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Permutation Factorizations and Prime Parking Functions

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Abstract. Permutation factorizations and parking functions have some parallel properties. Kim and Seo exploited these parallel properties to count the number of ordered, minimal factorizations of permutations of cycle type (n) and $(1, n-1)$. In this paper, we use parking functions, new tree enumerations and other necessary tools, to extend the techniques of Kim and Seo to the cases $(2, n-2)$ and $(3, n-3)$.

Keywords: permutation factorizations, trees, parking functions

1. Introduction

For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n, denoted $\lambda \vdash n$, let C_λ be the conjugacy class in $\mathfrak{S}_n$ consisting of elements of cycle type λ and let $\pi(\lambda)$ be the element $(1\,2 \cdots \lambda_1)(\lambda_1 + 1 \cdots \lambda_1 + \lambda_2) \cdots (\lambda_1 + \cdots + \lambda_{\ell-1} + 1 \cdots \lambda_1 + \cdots + \lambda_\ell)$ of C_λ. Let F_λ be the set of m-tuples of transpositions $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ such that

(1) $\sigma_1 \sigma_2 \cdots \sigma_m = \pi(\lambda)$,
(2) the σ_i generate $\mathfrak{S}_n$,
(3) $m = n + \ell - 2$; that is, m is minimal subject to (1) and (2) (see [4, Proposition 2.1]).

Elements of F_λ are called *minimal transitive factorizations* of $\pi(\lambda)$, or simply of λ by symmetry.

In [4], Goulden and Jackson prove

$$|F_\lambda| = m!\, n^{\ell-3} \prod_{i=1}^{\ell} \frac{\lambda_i^{\lambda_i}}{(\lambda_i - 1)!}. \tag{1.1}$$

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The proof, however, is not combinatorial. There have been several combinatorial proofs for the case $\lambda = (n)$ (see [5, 6, 9]). Recently, Kim and Seo [7] gave a combinatorial proof of the case $\lambda = (n)$ and $(1, n-1)$ using parking functions. Here, we use parking functions and develop the necessary enumerative methodology to extend the approach of Kim and Seo to give combinatorial proofs of (1.1) for $\lambda = (2, n-2)$ and $(3, n-3)$. In particular, we make heavy use of labelled tree enumerations that do not seem to have appeared elsewhere. The question of how far parking functions and other combinatorial objects can be used to enumerate minimal transitive factorizations naturally arises.

This paper is organized in the following way. The case $\lambda = (2, n-2)$ is considered first, with all details provided, in Sections 2 to 5. Then the case $(3, n-3)$ is considered in Sections 6 to 8. For this latter case, a number of details are omitted where they are similar to those given in the case $(2, n-2)$.

2. Preliminaries for the Case $(2, n-2)$

In this paper we use the convention that permutations are multiplied right to left. Let $\sigma = (i\,j)$ be a transposition and α a permutation in $\mathfrak{S}_n$. The product $\sigma\alpha$ can take one of two forms. If i and j are in the same cycle in α (in the disjoint cycle representation of α) then in $\sigma\alpha$ the elements i and j will be in different cycles. If i and j are in different cycles in α then in $\sigma\alpha$ the elements i and j will be in the same cycle. In the former case we call σ a *cut* of α and in the latter case we call it a *join* of α. In the case $\lambda = (2, n-2)$, we see $\pi(\lambda) = (1\,2)(3\,4\cdots n)$ and if $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in F_\lambda$ then $m = n$, since the factorization σ has precisely $(n-1)$ joins and a unique cut. Further, one can see the cut must be of the form $(1\,r)$ or $(2\,r)$, for $3 \le r \le n$. Let $\mathcal{G}_{n,k,r}$ be the subset of $F_{(2,n-2)}$ such that $(1\,r)$ is the cut and it is in the k-th position; that is, if $(\sigma_1, \sigma_2, \ldots, \sigma_m) \in \mathcal{G}_{n,k,r}$ then $\sigma_k = (1\,r)$ and σ_k is the cut. Similarly, let $\mathcal{H}_{n,k,r}$ be the subset of $F_{(2,n-2)}$ such that if $(\sigma_1, \sigma_2, \ldots, \sigma_m) \in \mathcal{H}_{n,k,r}$ then $\sigma_k = (2\,r)$ is the cut. The following is an easy observation.

Proposition 2.1. *For all* $3 \le r \le n$, $|\mathcal{G}_{n,k,r}| = |\mathcal{H}_{n,k,r}|$.

