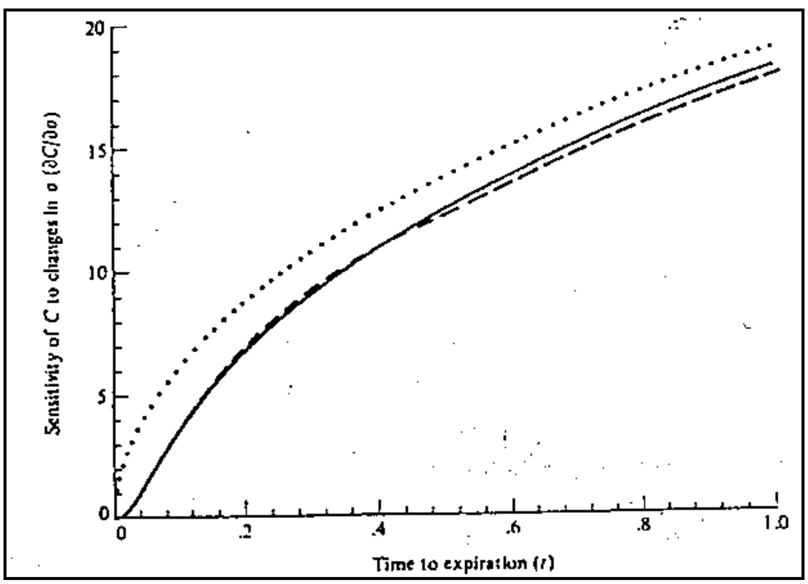
Figure 9.13 **The Sensitivity of *C* to Changes in σ as a Function of the Stock Price**



X = 50 *t** = .4

r =.06 σ = .3

Figure 9.14 **The Sensitivity of *C* to Changes in σ as a Function of the Time Expiration**



X = 50 *r* = .06

σ = .3 —— *S* = 45 ······ *S* = 50 – – – *S* = 55

$$\upsilon_C = \upsilon_P = \frac{\partial C}{\partial \sigma} = S\sqrt{t^*}\ N'[d_1] > 0$$

$$\rho_C = \frac{\partial C}{\partial r} = t^* X \exp\{-rt^*\}\ N[d_2] > 0$$

$$\rho_P = \frac{\partial P}{\partial r} = t^* X \exp\{-rt^*\}[N[d_2] - 1] < 0$$

The equivalence of vega for the put and call (with the same terms) is as expected, the more volatile is the underlying stock the higher will be the option price. As illustrated in Figures 9.11 and 9.12, the strength of this relationship will depend on both the time to expiration and whether the option is in or out of the money. Given this, the different sign for the put and call rho's requires some explanation. The impact on the call price follows immediately from considering Property 10 of Sec. 8.1, which can be interpreted as using a long call plus lending to approximate the return from holding the stock. A higher interest rate will lower the present value of the bond investment (that has expiration value equal to the exercise price). This frees up funds for investment in the call position, resulting in a higher call price. The impact on the put, which is unambiguous, follows from the same argument where it is recognized that the stock plus a purchased put exactly equals the call plus lending portfolio.

Verifying the Black-Scholes Solution

Given the analytical form of the derivatives for the Black-Scholes formula have been specified, it is now possible to demonstrate that the Black-Scholes formula is the solution to the fundamental PDE given in Sec. 8.2. In terms of delta, theta and gamma:

$$C_t = rC - rS\ C_S - \frac{1}{2}\sigma^2 S^2 C_{SS} \quad \Rightarrow \quad -\theta_t = rC - rS\ \Delta_C - \frac{1}{2}\sigma^2 S^2\ \Gamma_C$$

Substituting in the relevant derivatives gives:

$$-\frac{\partial C}{\partial t^*} = -[\frac{S\sigma}{2\sqrt{t*}} N'[d_1] + X \exp\{-rt^*\} r\ N[d_2]]$$

$$= rSN[d_1] - rX \exp\{-rt^*\}\ N[d_2]\} - rS\ N[d_1] - \frac{1}{2}\sigma^2 S^2 \frac{1}{S\ \sigma\ \sqrt{t^*}} N'[d_1]$$

$$= -rX \exp\{-rt^*\}\ N[d_2] - \frac{1}{2}\frac{S\sigma}{\sqrt{t^*}} N'[d_1]$$

Hence, the Black-Scholes formula satisfies the fundamental PDE. Verifying that the boundary conditions are satisfied is not difficult. *max[0, S(T) - X]* follows because when *S(T) > X*, dividing a positive numerator by *t* = 0* in d_1 and d_2 produces *N[4] = 1* and *C = S - X*. When *S(T) < X*, then dividing a negative numerator by *t* = 0* produces *N[-4]* = 0 and *C* = 0. Similarly, when *S* = 0, taking the log of zero in both d_1 and d_2 produces *N[-4]* = 0 and *C[0,J,X]* = 0 is also satisfied.

9.2 Hedge Portfolios, Spread Trades and Other Strategies

Riskless Hedge Portfolios

A riskless hedge portfolio is a theoretical construct. In general, it is a portfolio with a value that does not change when a specific random variable changes. While it is usually possible to construct a hedge portfolio, in some fashion, the riskless hedge condition may not always be attainable in practice. For example, consider a portfolio containing k securities with prices per unit of $\{X_1, X_2 ...., X_k\}$ and number of units held in the portfolio of $\{\alpha_1, \alpha_2 ...., \alpha_k\}$. Then the market value of the portfolio, V, is equal to: $V = \alpha_1 X_1 + \alpha_2 X_2 + ... + \alpha_k X_k$. If the X's are functions of a random variable, say, Y, then the riskless hedge portfolio condition is:

$$\frac{\partial V}{\partial Y} = \alpha_1 \frac{\partial X_1}{\partial Y} + \alpha_2 \frac{\partial X_2}{\partial Y} + ... + \alpha_k \frac{\partial X_k}{\partial Y} = 0$$

The variable Y could be a price of one security, e.g., X_j, or Y could be a non-priced variable, such as the weather. It is also possible for Y to be vector valued.

The statement that a hedge portfolio is ***riskless*** requires that the α's can be determined such that the value of the position will not change when Y changes. In effect, the position is ***delta neutral*** with respect to Y. This will depend on the precision with which the coefficients $\{\alpha\}$ can be determined. In turn, there may be considerable reliance on the assumption of continuous trading to achieve the hedge condition. In the Black-Scholes case, there are only two securities: $X_1 = S$ and $X_2 = C$, with $\alpha_1 = 1$, $\alpha_2 = -\{1/\Delta_C\}$ and $Y = X_1$. The role of continuous trading in the call option to achieve the riskless hedge portfolio condition is captured by the restriction on α_2. Insofar as $\alpha_2 = -\{1/\Delta_C\}$ can be achieved, changes in X_1 can be offset by altering the size of the call option position. The problem is not altered by letting $Y = X_3$. Riskless hedging now involves:

$$\frac{\partial V}{\partial X_3} = \alpha_1 \frac{\partial S}{\partial X_3} + \alpha_2 \frac{\partial C}{\partial X_3} = \{1 - \alpha_2 \frac{\partial C}{\partial S}\} \frac{\partial S}{\partial X_3} = 0$$

This produces the same continuous trading restriction on α_2.

As originally conceived by Black and Scholes, the riskless hedge portfolio involved a net investment of funds. This approach requires Black-Scholes to impose the further requirement that the funds invested in position earn the riskless rate of interest. A more modern approach is to construct the hedge portfolio with no net investment of funds. Yet, ***an arbitrage portfolio is a riskless hedge portfolio involving no net investment of funds***. More precisely, the arbitrage portfolio for the Black-Scholes model is:

$$V = S - \beta\ C - (S - \beta C) = 0 \quad \rightarrow \quad dV = dS - \beta dC - (S - \beta C)\ r\ dt = 0$$

Initial investment in the portfolio V = 0 because the difference between the cash outflow of purchasing the stock and the cash inflow from writing the options is made up by borrowing $(S - \beta C)$.

Delta neutrality for a position is one of many possible derivative properties. The usefulness of the derivative properties of Black-Scholes presented in Sec. 9.1 is well illustrated by applying these concepts to the options trading strategies discussed in Sec. 7.1. To accomplish this requires recognizing that all securities that can be included in a given portfolio also possess derivative properties. From linearity, this permits the delta, theta, gamma, etc. of a portfolio to be calculated. As discussed in Sec. 9.1, let V be the dollar value of the portfolio, V_i represent the price of security i and n_i be the number of units of security i held where $n < 0$ indicating a short (written) position and $n > 0$ a long (purchased) position. In general, for portfolios containing a large number of securities:

$$V = n_1\ V_1 + n_2\ V_2 + n_3\ V_3 + ...$$

For the applications considered in this Section, three types of securities will usually (but not always) be considered: the underlying nondividend paying stock as well as European call and put options on that stock. The stock position is generic, with appropriate adjustment any commodity position could be used.

The delta, theta and gammas for combinations follow appropriately. In the two security case, the delta of the portfolio can be derived:

$$\Delta_V = \frac{\partial V}{\partial S} = n_1 \frac{\partial V_1}{\partial S} + n_2 \frac{\partial V_2}{\partial S} = n_1\, \Delta_1 + n_2\, \Delta_2$$

Using this result, it is possible to solve for the n_1 and n_2 that are consistent with $)_V = 0$, creating a portfolio that is delta neutral. The riskless hedge portfolio requires the size of the call option position to be continuous rebalanced in order to maintain the delta neutrality condition. In other cases, such as purchasing an approximately at-the-money straddle, delta neutrality is achieved when the position is initiated but delta neutrality is lost as the stock price moves away from the exercise price.