Proof. One can verify the function $\theta\colon \mathcal{G}_{n,k,r} \longrightarrow \mathcal{H}_{n,k,r}$ defined by

$$\theta(\sigma_1, \sigma_2, \ldots, \sigma_m) = (\sigma'_1, \sigma'_2, \ldots, \sigma'_m),$$

where $\sigma'_i = (1\,2)\,\sigma_i\,(1\,2)$ is a bijection. ∎

We can also say something about the relationships of the cardinality of $\mathcal{G}_{n,k,r}$ for different r.

Proposition 2.2. $|\mathcal{G}_{n,k,r}| = |\mathcal{G}_{n,k,s}|$ *for all* $3 \le r, s \le n$.

Proof. Let $\theta = (3\,4\cdots n)$ and define $\Theta_{s-r}\colon \mathcal{G}_{n,k,r} \longrightarrow \mathcal{G}_{n,k,s}$ as

$$\Theta_{s-r}(\sigma_1, \sigma_2, \ldots, \sigma_m) = (\sigma'_1, \sigma'_2, \ldots, \sigma'_m),$$

where $\sigma'_i = \theta^{s-r}\,\sigma_i\,\theta^{r-s}$. One can easily verify that Θ_{s-r} is a bijection. ∎

A similar proof of a similar fact can be found in [7, p. 4].

At this time we note that for $\lambda = (2, n-2)$ the cardinality in (1.1) becomes

$$|F_\lambda| = 4(n-1)(n-2)^{n-1}. \tag{2.1}$$

Considering the $\mathcal{G}_{n,k,r}$ and $\mathcal{H}_{n,k,r}$, the parameter r can take any value between 3 and n, i.e., r can take any one of $n-2$ values implying there are a total of $2(n-2)$ sets $\mathcal{G}_{n,k,r}$ and $\mathcal{H}_{n,k,r}$. Thus, if we can count $\mathcal{G}_{n,k,r}$ for some fixed r, we can multiply this number by $2(n-2)$ to obtain the total number of factorizations in $F_{(2,n-2)}$. Here, we choose to enumerate $\mathcal{G}_{n,k,3}$ and we will drop the subscript '3', i.e., we use $\mathcal{G}_{n,k}$ to denote $\mathcal{G}_{n,k,3}$ in the rest of the paper. Therefore, setting $\mathcal{G}_n = \dot{\cup}_k \mathcal{G}_{n,k}$, where $\dot{\cup}$ denotes disjoint union, it follows from Propositions 2.1 and 2.2 that in order to prove (2.1) it suffices to show

$$|\mathcal{G}_n| = 2(n-1)(n-2)^{n-2}. \tag{2.2}$$

3. Parking Functions

A *parking function* is a sequence $(a_1, a_2, \dots, a_n)$ of positive integers such that there exists a permutation ω with $a_{\omega(i)} \le i$. Let $\mathcal{P}_n$ be the set of all parking functions of length n. A parking function $(a_1, a_2, \dots, a_n)$ is called *prime* if it satisfies the stronger condition that there is a permutation ω that satisfies $a_{\omega(1)} = 1$ and $a_{\omega(i)} \le i-1$ for all $i \ge 2$. In keeping with the notation in [7], let $\mathcal{Q}_{n,k}$ be the set of prime parking functions of length n where the left-most 1 appears in the k-th position. Further, in this paper we will need the set $\widehat{\mathcal{Q}}_{n,k}$, the set of prime parking functions in $\mathcal{Q}_{n,k}$ of length n where no 2 appears in the first k positions; that is, no 2 appears to the left of the left-most 1. Finally, set

$$\mathcal{Q}_n = \dot{\bigcup}_k \mathcal{Q}_{n,k},\ \widehat{\mathcal{Q}}_n = \dot{\bigcup}_k \widehat{\mathcal{Q}}_{n,k}. \tag{3.1}$$

Suppose that $(\sigma_1, \sigma_2, \dots, \sigma_m) \in \mathcal{G}_{n,k}$. Since 1 and 2 are in the same cycle in $(1\,2)(3\,4 \cdots n)$, some join, call it τ, will put them in the same cycle. There are two cases:

Case 1. the join τ occurs after the cut (that is, the join τ is one of $\sigma_1, \sigma_2, \dots, \sigma_{k-1}$). We will refer to this subset of $\mathcal{G}_{n,k}$ as $\mathcal{G}^1_{n,k}$.

Case 2. the join τ occurs before the cut (that is, the join τ is one of $\sigma_{k+1}, \sigma_{k+2}, \dots, \sigma_n$). We will refer to this subset of $\mathcal{G}_{n,k}$ as $\mathcal{G}^2_{n,k}$.

The seemingly reversed choice of words 'before' and 'after' are to keep consistent with our convention of multiplying permutations right to left.

These cases are dealt with separately. In the next proposition, we find a useful relationship between $\mathcal{G}^1_{n,k}$ and $\mathcal{Q}_{n,k}$.

Proposition 3.1. *The cardinality of the set* $\mathcal{G}^1_{n,k}$ *is* $(k-1) \cdot |\mathcal{Q}_{n-1,k-1}|$.

Proof. In Case 1 above, the join τ that unites 1 and 2 into the same cycle occurs after the cut. We can say a fair amount about the structure of such permutations using some simple observations. Notice that after the cut $(1\,3)$ has been applied to $\sigma_{k+1} \cdots \sigma_n$, the element 1 appears in a 1-cycle until it is joined to 2 by the transposition τ. From this we conclude $\tau = (1\,2)$. Also, since 2 only appears in τ after the cut and 2 never appears before the cut, the only transposition in which 2 appears is τ. Therefore, if $\sigma_1, \sigma_2, \ldots, \sigma_n \in \mathcal{G}^1_{n,k}$, then the sequence $(\sigma_1, \sigma_2, \ldots, \sigma_n) - \tau$ (that is, the sequence $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ with the transposition τ removed) is a factorization of $(1)(3\,4\cdots n)$ (it is minimal because there are $n-1$ factors and it is clearly transitive). Furthermore, τ is the only transposition after the cut $\sigma_k = (1\,3)$ where the elements 1 and 2 appear implying τ commutes with every transposition after the cut; that is, τ commutes with $\sigma_1, \sigma_2, \ldots, \sigma_{k-1}$. Hence, the number of factorizations in $\mathcal{G}^1_{n,k}$ is $k-1$ times the number of factorizations (minimal, transitive) of $(1)(3\,4\cdots n)$ with cut $(1\,3)$ in the $(k-1)$-th position. From [7, Theorem 5] the number of such factorizations of $(1)(3\,4\cdots n)$ is equal to the number of elements in $\mathcal{Q}_{n-1,k-1}$, completing the proof. ∎

We now deal with Case 2 above; we will find a similar expression for $\mathcal{G}^2_{n,k}$ in terms of $\widehat{\mathcal{Q}}_{n,k}$. Define $\mathcal{A}_{n,k}$ to be the factorizations $(\sigma_1, \sigma_2, \ldots, \sigma_{n-1})$ in $F_{(n)}$ such that

- $\sigma_1, \sigma_2, \ldots, \sigma_{k-1}$ does not contain a 2 or a 3, and
- $\sigma_k, \sigma_{k+1}, \ldots, \sigma_{n-1}$ does contain a 1.

Define a function $\phi\colon\ \mathcal{G}^2_{n,k} \longrightarrow \mathcal{A}_{n,k}$ by

$$\phi(\sigma_1, \sigma_2, \ldots, \sigma_n) = (\sigma'_1, \ldots, \sigma'_{k-1}, \sigma_{k+1}, \ldots, \sigma_n), \tag{3.2}$$

where $\sigma'_i = (1\,3)\,\sigma_i\,(1\,3)$

Lemma 3.2. *The function ϕ defined in* (3.2) *is a bijection, implying* $|\mathcal{G}^2_{n,k}| = |\mathcal{A}_{n,k}|$.

Proof. First, we verify that ϕ is well-defined. Suppose that $(\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathcal{G}^2_{n,k}$ and $(\sigma'_1, \ldots, \sigma'_{k-1}, \sigma_{k+1}, \ldots, \sigma_n) = \phi(\sigma_1, \sigma_2, \ldots, \sigma_n)$. Then

$$\begin{aligned}(1\,2\cdots n) &= (1\,3)(1\,2)(3\,4\cdots n)\\ &= (1\,3)\,\sigma_1\sigma_2\cdots\sigma_{k-1}(1\,3)\sigma_{k+1}\cdots\sigma_n\\ &= \sigma'_1\sigma'_2\cdots\sigma'_{k-1}\sigma_{k+1}\cdots\sigma_n.\end{aligned}$$

Hence, the product of $(\sigma'_1, \ldots, \sigma'_{k-1}, \sigma_{k+1}, \ldots, \sigma_n)$ is $(1\,2\cdots n)$ and contains $n-1$ factors, forcing the factorization to be transitive and minimal. Furthermore, since $(\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathcal{G}^2_{n,k}$, 1 must occur in one of $\sigma_{k+1}, \sigma_{k+2}, \ldots, \sigma_n$ (otherwise $\sigma_k = (1\,3)$ would be a join) and, clearly, 1 nor 2 appears in any of $\sigma_1, \sigma_2, \ldots, \sigma_{k-1}$. Thus, 2 nor 3 appears in $\sigma_1, \sigma_2, \ldots, \sigma_{k-1}$.