To see how delta neutrality can be achieved for a portfolio containing a combination calls and puts, let V_1 be a call and V_2 be a put, and recall that:

$$\Delta_C = \frac{\partial C}{\partial S} = N[d_1] > 0 \qquad \Delta_P = \Delta_C - 1 < 0$$

$$\Delta_S = 1 \qquad \Gamma_S = 0$$

Solving for a $)_V = 0$, delta neutral, position gives:

$$V = n_1\, C + n_2\, P$$

$$\Delta_V = 0 \quad \Rightarrow \quad \frac{n_1}{n_2} = -\frac{\Delta_2}{\Delta_1} \quad \Rightarrow \quad n_1 = -n_2 \frac{\Delta_2}{\Delta_1}$$

If the options are at the money then the put and call deltas will be approximately .5 and the straddle is delta neutral. Another application of delta neutrality occurs where V_1 refers to a long stock position and V_2 is either a put or a call. In this case, delta neutrality is a requirement for a hedge portfolio.[4] However, delta neutrality is only a property at a specific point. When the stock price changes, the portfolio must be adjusted to maintain the hedge, though it is not necessary to do this. The position can be left unadjusted and gains or losses will occur as the stock price and other variables change. In this case, delta neutrality has a different meaning. A straddle is one example of a ***possibly*** delta neutral position that is not rebalanced.

Anticipating the discussion in Sec. 9.3, a delta neutral portfolio can be contrasted with portfolio insurance. One possible method of insuring a stock position is to form a portfolio that combines purchased puts with a long stock position. In this case:

$$V = n_1 S + n_2 P \quad \Rightarrow \quad \Delta_V = n_1 + n_2 \Delta_P$$

For full insurance, the number of units of stock underlying the put equal the size of the long stock position, such that $n_1 = n_2$:

$$\Delta_V = n_1\, \{1 + \Delta_P\} > 0 \qquad where \quad \Delta_P > -1$$

The implication is that, in opposition to hedge portfolios and other positions that are delta neutral, portfolio insurance requires the position delta to be greater than zero, where the V exhibits a positive response to stock price increases.[5] In addition, the position is not rebalanced over time in order to maintain the starting delta value. As discussed in Sec. 9.3, the precise degree of delta sensitivity depends on the delta characteristics of the puts selected. Hence, the Greeks can be used to improve the construction of insured portfolios.

As discussed in Sec. 8.2, when a specific position partial derivative is set equal to a desired value, this imposes restrictions on the other partial derivatives of the fundamental PDE. To see this, consider the gamma of a delta neutral position involving two securities. Substituting in the n_2 previously derived for the two security portfolio gives:

$$\Gamma_V = n_1 \Gamma_1 + n_2 \Gamma_2 \quad \Rightarrow \quad \Gamma_V = n_1 \Delta_1\, \{ \frac{\Gamma_1}{\Delta_1} - \frac{\Gamma_2}{\Delta_2} \}$$

In the special case where V_1 refers to a stock position (or some other gamma zero security), then the portfolio gamma reduces to: ' $_V$ = -{ ' $_2$ /) $_2$}. Given this, compare the delta neutral strategies involving hedging a stock position with futures contracts and hedging a stock position using written calls.[6] For hedges involving futures, like stocks, ' = 0. Hence, this form of hedge portfolio is insulated against stock price changes. Now consider the strategy of creating a hedge portfolio by writing options against a long stock position. Setting $n_1 = 1$ and using the delta for a stock, n_2 is the same value as for the Black-Scholes hedge portfolio. However, unlike the futures based hedge portfolio, the gamma of this hedge portfolio is negative. This follows from the positive gamma and delta values for a call position. Recalling the interpretation of the sign of gamma, stable stock prices are required for the hedge portfolio to be successful. In addition, the size of the position gamma will depend on the relationship between the exercise price of the options and the current stock price.

Another practical example of using delta to create a hedge portfolio occurs with the ***ratio spread*** of Sec. 7.2. This trade involves the creation of a hedged position, where n_1 units of the overpriced (correctly priced) option are sold and n_2 units of the correctly priced (underpriced) option are purchased. More precisely:

$$V = n_1 \, C_1 + n_2 \, C_2 \qquad \Rightarrow \qquad \Delta_V = 0 \qquad \Rightarrow \qquad \frac{n_1}{n_2} = -\frac{\Delta_2}{\Delta_1}$$

Because the hedge ratio, n_1/n_2, ensures that there will be no change in the value of the position as S changes, the option trader is able to lock in the ***positive*** difference between the premiums received and premiums paid. Much as with other riskless hedge portfolios, this trade will require rebalancing in order to maintain) $_V$ = 0.

Delta Plus Gamma Hedge Portfolios

An important practical aspect of strategic portfolio design for portfolios containing derivatives involves the achievement of multiple partial derivative targets, e.g., delta ***and*** gamma neutrality. Typically the portfolio will contain a long stock position. Because of the relationship between the number of equations and the number of unknown derivative position sizes, achieving delta plus gamma targets will typically require the use of more than one derivative.[7] To see this, consider the strategy of selling call options in order to create a delta neutral position. Because the gamma of the stock is zero, this produces a position that is gamma negative. In order to get position gamma equal to zero, it is necessary to include a derivative with a positive gamma, such as a long put position. ***Assuming, for simplicity, that the puts and calls have the same exercise price and time to expiration.*** In this case:

$$\Delta_V = n_1 \Delta_C + n_2 \Delta_P + n_3 \Delta_S$$

Observing that the delta of the stock is equal to one, setting $n_3 = 1$ produces the following requirements for position delta neutrality per unit of the long stock position:

$$\Delta_V = 0 = n_2 \Delta_P + n_1 \Delta_C + 1 = n_2 \{\Delta_C - 1\} + n_1 \Delta_C + 1$$

$$\Rightarrow \qquad [n_1 + n_2]\Delta_C - n_2 = -1$$

In order to solve for the sizes of the put and call positions it is necessary to impose gamma neutrality:

$$n_1 \Gamma_P + n_2 \Gamma_C = 0 \qquad \Rightarrow \qquad -\frac{n_2}{n_1} = 1 \qquad \rightarrow \qquad -n_2 = n_1$$

Substituting this result back into the equation associated with) $_V$, reveals that $n_1 = -1$, $n_2 = 1$ and $n_3 = 1$.

This illustration of the additional use of a gamma restriction in the construction of a riskless hedge portfolio reveals the need to use a written call with a purchased put to hedge the stock position. Because the put and call have been assumed to have the same X and $t*$ and that an equal number of puts and calls are used in the hedge, it follows:

$$V = S - n_1 C + n_2 P = S - n_1(P - C)$$

$$\Delta_V = 0 = 1 - n_1(\Delta_C - 1 - \Delta_C) \quad \Rightarrow \quad n_1 = 1 \quad \Rightarrow \quad \Gamma_V = \Gamma_C - \Gamma_C = 0$$

This result is an extension of the put-call parity replication strategies derived in Sec. 7.3 using the expiration date profit diagram technique. More precisely, for the delta-gamma neutral portfolio: $S + P - C = X\ exp\{-rt^*\}$. Imposing a combination of delta neutrality and gamma neutrality on the hedge portfolio produces a portfolio with a payout identical to that of the riskless bond.

Another useful application of delta plus gamma targets can be illustrated with the design of the Black-Scholes riskless hedge portfolio. This example is used because it is familiar, the notions extend naturally to other important cases such as portfolio insurance. In deriving the Black-Scholes formula it was assumed that the riskless hedge portfolio involved a stock being combined with written call options featuring only one specific exercise price. Recalling that gamma can be interpreted as the frequency with which the hedge portfolio has to be rebalanced, it would be more appropriate to write calls at a number of different exercise prices in order to achieve a lower gamma. Writing options that feature only one exercise price may be convenient for purposes of deriving the Black-Scholes formula, but this approach is not to be recommended in practical situations where rebalancing frequency is a concern. To see this, consider hedging the stock position by writing an equal combination of in-the-money and out-of-the-money call options:

$$V = S - n_1 C_1 - n_2 C_2 = S - n_1(C_1 + C_2) \qquad \Delta_V = 0 \quad \Rightarrow \quad n_1 = \frac{1}{\Delta_1 + \Delta_2}$$

$$\Gamma_V = -n_1 (\Gamma_1 + \Gamma_2) = -\frac{\Gamma_1 + \Gamma_2}{\Delta_1 + \Delta_2}$$

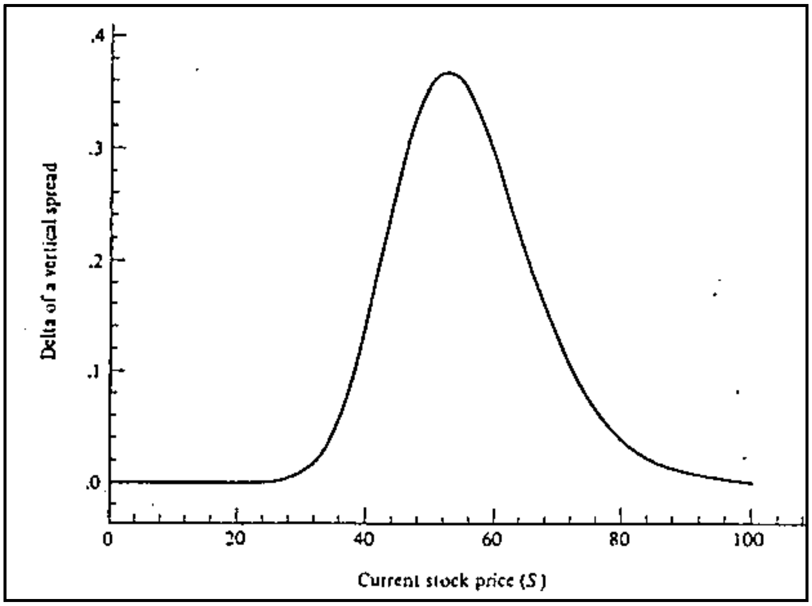
It follows that the gamma of hedge portfolio that uses at-the-money options will have the highest gamma and, as a consequence, the greatest need to rebalance.

To motivate this result it is useful to refer back to the delta and gamma diagrams of Sec. 9.1. Using .5 as the delta for the at-the-money option, then 2 at the money options have to be written to hedge the stock position. Taking .2 and .8 as the delta values for the in-the-money and out-of-the-money options, then one of each option has to be written. In both portfolios, two options are written. Assuming for simplicity that the sum of the in- and out- gammas are approximately equal to the gamma of the at-the-money, then it follows that the gamma of the hedge portfolio that involves writing options at two different exercise prices will be one-half the value of the portfolio using only one exercise price. Of course, this result depends on the 'moneyness' of the various options selected. In the extreme case, it would be possible to choose deep in- and out- options producing a gamma that is virtually zero, but this would be difficult to implement in practice. Similarly, it would be possible to use either an in- or out- option in the portfolio containing the one exercise price options. If that option was sufficiently in- or out- of the money it is possible to construct a case where the one exercise price option portfolio would have a lower gamma.

Vertical Spreads and Butterflies

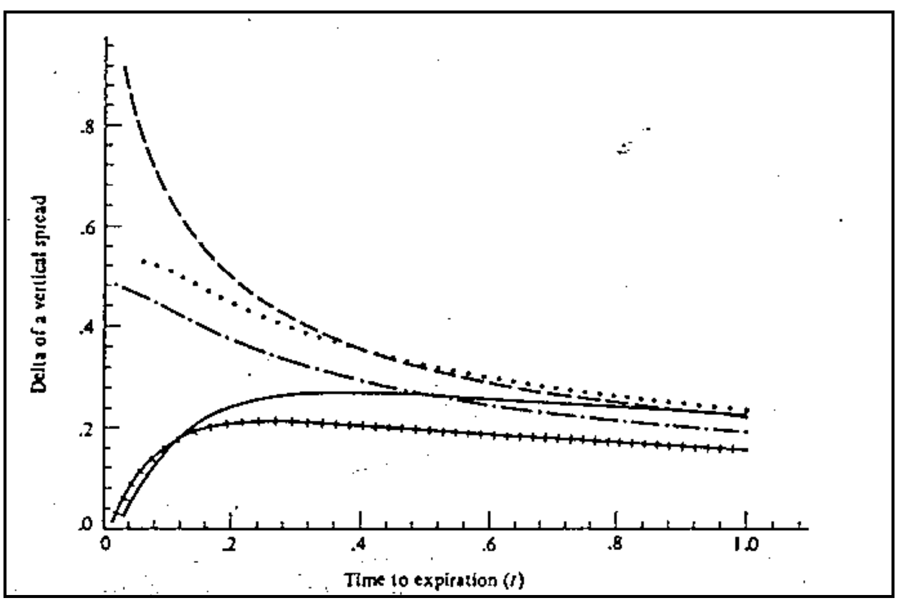
In addition to evaluating insured and hedged portfolios, it is also possible to use delta, theta and gamma to provide a more precise description of the payouts on the option spread trading strategies discussed in Sec. 7.2. Consider the general case for a vertical spread involving n_1 calls purchased (written) at exercise price X_1 and n_2 calls written (purchased) at X_2 where $X_1 < X_2$. For the purchased vertical spread using calls, also referred to as a bull spread, $n_1 = 1$ and $n_2 = -1$. It is straight forward to verify that the delta for the call with the lower exercise price is highest. This can be shown by considering the analytical values for the deltas on $C_1[S,\tau,X_1]$ and $C_2[S,\tau,X_2]$:

Figure 9.15 **The Delta of a Vertical Spread as a Function of the Stock Price**



X $_1$*= 50* *X*$_2$ *= 60* *t* = .4* *r* = .06 **F** = .3

Figure 9.16 **The Delta of a Vertical Spread as a Function of the Time to Expiration**

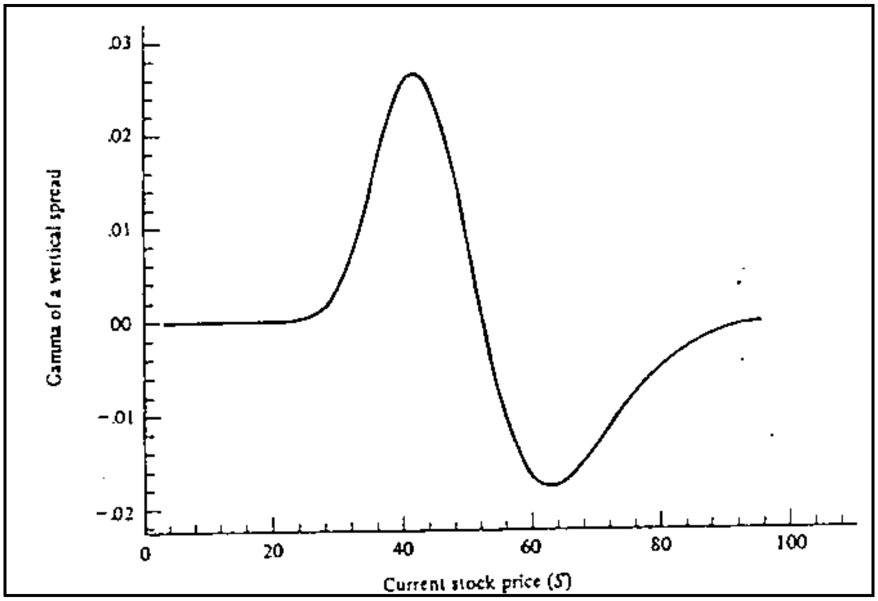


X $_1$*= 50*

X$_2$ *= 60* *r* = .06 **F** = .3

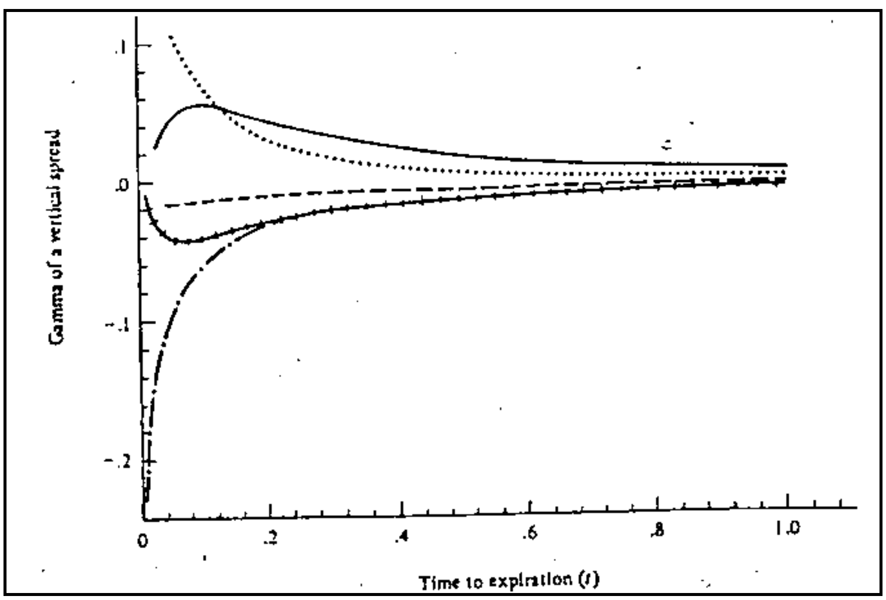
—— *S = 45* AAAAAAAS = 50 – – – *S = 55* – A – *S = 65* + + + + *S = 70*

Figure 9.17 **The Gamma of a Vertical Spread as a Function of the Stock Price**



$X_1 = 50$ $X_2 = 60$ $t^* = .4$ $r = .06$ F = .3

Figure 9.18 **The Gamma of a Vertical Spread as a Function ofthe Time to Expiration**



$X_1 = 50$ $X_2 = 60$ $r = .06$ F = .3

—— *S* = 45 AAAAAAA*S* = 50 – – – *S* = 55 – A – *S* = 65 ++++ *S* = 70

$$\frac{\partial C_1}{\partial S} = N[d_{1,1}] \qquad \frac{\partial C_2}{\partial S} = N[d_{1,2}]$$

where:

$$d_{1,1} = \frac{\ln\{\frac{S}{X_1}\} + (r + \frac{1}{2}\sigma^2)t^*}{\sigma\sqrt{t*}}$$

$$d_{1,2} = \frac{\ln\{\frac{S}{X_2}\} + (r + \frac{1}{2}\sigma^2)t^*}{\sigma\sqrt{t^*}}$$

Evaluation of the $N[\cdot]$ functions verifies the relationship between the deltas. Given this:

$$V = n_1 C_1 + n_2 C_2 \qquad \Rightarrow \qquad \Delta_V = n_1 \Delta_{C1} + n_2 \Delta_{C2}$$

$$when \quad n_1 = -n_2 = 1$$

$$\Delta_V = \Delta_{C1} - \Delta_{C2} > 0$$

As illustrated in Figures 9.15 and 9.16, the delta of the vertical spread will depend on the current stock price as well as the time to expiration. Using this result, it is possible to expand considerably the information provided by the expiration date profit diagram for the vertical spread given Sec. 7.2. Given this, it is straight forward to compute the gamma of the vertical spread. The sensitivity of gamma to both the current stock price and time to expiration is given in Figures 9.17 and 9.18.