Next, define $\hat{\phi}\colon\ \mathcal{A}_{n,k} \longrightarrow \mathcal{G}^2_{n,k}$ by

$$\widehat{\phi}(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}) = \left(\sigma'_1, \ldots, \sigma'_{k-1}, (1\,3), \sigma_k, \ldots, \sigma_{n-1}\right),$$

where $\sigma'_i = (1\,3)\,\sigma_i\,(1\,3)$. First, we note that $\widehat{\phi}$ is well defined. Indeed, if

$$\left(\sigma'_1, \ldots, \sigma'_{k-1}, (1\,3), \sigma_k, \ldots, \sigma_{n-1}\right) = \widehat{\phi}(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}),$$

then

$$(1\,3)\,\sigma_1'\cdots\sigma_{k-1}'\,(1\,3)\,\sigma_k\cdots\sigma_{n-1} = \sigma_1\sigma_2\cdots\sigma_{n-1} = (1\,2\cdots n)$$

implying

$$\sigma_1'\cdots\sigma_{k-1}'\,(1\,3)\,\sigma_k\cdots\sigma_{n-1} = (1\,2)(3\,4\cdots n).$$

Further, since $\big(\sigma_1',\ldots,\sigma_{k-1}',(1\,3),\sigma_k,\ldots,\sigma_{n-1}\big)$ has n factors, one of which is $(1\,3)$, it must be minimal and transitive. Also, since 2 nor 3 occurs in $\sigma_1,\sigma_2,\ldots,\sigma_{k-1}$, 1 nor 2 occurs in $\sigma_1',\sigma_2',\ldots,\sigma_{k-1}'$, then 1 and 2 are joined in $\sigma_k,\ldots,\sigma_{n-1}$ and the unique cut cannot be in $\sigma_1',\sigma_2',\ldots,\sigma_{k-1}'$. Finally, the fact that $(\sigma_1,\sigma_2,\ldots,\sigma_{n-1})\in F_{(n)}$ leads to the transpositions $\sigma_k,\ldots,\sigma_{n-1}$ are not cuts. Thus, $(1\,3)$ must be the cut in $\big(\sigma_1',\ldots,\sigma_{k-1}',(1\,3),\sigma_k,\ldots,\sigma_{n-1}\big)$.

An easy computation will verify that $\widehat{\phi}$ is the inverse of ϕ, so ϕ is a bijection. ∎

In [7, Corollary 4] Kim and Seo prove the map from $\Phi\colon F_{(n)}\longrightarrow \mathcal{P}_{n-1}$ (the set of parking functions of length $n-1$) defined by

$$\Phi(\sigma_1,\sigma_2,\ldots,\sigma_{n-1}) = (a_1,a_2,\ldots,a_{n-1}), \tag{3.3}$$

where $\sigma_i=(a_i\,b_i)$ and $a_i<b_i$, is a bijection. Define $\psi\colon \mathcal{A}_{n,k}\longrightarrow \widehat{\mathcal{Q}}_{n,k}$ by

$$\psi(\sigma_1,\sigma_2,\ldots,\sigma_{n-1}) = (\widehat{a}_1,\ldots,\widehat{a}_{k-1},1,a_k,\ldots,a_{n-1}), \tag{3.4}$$

where

(1) $\sigma_i=(a_i\,b_i)$ and $a_i<b_i$,
(2) $\widehat{a}_i=(1\,3)\,a_i$, i.e.,

$$\widehat{a}_i=\begin{cases}1, & \text{if } a_i=3,\\ 3, & \text{if } a_i=1,\\ a_i, & \text{otherwise.}\end{cases}$$

Proposition 3.3. *The function ψ defined in* (3.4) *is a bijection. Furthermore,*

$$\big|\mathcal{G}^2_{n,k}\big| = \Big|\widehat{\mathcal{Q}}_{n,k}\Big|,$$

i.e., the number of elements in $\mathcal{G}^2_{n,k}$ is the same as the number of prime parking functions of length n where the left most 1 *occurs in the k-th position and no* 2 *appears in the first k positions.*

Proof. Suppose that $(\sigma_1,\sigma_2,\ldots,\sigma_{n-1})\in\mathcal{A}_{n,k}$ and

$$(\widehat{a}_1,\ldots,\widehat{a}_{k-1},1,a_k,\ldots,a_{n-1}) = \psi(\sigma_1,\sigma_2,\ldots,\sigma_{n-1}).$$

If $\sigma_i=(a_i\,b_i)$ and $\widehat{a}_i=(1\,3)\,a_i$, by [7, Corollary 4] we have

$$\begin{aligned}
&\Longrightarrow (a_1,a_2,\ldots,a_{n-1})\in\mathcal{P}_{n-1}, && 2,3\notin a_1,\ldots,a_{k-1}\\
&\Longrightarrow (\widehat{a}_1,\ldots,\widehat{a}_{k-1},a_k,\ldots,a_{n-1})\in\mathcal{P}_{n-1}, && 1,2\notin \widehat{a}_1,\ldots,\widehat{a}_{k-1}\\
&\Longrightarrow (\widehat{a}_1,\ldots,\widehat{a}_{k-1},1,a_k,\ldots,a_{n-1})\in\widehat{\mathcal{Q}}_{n,k},
\end{aligned}$$

so ψ is well defined. Now, define $\widehat{\psi}\colon \widehat{\mathcal{Q}}_{n,k} \longrightarrow \mathcal{A}_{n,k}$ to be

$$\widehat{\psi}(a_1,\dots,a_{k-1},1,a_{k+1},\dots,a_n) = \Phi^{-1}(\widehat{a}_1,\dots,\widehat{a}_{k-1},a_{k+1},\dots,a_n),$$

where $\widehat{a}_i = (1\,3)a_i$ and Φ is defined in (3.3). It is clear that $\widehat{\psi}$ is the inverse of ψ, leading to ψ is bijective.

To prove $|\mathcal{G}^2_{n,k}| = |\widehat{\mathcal{Q}}_{n,k}|$, it follows from Lemma 3.2 that the composition $\psi \circ \phi$ is a bijection from $\mathcal{G}^2_{n,k}$ to $\widehat{\mathcal{Q}}_{n,k}$. ∎

Define

$$\mathcal{G}^i_n = \dot{\bigcup}_k \mathcal{G}^i_{n,k}, \tag{3.5}$$

for $i = 1, 2$. It is clear from the definitions that

$$\mathcal{G}_n = \mathcal{G}^1_n \,\dot{\bigcup}\, \mathcal{G}^2_n. \tag{3.6}$$

Propositions 3.1 and 3.3 give us alternative combinatorial interpretations for $\mathcal{G}^1_{n,k}$ and $\mathcal{G}^2_{n,k}$. In Sections 4.1 and 4.2 we will use these alternative interpretations for $\mathcal{G}^1_{n,k}$ and $\mathcal{G}^2_{n,k}$ to determine the quantities $|\mathcal{G}^i_n|$ for $i = 1, 2$ and use these values to determine $|\mathcal{G}_n|$.

4. Enumerating the Classes for the Case $(2, n-2)$

In this section, we enumerate the two sets $\mathcal{G}^1_n$ and $\mathcal{G}^2_n$. This is done in the following two subsections.

4.1. The Number of Elements in $\mathcal{G}^1_n$

We begin this section with a lemma that will be useful to enumerate $\mathcal{G}^1_{n,k}$.

Lemma 4.1. *The number of elements in* $\mathcal{Q}_{n,k}$ *is*

$$(n-k)n^{n-k-1}(n-1)^{k-2}.$$

Proof. First note that any prime parking function with i 1's is a permutation of a sequence of the form

$$(\underbrace{1,\dots,1}_{i},\underline{a}),$$

where $\underline{a} = a_1, a_2, \dots, a_{n-i}$ has the property that there exists a permutation ω such that $2 \le a_{\omega(j)} \le i + j - 1$. Using an argument due to Pollack (see [10, Solutions to Exercise 5.49b]. The argument was originally used by Pollack to enumerate the number of parking functions of length n), one finds the number of sequences $\underline{a}$ of length $n-i$ is $(i-1)(n-1)^{n-i-1}$. If we restrict the i 1's to be in the positions $k, k+1, \dots, n$ (one of the 1's is necessarily in the k-th position, guaranteeing that we have a prime parking function in $\mathcal{Q}_{n,k}$), then we see that the number of prime parking functions of this type of length n is

$$\binom{n-k}{i-1}(i-1)(n-1)^{n-i-1}. \tag{4.1}$$

Clearly, the index i can take the values from 2 to $n-k+1$. Summing (4.1) from $i = 2$ to $n-k+1$ gives the desired result. ∎

From Proposition 3.1 and Lemma 4.1 we see that

$$\begin{aligned}
\left|\mathcal{G}_n^1\right| &= \sum_{k=1}^{n-1} \left|\mathcal{G}_{n,k}^1\right| \qquad \text{(definition of } \mathcal{G}_n^1\text{)} \\
&= \sum_{k=1}^{n-1} (k-1)\cdot \left|Q_{n-1,k-1}\right| \\
&= \sum_{k=0}^{n-2} k(n-k-1)(n-1)^{n-k-2}(n-2)^{k-2}. \qquad (4.2)
\end{aligned}$$

We now obtain a closed form for (4.2).