The butterfly spread provides a useful illustration of gamma. Consider the case where purchased options at X_1 and X_3 are combined with two written options at X_2 with $X_1 < X_2 < X_3$ with $S = X_2$:

$$V = n_1 V_1 + n_2 V_2 + n_3 V_3$$

$$where: \quad n_1 = 1 \quad n_2 = -2 \quad n_3 = 1$$

$$\Delta_V = \Delta_{C1} + \Delta_{C3} - 2\Delta_{C2}$$

$$\Gamma_V = \Gamma_{C1} + \Gamma_{C3} - \Gamma_{C2}$$

Evaluating these derivatives using the results of Sec. 9.1, if the X_1 option is deep in the money its delta will be close to 1. If this position is combined with a deep out of the money option at X_3 that has a delta of close to 0 this produces an approximately delta neutral position because the two at the money options will have a delta of 1/2. This result holds whether the butterfly is being written or purchased. Based on this, it should be apparent that with appropriate selection of the exercise prices of the options included in the butterfly, it is not necessary for the middle exercise price to be at the money in order to produce a delta neutral position. Given this, differences in gamma have to be recognized. Unlike delta, gamma will be non-zero and will be negative for a purchased butterfly and positive for a written butterfly.[8] The importance of selecting the at the money option for the middle two positions is apparent when gamma is considered: in order to get Γ_V as negative (positive) as possible, an at the money option is required.

Straddles, Straps, and Strangles

The straddle provides another practical application of the partial derivatives of Black-Scholes. Consider the delta for a straddle:

$$\Delta_V = n_1 \Delta_C + n_2 \Delta_P \qquad for \quad n_1 = n_2 = 1 \qquad \Delta_V = \Delta_C + \{\Delta_C - 1\}$$

$$= 2\Delta_C - 1$$

In Sec. 9.1 is was demonstrated that for short dated options d_1 for $S = X$ is small, permitting the approximation $N[0] = .5$. In this case, when the put and call options are at-the-money, both the purchased and written straddles are approximately delta neutral. Examination of delta further reveals that once the stock price moves away from the money, the straddle is no longer delta neutral. Consider the delta when $S > X$, and the C is in-the-money with the put out-of-the-money. In this case evaluation of Δ_V reveals that the $\Delta_C > .5$ that produces a positive position delta. To see the intuition of the impact of 'moneyness' on delta, consider the case where the call option is deep in- and the put option is deep out-. In this case, increases in the stock price will increase the call price almost one-for-one, without having any impact on the put price. Similarly, for C out-of-the-money and P in-the-money, $\Delta_C < .5$ that produces a negative position delta. Decrease in stock price will have a much greater positive impact on the put than the negative impact on the call price.

Evaluation of gamma reveals different signs for purchased and written straddle positions. The purchased position has a positive gamma, indicating that volatile stock price movements are required for profitability while the written position has a negative gamma, indicating that a stable stock price is needed for profitability. This much was already known from the expiration date profit diagram analysis of Sec. 7.1. What gamma reveals is how sensitive the position profitability will be to changes in stock prices. When $S > X$ or $S < X$, the position gamma will be significantly less than when $S = X$ to the point where gamma approaches zero when deep in- and deep out- options are used to create the straddle. It follows that while it is possible to tailor a straddle to combine directional (delta not equal zero) and volatility bets by using options that are not at the money, there is a tradeoff. The greater is the delta of the position, the less sensitive the profitability of the position is to the volatility of stock prices. This raises the question of how a straddle, constructed using options that are not at-the-money, compares to straps and strips that are designed to combine directional and volatility bets.

Consider the strap, which combines two calls and one put, all with the same exercise price and time to expiration. The delta of this position is:

$$\Delta_V = n_1 \Delta_C + n_2 \Delta_P \qquad for \quad n_1 = 2 \quad n_2 = 1 \qquad \Delta_V = 2\Delta_C + \{\Delta_C - 1\}$$

$$= 3\Delta_C - 1$$

When the options are at-the-money, the strap is delta positive. If the exercise price is selected such that $\Delta_C = .5$, the Δ_V for the strap would be .5. Because of the different costs associated with the straddle and the strap, it is useful to provide specific option prices. For this purpose, assume that:

$$X = 40 \quad \sigma^2 = .25 \quad r = .05 \quad t^* = .25$$

Using these parameter values, the Black-Scholes option prices, consistent with the simplifying assumption that $d_1 = 0$, are C = \$3.29 and P = \$4.51 making the price of the delta neutral straddle \$7.80 for an initial stock price \$38.29. At these values the strap would cost \$11.09.

Recognizing that it is possible to make the strap delta neutral by combining an in-the-money put with two out-of-the-money calls, it follows that this can be done precisely when the call options have a delta of 1/3. For the volatility, interest rate and time to expiration values given, this occurs with an exercise price of $X = 44.55$ when the stock price is \$38.29. The corresponding put price of \$7.56 and call price of \$1.85 produces a strap that costs \$11.26. At .11395, the gamma of this position is considerably higher than the gamma of the delta neutral straddle that is 2(.4168) = .08336, indicating that greater stock price volatility is required for profitability of the strap. This is reflected in the different expiration date payoffs. For example, consider the expiration date values of the strategies, assuming that interest on the premium is ignored. If $S(T)$ = 40, the straddle loses the maximum of \$7.80 but the strap only loses (-11.26 + (44.55-40)) = -\$6.71. For S(T) = 44.55, the strap loses the maximum of \$11.26 and the straddle loses (-7.80 + (44.55 - 40)) = -\$3.25. Similarly, at S(T) = 50, the straddle provides (-7.80 + (50-40)) = \$2.20 and the strap (-11.26 + 2(50-44.55)) = -\$.36 and at S(T) = 55, the straddle earns \$7.70 and the strap \$9.64. Hence, a strap that is initiated as delta neutral requires a larger upward increase in the stock price to outperform the straddle, but will lose slightly less if the stock price remains unchanged or falls.

A similar comparison can be made between the strangle and the straddle. Recalling that the strangle combines put and call options with different exercise prices and the same time to maturity, the delta for a strangle is:

$$\Delta_V = n_1\Delta_C + n_2\Delta_P \qquad \textit{for} \quad n_1 = n_2 = 1 \qquad \Delta_V = \Delta_{C1} + \{\Delta_{C2} - 1\}$$

$$= \{\Delta_{C1} + \Delta_{C2}\} - 1$$

If the stock price lies approximately halfway between the two exercise prices, then the strangle will be delta neutral. It is also possible to create strangles that are delta positive, e.g., using at-the-money calls and out-of-the-money puts, or delta negative, e.g., using at-the-money puts and out-of-the-money calls. In general, if the straddle is delta neutral, even with discrete exercise price intervals it is possible to specify a strangle that is close to being delta neutral. To see this, consider the previous example of the delta neutral straddle with X = 40 and $S(0)$ = \$38.29. With \$5 exercise intervals, there are three comparable {$X1$ for C, $X2$ for P} strangle variations: {$X1$ = 40, $X2$= 35}, {$X1$=45, $X2$ = 40} and {$X1$ = 45, $X2$ = 35}.

Consider the last case, where a call, $C(X1$=45), with price = \$1.74, delta = .318772 and gamma = .0373 is combined with a put, $P(X2$=35), with price = 2.06, delta = (.7034 - 1) and gamma = .0361. The cost of the position is \$3.80 with a delta approximately equal to the straddle value of zero and a ***smaller*** gamma of .0734. As for the straddle, delta neutral implies that the position profitability is the same whether stock price goes up or down. The smaller gamma implies that the delta of the position does not (locally) change as much as the delta for a straddle. The delta and gamma results change when different options are used. Consider the case of $C(X1$=40) and $P(X2$=35), premiums paid would be (\$3.29+\$2.06) = \$5.35 with Δ_V = .5 + (.7034-1) = .2034 and Γ_V = .05224 + .0361 = .08885. In this case, the positive delta indicates that the position is more profitable when prices rise. The higher gamma, compared to the case where both the call and put are out-of-the-money, indicates the change is the value of this position is more sensitive to stock price changes.

9.3 Portfolio Insurance

> "Portfolio insurance, in its purest and simplest form, is equivalent to a securities position comprised of an underlying portfolio plus an insurance policy that guarantees the insured portfolio against loss through a specified policy expiration date. Should the underlying portfolio (including any income earned and reinvested in the portfolio but deducting the cost of buying the insurance) experience a loss by the policy expiration date, the insurance policy can be used to refund the amount of the loss. On the other hand, should the underlying portfolio show a profit, all profit net of the cost of insurance is retained." *Rubinstein (1985, p.42)*

The History of Portfolio Insurance

In relatively recent history, portfolio insurance has had a significant impact in both practical and theoretical areas. Many of these developments have been associated with the emergence of financial engineering. However, heuristic forms of portfolio insurance have been used for decades. For example, a form of portfolio insurance can be achieved with the systematic use of order placement strategies, such as stop-loss and limit orders that have been acceptable market practice at least since the 19th century. As discussed below, these types of trading dependent strategies suffer from the defect of being path dependent, an undesirable property of insurance schemes. In addition to trading related techniques, option replication strategies using stock/bond combinations were also likely in use, though in the realm of proprietary management practices. These techniques also suffer from the defect of path dependence and, in the absence of Greek information, would probably have been imprecise. The application of option replication to specifying dynamically traded stock/bond portfolios was not of academic interest until much later, after the development of the Black-Scholes formula.