Proposition 4.2.

$$\left|\mathcal{G}_n^1\right| = 2(n-1)(n-2)^{n-2} - (n-1)^{n-1} + (n-2)^{n-1}.$$

Proof. First notice that $(n-2)$ times the right hand side of (4.2) is

$$\frac{\partial^2}{\partial x \partial y} \sum_{k=0}^{n-1} x^{n-1-k} y^k \Bigg|_{\substack{x=n-1\\ y=n-2}}. \qquad (4.3)$$

However, the sum in (4.3) is also

$$\left[z^{n-1}\right] \frac{1}{(1-xz)(1-yz)},$$

where $[z^m]\, f(z)$ is the coefficient of z^m in the Taylor series expansion of $f(z)$ about $z=0$. Thus,

$$\begin{aligned}
(n-2)\cdot \mathcal{G}_n^1 &= \frac{\partial^2}{\partial x \partial y} \left[z^{n-1}\right] \frac{1}{(1-xz)(1-yz)} \Bigg|_{\substack{x=n-1\\ y=n-2}} \\
&= \left[z^{n-1}\right] \frac{z^2}{(1-(n-1)z)^2(1-(n-2)z)^2} \\
&= \left[z^{n-3}\right] \left(\frac{(n-1)^2}{(1-(n-1)z)^2} - \frac{2(n-1)^2(n-2)}{1-(n-1)z} \right. \\
&\qquad \left. + \frac{(n-2)^2}{(1-(n-2)z)^2} + \frac{2(n-2)^2(n-1)}{1-(n-2)z} \right) \\
&= 2(n-1)(n-2)^{n-1} - (n-2)(n-1)^{n-1} + (n-2)^n,
\end{aligned}$$

and the result follows. ∎

4.2. The Number of Elements in $\mathcal{G}_n^2$

In order to enumerate $\mathcal{G}_{n,k}^2$ we find a bijection from prime parking functions to certain types of trees. In particular, prime parking functions are in 1-1 correspondence with rooted trees $\big($say on vertices $\{1,2,\ldots,n\}\big)$ such that the root vertex is less than all of its children. We refer to such trees as $\mathcal{T}_n$ and the trees in $\mathcal{T}_n$ with root k we call $\mathcal{T}_{n,k}$.

Recall the number of parking functions is $(n+1)^{n-1}$. There are many bijections from parking functions of length n to rooted forests on n vertices, proving this fact (see [1, 2, 8, 11] for some examples of such bijections). In particular, in [2, Section 3] Foata and Riordan give a bijection from parking functions to rooted forests mapping parking functions with j 1's to rooted forests with j components where the positions of the 1's in the parking functions become the roots in the forest (that is, if a parking function has five 1's in position 2, 4, 9, 12 and 21, then this parking function is mapped to a rooted forest with 5 components whose roots are 2, 4, 9, 12 and 21). Suppose that we are given a prime parking function p in $\mathcal{Q}_{n,k}$, i.e., the left most 1 in p is in the k-th position. If we remove this 1 from the p, we are left with a parking function, call it $\widehat{p}$. The remaining parking function can be mapped to a rooted forest (via the map in [2, Section 3]) with root labels all greater than k (since all the remaining 1's are to the right of the k-th position). We may assume the labels on the vertices of the rooted forest are $\{1,2,\ldots,n\}\backslash\{k\}$. We may now add a vertex labelled k to the rooted forest and attach this vertex to all the roots of the rooted forest, giving us a rooted tree in which the root vertex is less than all its children.

It is not hard to see that this mapping is a bijection. In fact, from the above description, one can see that $\mathcal{Q}_{n,k}$ is in 1-1 correspondence with $\mathcal{T}_{n,k}$.

Using the above bijection from prime parking functions to rooted trees with the root smaller than all its children, one can see the subset $\widehat{\mathcal{Q}}_n$ of $\mathcal{Q}_n$ (defined in (3.1)) corresponds to the subset of $\mathcal{T}_n$ where the smallest child of the root has no children smaller than the root (this, in essence, reflects the fact that no 2 appears to the left of the left most 1). This subset of $\mathcal{T}_n$ will be denoted as $\widehat{\mathcal{T}_n}$. Thus, from (3.1), (3.5) and Proposition 3.3 we see

$$\left|\mathcal{G}_n^2\right| = \left|\widehat{\mathcal{Q}}_n\right| = \left|\widehat{\mathcal{T}_n}\right|. \tag{4.4}$$

Hence, we will now enumerate $\widehat{\mathcal{T}_n}$.

Let $T(x) = \exp(R(x))$, where $R(x) = x\exp(R(x))$ (i.e., $R(x)$ and $T(x)$ are the exponential generating functions of rooted trees and rooted forests, respectively, with respect to the number of vertices). Define the exponential generating function

$$\widehat{\mathcal{T}}(x) = \sum_{n\geq 2}\left|\widehat{\mathcal{T}_n}\right|\frac{x^n}{n!}.$$

In the next theorem we use the abbreviations $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}(x)$, $T = T(x)$ and $R = R(x)$.