As for the history of insurance related financial products, some of the insurance schemes of the late 17th and 18th century did offer payouts based on specific outcomes associated with joint stock performance. Being introduced prior to the development of actuarial science, these insurance schemes were more like gambling than insurance. In more recent history, Benninga and Blume (1985) report the selling of insurance against investment losses in the UK as early as 1956. In the US, Gatto, Geske, Litzenberger and Sosin (1980) report on portfolio insurance plans offered to individuals by both the Harleysville Mutual Insurance Company and Prudential Insurance Company of America. Brennan and Schwartz (1987) observe that the Harleysville plan was the first without any element of mortality insurance. Academically, Brennan and Schwartz (1976) were the first to make the connection between the potential for integrating insurance and equity returns. Leland, O'Brien, Rubinstein and Associates were important proponents in the marketing of dynamically traded option replication strategies to institutional clients.

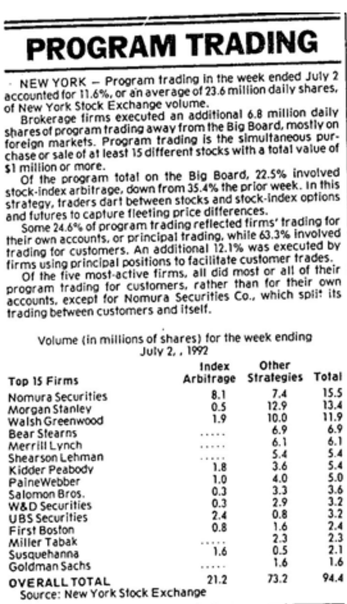
The explosion in the use of the various types of portfolio insurance techniques can be traced to the introduction of exchange trading in options. Liquid options markets made possible the implementation of numerous portfolio insurance strategies. Even more strategies were permitted with the development of futures and options markets for stock indices. Analytical contributions based on Black-Scholes resulted in further portfolio insurance strategies being introduced. Many "alternative paths to portfolio insurance" (Rubinstein 1985) were proposed and implemented. The widespread use of dynamically traded portfolio insurance techniques has been identified as an important contributing factor in the Oct. 1987 stock market "crash", e.g., Tosini (1988)(see Sec. 1.2). Academic understanding of notions associated with portfolio insurance have expanded considerably since the early work by Leland (1980) and Rubinstein and Leland (1981). The 1987 "crash" provided a textbook illustration of the inadequacies of the academically inspired option replication strategies; sizable unexpected losses were experienced by investors holding what were expected to be "insured" portfolios.

One of the fundamentals driving institutions to use dynamic trading strategies was the absence of risk management products with maturities and other characteristics that captured the time profile of their particular risk exposures. Since the crash, an array of OTC and exchange traded risk management products have been introduced that greatly enhance the ability to implement path independent strategies. Included in the list of such new products would be: long dated exchange traded option products, such as LEAPS for individual stocks and long dated index options and equity swaps. Despite these improvements, the bulk of contract liquidity on both the exchanges and OTC is still concentrated in short dated contracts. The relative absence of strict mark-to-market rules in OTC contracts provides a strong incentive to use short dated contracts.

Properties of Insured Portfolios

Before describing the various forms of portfolio insurance, it is useful to introduce two features associated with insured portfolios: ***path independence*** and ***time invariance***. The first concept has already been introduced in Sec. 7.1. In effect, true portfolio insurance strategies should be path independent. "A strategy that is not path independent gives an uncertain payoff, and therefore violates the very premise of portfolio insurance: giving a known payoff." (Bookstaber and Langsam 1988). Time invariance requires that the insurance strategy does not depend on the time remaining in the program (or on the use of a fixed time horizon). The primary practical difficulty arises because of the option price convexity with respect to time, e.g., an option with six months to expiration will cost less than twice what a three month option costs. Until recently, a strong argument in favor of so-called dynamic replication strategies was that the short dated maturities available for liquid exchange-traded options have insufficient time convexity for typical portfolios with long term investment horizons. The recent introduction of LEAPs and (longer dated) futures and options on stock indices has substantively reduced this problem.

Given this background, the basic idea behind portfolio insurance is to provide a rate of return that will not fall below a given floor. As was illustrated in Sec. 7.1, if this is done by purchasing puts in conjunction with a long stock position, this will replicate the payoff on a purchased call position. However, while this strategy will typically be path independent, it will not be time invariant. Based on application of the ***binomial option pricing formula***, e.g., Ritchken (1987, ch. 9), Rubinstein and Leland (1981), it is also possible to replicate the payoff on a call option by dynamically adjusting a portfolio of stocks and bonds. While the dynamic replication strategies are path dependent, there are other desirable features that can be achieved by these strategies. The widespread use of these strategies is reflected in Figure 9.19 where the sizable volume attributed to Other



PROGRAM TRADING

NEW YORK – Program trading in the week ended July 2 accounted for 11.6%, or an average of 23.6 million daily shares, of New York Stock Exchange volume.

Brokerage firms executed an additional 6.8 million daily shares of program trading away from the Big Board, mostly on foreign markets. Program trading is the simultaneous purchase or sale of at least 15 different stocks with a total value of $1 million or more.

Of the program total on the Big Board, 22.5% involved stock-index arbitrage, down from 35.4% the prior week. In this strategy, traders dart between stocks and stock-index options and futures to capture fleeting price differences.

Some 24.6% of program trading reflected firms' trading for their own accounts, or principal trading, while 63.3% involved trading for customers. An additional 12.1% was executed by firms using principal positions to facilitate customer trades.

Of the five most-active firms, all did most or all of their program trading for customers, rather than for their own accounts, except for Nomura Securities Co., which split its trading between customers and itself.

Volume (in millions of shares) for the week ending July 2, , 1992

Top 15 Firms	Index Arbitrage	Other Strategies	Total
Nomura Securities	8.1	7.4	15.5
Morgan Stanley	0.5	12.9	13.4
Walsh Greenwood	1.9	10.0	11.9
Bear Stearns	.....	6.9	6.9
Merrill Lynch	.....	6.1	6.1
Shearson Lehman	.....	5.4	5.4
Kidder Peabody	1.8	3.6	5.4
PaineWebber	1.0	4.0	5.0
Salomon Bros.	0.3	3.3	3.6
W&D Securities	0.3	2.9	3.2
UBS Securities	2.4	0.8	3.2
First Boston	0.8	1.6	2.4
Miller Tabak	.....	2.3	2.3
Susquehanna	1.6	0.5	2.1
Goldman Sachs	.....	1.6	1.6
OVERALL TOTAL	21.2	73.2	94.4

Source: New York Stock Exchange

Strategies is largely due to dynamic replication. Required reporting of such positions to the NYSE was introduced following the 1987 crash. Unfortunately, as discovered by market participants involved in the use of these strategies during the crash of Oct. 1987, at times when the market is moving down these strategies involving selling, thereby adding to the downward market pressure.

Types of Portfolio Insurance

The basic mechanics of portfolio insurance can be isolated from the put-call parity arbitrage condition for a non-dividend paying stock: $S + P = C + X\, e^{-rt^*}$. For portfolio insurance, instead of an individual stock S now refers to a portfolio of stocks and dividends have been ignored for simplicity of exposition. As stated, put-call parity provides two path independent insurance strategies. One strategy is $S + P$, buy puts against the portfolio. If S is an index portfolio, relevant exchange traded puts may be available. Another strategy is $C + X\, e^{-rt^*}$, buy calls and invest the remainder in appropriately dated bonds. Again, if the portfolio is an index portfolio, exchange traded calls may be available. One important advantage of this strategy is that transactions costs in bond markets are typically lower than transactions costs for stocks and the bond portfolio can be actively managed, e.g., by riding the yield curve, to earn potentially higher returns than the $S + P$ approach.

Different Forms of
Equity Portfolio Insurance(a)

Strategy	Advantages	Disadvantages
Buying index puts against a portfolio	Insurance cost determined in advance. Investor captures portfolio nonmarket return.	Listed puts do not trade with expirations greater than 4 months. Must accept the pricing risk of subsequent options purchases.
Buying puts on individual stocks	Portfolio positions protected against decline on a stock-by-stock basis.	Premiums greater than for index puts. Not every stock has a listed put.
Buying index calls and money market securities	Can vary fixed income strategy around the call position; call performance tied to a diversified index.	Cannot capture nonmarket return on a portfolio of stocks. Must accept the pricing risk of subsequent options purchases.
Buying calls on individual stocks and money-market securities	Full participation in all gains from individual stock movement.	Premiums greater than for index calls. Not every stock has a listed call.
Selling stock index futures to create a synthetic put	Can create strike price and expiration date. Will capture portfolio alpha.	Actual cost cannot be predetermined. Must accept pricing risk of the futures contract.
Raising cash by selling stocks to create a synthetic put	No futures pricing risk.	Higher transaction costs and market impact costs in most instances.
Buying stock index futures to own a synthetic call	Can vary fixed income investment. Equity performance tied to a common index such as the S&P 500.	Cost cannot be predetermined. Position is exposed to index futures pricing risk.

(a)In addition to the insurance strategies using listed options and stock index futures, it is possible to create an over-the-counter European or American index option with a longer life than that available in the listed markets.

While the path independent strategies have some desirable features, there are some drawbacks. One disadvantage is the inability to accurate replicate insurance for portfolios that do not track an index; the relevant options are not available. Another disadvantage is that the maturity dates for options may not be long enough to match the portfolio's investment horizon, insufficient time invariance. To handle these types of problems, dynamic trading strategies have been developed that involve actively trading portfolios composed of stocks and bonds in order to replicate the payoff on an insured stock portfolio. These strategies can be illustrated by substituting the Black-Scholes formula into the put-call parity condition:

$$S + P = S\, N[d_1] - X\, e^{-rt^*} N[d_2] + X\, e^{-rt^*} = S\, N[d_1] + X\, e^{-rt^*} (1 - N[d_2]) = w_1\, S + w_2\, X\, e^{-rt^*}.$$

The weights w_1 and w_2 indicate the proportions of the portfolio held in stock and bonds in order to achieve insurance with and exercise price of *X* and time to maturity of *t**. Unlike the portfolio optimization models, the weights here will not sum to one, as the relationship is derived to equate values on the rhs and lhs. The weights will be close to one but, except coincidentally, not equal to zero.

As illustrated in Figure 9.20, there are a number of potential path dependent and path independent portfolio insurance strategies. While the information about portfolio insurance given in this figure is not overly novel, the source and date of the publication are novel. This figure was taken from a Goldman Sachs publication, “The Different Forms of Portfolio Insurance”, authored by Fischer Black, the head of investment research at Goldman, and published in the summer of 1987 as part of Goldman’s concentrated effort to capture market share in the lucrative portfolio insurance business. This was not an isolated effort, as there was a range of Goldman Sachs, authored by Black and other from Goldman.

From a practical perspective, it is important to for the potential portfolio insurer to identify why dynamic replication strategies, i.e., strategies dynamically replicating a call option payoff using stock/bond positions, should be used. Related to this are subsidiary issues concerning how to replicate and when to replicate. In this vein, large fund managers would consider the liquidity needed to establish large enough positions using derivatives and whether there are suitable *X* and **J** available. For example, while a well-diversified fund (e.g., an index fund) could make use of options or futures written on the appropriate index, funds targeted at non-systematic risk are more likely to be obligated to use dynamic replication strategies. However, even a well-diversified fund may find that available expiration dates on traded derivatives are not long enough, i.e., sufficient "time convexity" cannot be achieved. Because the dynamic replication strategies can be designed to theoretically achieve almost any desired expiration date and exercise price, this provides another reason for the use of these strategies.

To illustrate the use of dynamic replication, consider the creation of a synthetic put option for an index portfolio. Given that the dividend yield on the index is q, Hull (1987, p. 204) shows that the delta of a European put on the index is:

$$\Delta_P = \exp\{-qt^*\}\ [N[d_1] - 1] = \exp\{-qt^*\}\ (\Delta_c - 1)$$

where:

$$d_1 = \frac{\ln\{\frac{S}{X}\} + (r - q + \frac{\sigma^2}{2})t^*}{\sigma\sqrt{t^*}}$$

Assuming that *S = 300, X = 290, r = .09, q = .03,* ***F*** *= 0.25* and *t* = .5*, evaluation of the delta of the put gives **)** = -0.322. It follows that if dynamic replication of a put is being used that 32.2% of the index fund should be sold and invested in (riskfree) fixed income securities. From the properties of the put delta discussed in Sec. 9.1, as the value of the index fund drops, the delta of the put will become more negative, indicating that a larger proportion of the index fund has to be sold, i.e., a large fraction of the portfolio will be invested in fixed income securities. A similar result would hold where the value of the index was increasing. In this case the delta of the put would be less negative, indicating that fixed income securities should be sold to purchase more units of the index fund. In this case, the proportion of the portfolio invested in the index fund would increase.

Figure 9.21 **Examples of Portfolio Insurance**

Note: To value the European put, it is assumed: the index pays no dividends; $r = .08$; $F = .2$; $X = 100$; and $t^* = .5$

Insured stock portfolio value at alternative stock index levels, using static portfolio insurance.[a]

Index Level S	Put Option p	Portfolio Value $S + p$
59.87	36.21	96.08
63.02	33.06	96.08
66.34	29.75	96.09
69.83	26.29	96.13
73.51	22.70	96.21
77.38	19.03	96.41
81.45	15.38	96.83
85.74	11.87	97.61
90.25	8.67	98.92
95.00	5.94	100.94
100.00	3.79	103.79
105.00	2.29	107.29
110.25	1.27	111.52
115.76	0.65	116.42
121.55	0.31	121.86
127.63	0.13	127.76
134.01	0.05	134.06
140.71	0.02	140.73
147.75	0.01	147.75
155.13	0.00	155.13
162.89	0.00	162.89

Insured stock portfolio value at alternative index levels, using dynamic portfolio insurance with continuous rebalancing.[a]

Index Level S	T-Bill Price Xe^{-rT}	Stock Portfolio Weight w_1	T-Bill Weight w_2	Portfolio Value
59.87	96.08	0.001	1.000	96.08
63.02	96.08	0.002	0.999	96.08
66.34	96.08	0.005	0.996	96.09
69.83	96.08	0.014	0.990	96.13
73.51	96.08	0.034	0.975	96.21
77.38	96.08	0.072	0.945	96.41
81.45	96.08	0.136	0.892	96.83
85.74	96.08	0.231	0.809	97.61
90.25	96.08	0.355	0.696	98.92
95.00	96.08	0.496	0.560	100.94
100.00	96.08	0.638	0.416	103.79
105.00	96.08	0.758	0.289	107.29
110.25	96.08	0.852	0.183	111.52
115.76	96.08	0.918	0.106	116.42
121.55	96.08	0.959	0.056	121.86
127.63	96.08	0.981	0.026	127.76
134.01	96.08	0.992	0.011	134.06
140.71	96.08	0.997	0.004	140.73
147.75	96.08	0.999	0.001	147.75
155.13	96.08	1.000	0.000	155.13
162.89	96.08	1.000	0.000	162.89

Note: The discrete rebalancing case assumes a trigger value of 5%.

Insured stock portfolio value at alternative stock index levels, using dynamic portfolio insurance with no rebalancing.[a]

Index Level S	T-Bill Price Xe^{-rT}	Stock Portfolio Weight w_1	T-Bill Weight w_2	Portfolio Value
59.87	96.08	0.638	0.416	78.18
63.02	96.08	0.638	0.416	80.19
66.34	96.08	0.638	0.416	82.31
69.83	96.08	0.638	0.416	84.53
73.51	96.08	0.638	0.416	86.88
77.38	96.08	0.638	0.416	89.35
81.45	96.08	0.638	0.416	91.95
85.74	96.08	0.638	0.416	94.68
90.25	96.08	0.638	0.416	97.56
95.00	96.08	0.638	0.416	100.59
100.00	96.08	0.638	0.416	103.79
105.00	96.08	0.638	0.416	106.98
110.25	96.08	0.638	0.416	110.33
115.76	96.08	0.638	0.416	113.84
121.55	96.08	0.638	0.416	117.54
127.63	96.08	0.638	0.416	121.42
134.01	96.08	0.638	0.416	125.49
140.71	96.08	0.638	0.416	129.76
147.75	96.08	0.638	0.416	134.25
155.13	96.08	0.638	0.416	138.97
162.89	96.08	0.638	0.416	143.92

Insured stock portfolio value at alternative stock index using dynamic portfolio insurance with discrete rebalancing.[a]

Index Level S	T-Bill Price Xe^{-rT}	Stock Portfolio Weight w_1	T-Bill Weight w_2	Portfolio Value
59.87	96.08	0.001	0.985	94.65
63.02	96.08	0.002	0.984	94.66
66.34	96.08	0.005	0.982	94.67
69.83	96.08	0.014	0.976	94.72
73.51	96.08	0.034	0.961	94.85
77.38	96.08	0.071	0.933	95.12
81.45	96.08	0.135	0.882	95.67
85.74	96.08	0.229	0.802	96.65
90.25	96.08	0.353	0.691	98.24
95.00	96.08	0.495	0.558	100.59
100.00	96.08	0.638	0.416	103.79
105.00	96.08	0.755	0.288	106.98
110.25	96.08	0.847	0.183	110.94
115.76	96.08	0.911	0.105	115.61
121.55	96.08	0.951	0.055	120.89
127.63	96.08	0.973	0.026	126.67
134.01	96.08	0.984	0.011	132.87
140.71	96.08	0.988	0.004	139.46
147.75	96.08	0.990	0.001	146.42
155.13	96.08	0.991	0.000	153.73
162.89	96.08	0.991	0.000	161.41

Consideration of the strategies outlined in Figure 9.20 reveals that there are a number of different methods of approaching the portfolio insurance problem. In addition to strategies already examined such combining stocks with purchased puts or dynamically replicating such positions, a number of the remaining attempt to achieve the same objectives by using derivative positions as surrogates for stock ownership, on the presumption that there are execution advantages (e.g., greater liquidity and lower transactions costs) to using derivatives. For example, instead of owning stock, it is possible to form portfolios composed of purchased call options and fixed income securities (recall Property 10 of Sec. 7.1). Another strategy would alter the dynamic replication strategies by substituting short futures positions for the stock sales required when the value of the stock position declines. However, if this approach is used, for example, to insure an index fund, it is important to recognize that the number of stock index futures to be shorted for a given long index fund will be different than in the case where the stock position is being sold directly (and invested in bonds). In addition, there is the mechanical problem of calculating the dollar equivalency hedge ratio for the futures and cash positions.

The relevant issues surrounding which is the most appropriate portfolio insurance technique to use in a given situation is addressed in a number of sources, e.g., Bookstaber and Langsam (1988). For example, in considering whether to short the futures or write an option, both of which will require periodic rebalancing, the following Proposition applies: "An option will be more efficient than a futures if the gamma of the hedge program is greater than 1/2 the gamma of the hedging instrument." If the stock position is sufficiently large, there are distinct advantages to combining a number of different approaches to portfolio insurance, in order to exploit the benefits of each approach. Further, in keeping with the discussion in Sec. 9.2 about achieving multiple delta-theta-gamma targets, the use of more than one type of insurance strategy will likely be required in order to achieve the desired objectives. As mentioned in Sec. 8.4, for the case of hedging foreign currency options, delta + vega positions may provide better results than delta + gamma positions. This result almost certainly extends to insuring stock positions, though that point will not be developed here.