Theorem 4.3. *The following functional equation holds.*

$$\widehat{\mathcal{T}} = \frac{1}{2} - \frac{1}{T} + \frac{1}{2T^2}.$$

Proof. Consider the triple $(X, V, \mathcal{T})$ where $X = \{x_1, x_2, \ldots, x_{i+j+1}\}$ is a set of vertices with $i \geq 1$ and $j \geq 0$ (we assume that x_1 is the smallest element of X), $V = \{v_1, v_2, \ldots, v_i\}$ is a subset of X of size i with $x_1 \notin V$ and $\mathcal{T} = (T_1, T_2, \ldots, T_{i+j-1})$ is a sequence of rooted forests. Let $W = \{w_1, w_2, \ldots, w_j\}$ be the remaining elements of X, that is, W is the set of element of X excluding x_1 and the set V. Thus, $X = \{x_1\} \dot\cup V \dot\cup W$. From $(X, V, \mathcal{T})$ we can construct a tree with the desired properties by

(1) making x_1 the root of the tree, having the members of V as children (thus forcing the root to be smaller than all its children),

(2) assuming that v_1 is the smallest element of V, let W be the set of children of v_1. Thus, the root is smaller than all the children of its smallest child,

(3) for $1 \leq \ell \leq i-1$ attaching all the roots of T_ℓ to w_ℓ,

(4) for $i \leq \ell \leq i+j-1$ attaching the roots of T_ℓ to $v_{\ell-j+1}$.

This constructs a tree as in Figure 1. It is not difficult to see that this process in

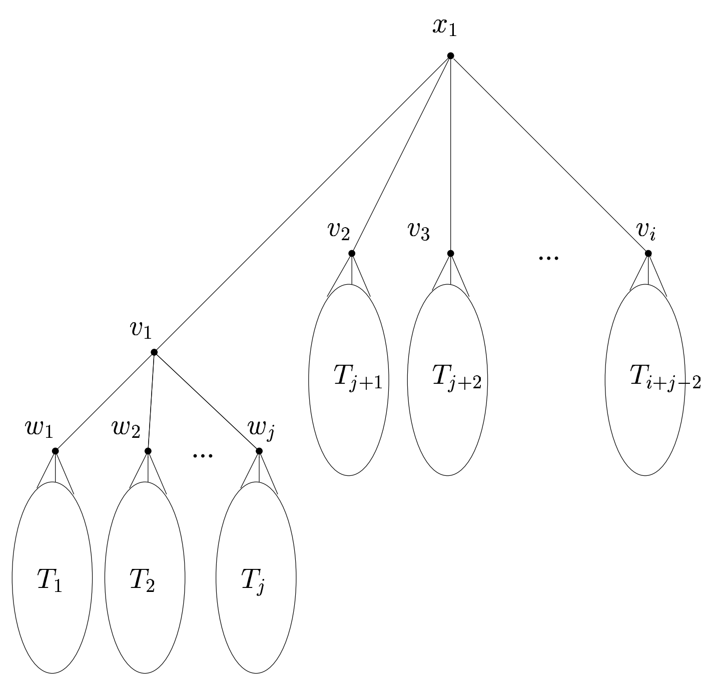

Figure 1: The tree obtained from a set $(X, \mathcal{T})$.

fact gives a bijection from ordered triples $(X, V, \mathcal{T})$ with the properties listed above and rooted trees with the root smaller than all its children and all the children of its smallest child. Therefore, we see from the theory of exponential generating functions that

$$\begin{aligned}\widehat{\mathcal{T}} &= \sum_{i\geq 1}\sum_{j\geq 0}\binom{i+j}{i}\frac{x^{i+j+1}}{(i+j+1)!}T^{i+j-1} \qquad (4.5)\\ &= \frac{x}{T}\sum_{i\geq 1}\sum_{j\geq 0}\frac{1}{i+j+1}\frac{(xT)^i}{i!}\frac{(xT)^j}{j!}\\ &= \frac{x}{T}\frac{1}{xT}\int_0^{xT}\exp(y)\left(\exp(y)-1\right)dy\\ &= \frac{1}{2}-\frac{1}{T}+\frac{1}{2T^2},\end{aligned}$$

completing the proof. ∎

Corollary 4.4. *For $n \geq 2$, we have $|\mathcal{G}_n^2| = |\widehat{\mathcal{T}}_n| = (n-1)^{n-1} - (n-2)^{n-1}$.*