Whalley and Stoll (1991, p.361-4) has a helpful tabular presentation of how various methods of portfolio insurance would perform to insurance a stock index across a range of index levels (see Figure 9.21). The first table gives payoff for the path independent strategy, $S + P$. Comparing the distribution of S with $S + P$ reveals, in a simple tabular form, what Bookstaber and Clark (1983) have examined in much more general detail using distributional plots. The addition of a put transforms the symmetric S distribution to a positively skewed $S + P$ distribution. In terms of the return distribution, the additional cost of the put will result is a lower mean value for the $S + P$ distribution.

Theoretical application of the dynamic replication portfolio insurance strategy derived from $S + P = S\,N[d_1] + X\,e^{-rt^*}\ (1 - N[d_2])$ requires continuous trading. In practice, dynamic replication faces substantive implementation issues. Trading cannot be conducted under the perfect markets, continuous trading assumptions required for the Black-Scholes formula to capture the price of the option. Nevertheless, assuming that the Black-Scholes assumptions apply, permits the decomposition of $S + P$ into the exact holdings stocks and bonds to hold in order to precisely replicate the $S + P$ payoff. As the stock index level falls away from $S = 100$, the stock index position will be continuously reduced to the point where the stock position is nearly zero at $S = 59.87$. Similarly, as the stock index rises, the bond is sold to the point where at $S = 162.89$, there are no funds left in bonds. From this, it is apparent how dynamic portfolio insurance strategies, if applied by a large enough fraction of market traders, would amplify market movements.

In practice, dynamic trading strategies have to deal with the realities of discrete trading. Rules have to be determined about how large a movement in S is required before the rebalancing decision is executed. There are a number of possible methods of specifying a rebalancing trigger value. The Whalley and Stoll example assumes that the trigger value is 5%. From this point, the tabular presentation method can only provide an accurate picture of the distribution of weights, and the associated impact on portfolio value. For example, upside movements of S will produce increasing weights for S that lag the continuously rebalanced weights, resulting in a slight reduction in portfolio value. A similar result happens for downside movements of S where the reduction in S weights lacks the continuously rebalanced weights, again resulting in a slight reduction in portfolio value. Hence, the simple introduction of discrete rebalancing results in a deterioration of the performance of the dynamic replication strategy.

As it turns out, the discretely rebalanced case has considerably more complications than can be captured in one table. Being ***path dependent***, the terminal portfolio value can take a range of values, depending on the particular time path realized by *S*. For the ***path independent*** cases, $S + P$ and continuous rebalancing, the distribution of portfolio value can be determined precisely because the terminal portfolio value does not depend on the particular time path realized by *S*. This does not happen with discrete rebalancing. For example, a price path that starts at 100 and goes to 95 generates a rebalancing involving a sale of *S* to produce a weight change of .638 to .495. If the next step is back to 100, the rebalancing involves the weight returning from .495 to .638. The resulting portfolio value will now be less than a portfolio value along a price path where *S* was unchanged and no rebalancing happened.

Whalley and Stoll also provide results for the case where a stock/bond portfolio is created using the dynamic portfolio weights but no rebalancing is done along the time path. This is another type of path independent strategy. Though not immediately apparent from the tabular presentation, the distribution of the portfolio value for the no rebalancing case is not unlike the *S* distribution. Unlike the dynamically traded portfolio, the no rebalancing distribution retains the symmetric shape of the *S* distribution, though there is less dispersion due to the presence of a long investment in the riskless asset.[9] In all of this, the no rebalancing and static portfolio insurance cases are being unfavorably compared with the discrete rebalancing case because there are no transactions costs factored into the various calculations. In the limit, continuous rebalancing with transactions costs, the dynamic strategies may produce infinite losses.

In practice, there will be a tradeoff between rebalancing frequency, the various transactions and execution changes and the terminal value of the insured portfolio. Wider rebalancing frequencies will permit greater deviation from the path independent static portfolio insurance case, but this loss of precision will be balanced out with a savings in transactions costs due to reduced trading frequency. All this leads back to Sec. 6.2 and the problem of specifying an optimal trigger value for speculative trading strategies. This connection between speculative trading strategies and dynamic traded portfolio insurance strategies is revealing, providing another illustration of the systemic connection between risk management and speculation in the analysis of derivative security applications.

Insuring Portfolios with Foreign Assets

There is nothing unique about a portfolio of domestic stocks. The notions of portfolio insurance can be applied to any commodity. One useful extension involves insuring the domestic currency value of a foreign bond position. Much as with dynamic portfolio insurance for stocks, dynamic portfolio insurance for foreign bonds can be derived using put-call parity for currency options. The objective is to dynamically trade a portfolio composed of domestic bonds and foreign bonds in order to achieve the same payout as a path independent portfolio composed of a foreign bond plus a currency put option. If the exchange rate increases, the value of the domestic currency rises relative to foreign currency, then the dynamic strategy involves selling foreign bonds and buying domestic bonds. If the exchange rate deteriorates, the domestic bond is sold in favour of buying the foreign bond. As before, the Black-Scholes formula for a call can be substituted into the put-call parity condition to derive the appropriate portfolio weights.

To see this consider the path independent value of a portfolio that contains a foreign currency bond which has the domestic currency value protected with a currency put option. The associated dynamic replication portfolio can new be derived:

$$V = S \exp\{-r_f\, t^*\} + P = C + X \exp\{-r\ t^*\}$$

$$= S \exp\{-r_f\, t^*\}\ N[d_1] - X \exp\{-r\ t^*\}\ N[d_2] + X \exp\{-r\ t^*\}$$

$$= S \exp\{-r_f\, t^*\}\ N[d_1] + X \exp\{-r\ t^*\}\ (1 - N[d_2])$$

In this formulation, $S\ exp\{-r_f t^*\}$ is the domestic currency value of the foreign bond position and $X\ exp\{-rt^*\}$ is the domestic currency value of the domestic bond. More precisely, $exp\{-r_f t^*\}$ is the foreign currency value of a continuously compounded zero coupon bond that matures to one unit of domestic currency; multiplying this foreign bond price by *S*, the time t spot exchange rate expressed in domestic direct terms, converts the foreign bond price to units of domestic currency. Similarly, *X exp{-rt*}* is the domestic currency value of a continuously compounded zero coupon bond that matures to the number of units of domestic currency reflected in the exercise price for the currency option.

As with portfolio insurance for stocks, portfolio insurance for foreign bonds involves dynamic trading of the position. When the value of domestic currency rises relative to foreign currency, S will fall and the dynamic strategy requires selling a portion of the foreign bond and using the funds to purchase domestic bonds. The dynamic replication formulation identifies the precise amount of foreign bonds that have to be sold in order to maintain the same payout as the path independent portfolio $[S \exp\{-r_f t^*\} + P]$. While much the same theoretically, dynamic replication for foreign bonds can differ in practice. Unlike the dynamic strategies for stock portfolios, which can suffer from inaccurate replication due to illiquidity in the underlying stocks, the cash markets involved in the dynamic insurance for foreign bonds are typically liquidity. The foreign exchange market, as well as the domestic and foreign bond markets, are unlikely to be subject to the types of pricing discontinuities that precipitated the October 1987 market break.

9.4 Optimal Stopping and Perpetual Options[10]

Perpetual Options

A perpetual option has two features of interest. There is no stated expiration date and, because of this, the option must be American. A perpetual option is alive until it is exercised. As discussed in Sec. 7.4, practical applications of perpetual options have been made to real investment decisions, such as real estate development, e.g., Capozza and Li (1994). The perpetual option is of analytical significance because, as first demonstrated by McKean (1965), it is an American option with a valuation formula. This occurs because the infinite expiration date produces a free boundary condition for the American option valuation problem. This permits the intractable PDE problem to be converted to a tractable ODE problem. A number of extensions of the perpetual option have been proposed, such as the Russian option that has a payoff determined by the difference between the exercise value and the maximum value that the stock price achieves over the life of the option. As discussed by Gerber and Shu (1994) and Gerber (1999), there are a number of possible methods to derive the price formula for the perpetual option.

Stopping Rules

A stopping rule problem is a type of optimal control problem where the decision maker has to decide whether to "stop" or "continue" a particular activity. Stopping rule problems occur in many forms in Finance. The simplest forms occur in simple gambling problems. For example, in a coin-flipping game, the stopping rule problem is concerned with when is the best time to leave the game in order to maximize the expected reward from playing the game. The valuation of contingent claims, such as options, provides a number of stopping rule problems. American options pose an obvious stopping rule problem: when is the best time to exercise an American put or call option? Because the concern is typically with identifying the best time to stop, the theory of ***optimal*** stopping was developed.

There are a number of possible approaches to motivate optimal stopping problems. One approach is to consider optimal stopping to a type of decision making under uncertainty problem, e.g., Dixit and Pindyck (1994). In this context, optimal stopping can be formulated as a dynamic programming problem involving a binary choice, stop or continue, over all future decision dates. Optimal stopping solutions can then be solved using the appropriate Bellman equation. Another approach considers optimal stopping as an applications of martingale methods, e.g., Karlin and Taylor (v.1). This leads to introduction of ***Markov time*** and application of the ***optional sampling theorem*** and the related ***optional stopping theorem*** (Karlin and Taylor, v.1, Sec. 6.3 and 6.4) to solve optimal stopping problems. In some cases, solutions can be further refined by assuming diffusions and computing the appropriate derivatives of functionals of Brownian motion.

An important feature of optimal stopping problems typically encountered in Finance is the ***smooth pasting condition*** (Dixit 1993). In various sources, this condition is also referred to as the ***high contact condition*** (Samuelson 1965) or ***principle of smooth fit*** (Shepp and Shirayev 1983). The smooth pasting condition is derived by recognizing that the optimal stopping problem requires the identification of values of the state variable that involve either stopping or continuing. This leads to the derivation of an ***optimal exercise boundary*** using first order conditions of the value functions associated with the exercise decision. These first order conditions form the smooth pasting conditions that the optimal stopping problem must satisfy. In the case of an American option, the valuation problem that has to be solved is subject to this optimal exercise boundary. Because the applicability of this boundary condition usually depends on time, a so-called ***free boundary***, the American option pricing problem is a free boundary value problem. Such problems almost always are not possible to solve in closed form.

Pricing Perpetual Options

The perpetual option is an optimal stopping problem that does permit the derivation of a closed form solution for a type of American option. This is because in the perpetual case the optimal exercise boundary does not vary with time. While a number of possible solution procedures have been proposed for the perpetual option, e.g., Ingersoll (1987, p.375), Kim (1990), Jacka (1991), the martingale approach is both expedient and revealing, e.g., Gerber and Shu (1994). The solution depends on evaluating the Laplace transform of the first passage time to an appropriate exercise boundary. The smooth pasting condition is then derived by maximizing the conditional expected value of the profit function with respect to the stock price that defines the exercise boundary. It is this derivative that can be evaluated precisely because the exercise boundary for the perpetual is not dependent on time. The solution to this optimization is the perpetual option price.

The optimal stopping problem for the perpetual call starts by specifying a price *S* (> X)* where exercise of the perpetual option will occur. The optimal *S** is a value to be determined later. Initially, the value of *S** is only restricted to be > *X*. Let the ***first passage time*** to the barrier imposed by *S** be specified as *T[S*]*. In other words, if the process is initially at *S(0)*, then *T[S*]* is the time that the process takes to first get to *S**. This first passage time will have a distribution that will depend on *S(0), S** and the stochastic process assumed for *S(t*). If the underlying stock ***price*** process (where *dS/S* = *: dt* + *FdW*) grows at the per-unit-time rate *" = : + 1/2 F² > 0*, then let *2 = 8: + 1/2 8² F² > "*.

The parameter *2* can be interpreted as the discount rate associated with funds invested in the perpetual option. It is the discount rate that is used to calculate the discounted value of expected profit from exercising the option. The condition *2* > *"* follows because direct ownership of the stock is preferable to indirect ownership through an option. In particular, direct ownership of the stock entitles the holder to receive dividends. The case where the stock pays no dividends produces a limiting solution where the optimal exercise boundary is at infinity and the perpetual call option is never exercised.

The solution to the optimal stopping problem proceeds by applying the following Laplace transform solution applicable to the first passage of a Brownian motion process with drift *: $ 0* and variance *F²*. *T[z*] is the first passage time for the Brownian process, starting from *x $ 0* to reach the level *z* > *x* (Karlin and Taylor, v.1, p.362):

$$E[e^{-\theta\, T[z]}] = e^{-\frac{z-x}{\sigma^2}\left(\sqrt{\mu^2 + 2\sigma^2\,\theta} - \mu\right)}$$

The definition of *2* is the same as that given above. To apply this Laplace transform solution to the case of a stock price that follows geometric Brownian motion, make the substitutions *z = ln[S*] and x = ln[S(0)]* to produce the result:

$$E[e^{-\theta\, T[S^*]}] = \left[\frac{S(0)}{S^*}\right]^{\rho} \qquad \textit{where}: \quad \rho = \frac{1}{\sigma^2}\left(\sqrt{\mu^2 + 2\sigma^2\,\theta} - \mu\right)$$

This first passage result can now be applied to calculating the discounted expected profit.

The risk-neutral valuation approach to pricing options now has an immediate application:

$$C_{\perp}[S(t),S^*] = E\left[e^{-\theta\, T[S^*]}\,[S^* - X]\right] = [S^* - X]\; E\left[e^{-\theta\, T[S^*]}\right] = [S^* - X]\left[\frac{S(0)}{S^*}\right]^{\rho}$$

This valuation is aided considerably by the arbitrary *S**. This condition can now be used to solve for the smooth pasting condition. This is done by differentiating *C[S(t),S*]* with respect to *S**:[11]

$$\frac{\partial C}{\partial S^*} = -\rho \; (S^{*+} - X) \left[\frac{S(0)}{S^{*+}}\right]^{\rho + 1} \frac{1}{y} + \left[\frac{S(0)}{S^{*+}}\right]^{\rho}$$

where S^{*+} is the optimal price to exercise the option and, as before, *S(0)* is the current stock price. Solving for S^{*+} and substituting back into the $C[S^{*+},S(0)]$ gives the desired solutions:

$$S^{*+} = \frac{X \rho}{\rho - 1}$$

$$C[S^{*+}, \; S(0)] = \frac{X}{\rho - 1} \left[\frac{S(0) \; (\rho - 1)}{X \rho}\right]^{\rho}$$

This is the solution for the price of the perpetual call.

To see the types of solutions produced by the perpetual call formula consider the following examples:

S(0) = 50 X = 50 **:** *= 0* ***F*** *= .1* ***2*** *= .06* ---> S^{*+} *= 70.2914* C_{perp} *= 6.23534*
S(0) = 50 X = 50 **:** *= .3* ***F*** *= .1* ***2*** *= .06* ---> S^{*+} *= 135.826* C_{perp} *= 17.65*
S(0) = 50 X = 50 **:** *= .055* ***F*** *= .1* ***2*** *= .06* ---> S^{*+} *=> infinity* C_{perp} *= 50.*

It is possible to show cases where *S(0)* is above the optimal exercise boundary, in which case the option would be exercised immediately and C = *S(0) - X.*

Questions

1. Explain how the Black-Scholes model can be used to structure to structure portfolios containing options. What is meant by the delta, theta and gamma of a position and how are these concepts used in portfolio design?

2. What is portfolio insurance and what role do stock index derivatives play in insuring portfolios? What role did stock index derivatives play in the October 1987 market break? Identify and explain some factors that restrict the execution of stock index futures and option arbitrages.

3. If the risk of a stock increases, what is likely to happen to the prices of call options on the stock? To the prices of put options? What happens if interest rates change? Explain.

4. Explain how to construct an insured stock portfolio that uses a combination of long in-the-money, at-the-money, and out-of-the-money put options, where **)**$_v$ > 0 and **'**$_v$ = 0.

5. Calculate the **)**, **'** and **2** for the long/short stock replication strategies discussed in Chapter 7. Consider the cases, X < S, X = S and X > S.

6. Derive the closed form for theta, the sensitivity of the call option price to changes in time. Evaluate theta for a wide range of values of t.

7. Using the results provided in Appendix III, derive the appropriate formulae for the Greeks relevant for puts.

8. Derive the delta plus gamma conditions for the ratio spread and discuss the advantages of constructing the trade with more than two options.

NOTES

Applications of Option Valuation Concepts

1. The material in this Section is covered in more detail in Cox and Rubinstein (1985, Chap.6), Hull (1989, Chap.8) and Gibson (1991, Chap.4). The former reference, provides an important special case of the calculus chain rule that is required to simplify the evaluation of the derivatives:

$$\frac{\partial N[z]}{\partial v} = N'[z] \, \frac{\partial z}{\partial v}$$

where N'[z] is the standard normal density evaluated for the argument z. More precisely:

$$N'[z] = \frac{1}{\sqrt{2\pi}} \exp\{\frac{-z^2}{2}\}$$

2. In some presentations, the elasticity rather than the derivative is of interest, e.g., the call price elasticity with respect to stock price changes is:

$$\eta_{C,S} = \frac{S}{C}\frac{\partial C}{\partial S}$$

The advantage of using elasticity is that the elasticity value is normalized which the derivative is not. The elasticity form makes it easier to interpret a calculated value. This approach will not be used here.

3. For options written on spot commodities (futures), the stock position can be replaced the spot (futures) and, as with stocks, the delta is one in this case also.

4. One immediate method of creating a hedge portfolio is to combine a short (long) stock index futures position with a long (short) position in the stock index. When the size of the futures and stock index position are equal, this produces:

$$\Delta_V = 1 - \frac{\partial F}{\partial S}$$

Provided the stock position of interest is the deliverable, this position will be delta neutral. Where cross hedges and other sources of slippage are present, adjustment would have to be made to achieve delta neutrality.

5. In effect, insured long positions required the delta of the portfolio to be positive and the gamma to be equal to zero. Over time, this will require rebalancing the derivative positions.

6. This example is stylized because, in general, it is not possible to hedge a given stock position with a futures contract. This is because futures contracts are only available for stock indices. However, in addition to positions in the stock index, the analysis does apply where a relevant spot position is being hedged.

7. Derivatives here does not include the possibility of using futures contracts where, in combination with a long stock position, position deltas and gammas are zero.

8. The terms written and purchased in this case refer to whether a net cash flow is derived from premiums. Typically, the bulk of the premium income is associated with the option exhibiting the lower exercise price.

9. This observation provides a window into the various complications that non-linear payoffs, such as options, can have for mean-variance optimization analysis.

10. Early seminal contributions on this subject are Chow, Robbins and Siegmund (1971) and de Groot (1970). A useful summary of the theory can be found in Karlin and Taylor, v.1 and a treatment oriented to Economics in Dixit (1993) and Dixit and Pindyck (1994).

11. To do this derivative it is helpful to observe that the discounted expected profit can be reformulated as:

$$C[S^*, \, S(0)] = S(0)^{\rho} \; S^{*(1 - \rho)} - X \; (S(0)^{\rho} \; S^{*-\rho})$$