Proof. The first equality is in (4.4). For the second equality we again use the notation $T = \exp(R)$ where $R = x\exp(R)$ and $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}(x)$. Applying the Lagrange Inversion Formula (see [3, Section 1.2] and [10, Theorem 5.4.2]) to the resulting generating function in Theorem 4.3 we obtain for $n \geq 2$

$$\begin{aligned}n!\,[x^n]\left(\frac{1}{2}-\frac{1}{T}+\frac{1}{2T^2}\right) &= -n!\,\left[x^{n-1}\right]\frac{1}{R}+n!\,\left[x^{n-2}\right]\frac{1}{2R^2}\\ &= (n-1)^{n-1}-(n-2)^{n-1},\end{aligned}$$

giving the desired result. ∎

5. Main Theorem for the $(2, n-2)$ Case

Theorem 5.1. [Main Theorem for the $(2, n-2)$ case] *The equality in* (2.1) *holds*; *that is,*

$$\left|F_{(2,n-2)}\right| = 4(n-1)(n-2)^{n-1}.$$

Proof. As discussed at the end of Section 2, to prove (2.1) it suffices to show (2.2), that is, it suffices to show $|\mathcal{G}_n| = 2(n-1)(n-2)^{n-2}$. However, from (3.6) we see that

$$|\mathcal{G}_n| = \left|\mathcal{G}_n^1\right| + \left|\mathcal{G}_n^2\right|.$$

The cardinalities $\mathcal{G}_n^1$ and $\mathcal{G}_n^2$ are evaluated in Proposition 4.2 and Corollary 4.4 and their sum gives the desired value of $|\mathcal{G}_n|$. ∎

6. Preliminaries for the Case $(3, n-3)$

We use the solution for the case $(2, n-2)$ as a guide for solving the $(3, n-3)$ case; in fact, many of the proofs are similar to the case $(2, n-2)$, and in those cases we omit the proof and simply refer to the previous given proof.

Here, we are finding all transitive minimal factorizations of $(1\,2\,3)(4\,5\cdots n)$. Note, in this case (1.1) becomes

$$F_{(3,n-3)} = \frac{27}{2}(n-1)(n-2)(n-3)^{n-2}. \tag{6.1}$$

Note also that in a minimal transitive factorization of $(1\,2\,3)(4\,5\cdots n)$ there is exactly one cut. Further, we may once again take advantage of symmetry; it is clear that the cut can be of the form $(r\,s)$, where $r = 1, 2$ or 3 and $s = 4, 5, \ldots, n$ and, therefore, it suffices to enumerate all the minimal, transitive factorizations where $(1\,4)$ is the cut and multiply the result by $3(n-3)$. Namely, it suffices to show that the number of factorizations of $(1\,2\,3)(4\,5\cdots n)$ with $(1\,4)$ as the cut is

$$\frac{9}{2}(n-1)(n-2)(n-3)^{n-3}. \tag{6.2}$$

We use the symbol $\mathcal{D}_{n,k}$ to denote the set of minimal, transitive factorizations $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of $(1\,2\,3)(4\,5\cdots n)$ with $\sigma_k = (1\,4)$ as the cut.

The factorizations $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ in $\mathcal{D}_{n,k}$ fall into three cases:

Case 1. 1, 2 and 3 are all fixed points after that cut, that is, in the product $\sigma_k \cdots \sigma_n$ the elements 1, 2 and 3 are all fixed.

Case 2. Exactly one of 1, 2 or 3 is a fixed point after the cut, that is, in the product $\sigma_k \cdots \sigma_n$ exactly one of 1, 2 or 3 is a fixed point.

Case 3. 1, 2 and 3 are in one cycle after the cut, i.e., in the product $\sigma_k \cdots \sigma_n$, none of 1, 2 or 3 is a fixed point.

Case 1. In this case we can conclude that the transpositions containing 2 or 3 occur after the cut, i.e., the only transpositions that contain 2 or 3 are amongst $\sigma_1, \sigma_2, \ldots, \sigma_{k-1}$. Further, there are exactly two transpositions that contain 2 and 3 in $\sigma_1, \sigma_2, \ldots, \sigma_{k-1}$ and these two transpositions join 1, 2 and 3 into a three cycle. Suppose that these two transpositions are σ_i and σ_j. Clearly, the product of $\sigma_1, \sigma_2, \ldots, \sigma_n$ with σ_i and σ_j removed is $(1)(4\ldots n)$ and it is a factorization of $(1)(4\cdots n)$ with $(1\,4)$ as the cut in the $(k-2)$-th position (as both σ_i and σ_j occur after the cut), which is enumerated in [7, Theorem 5] as $|\mathcal{Q}_{n-2,k-2}|$. The two transpositions σ_i and σ_j can be placed after the cut in $\binom{k-1}{2}$ ways and there are three choices for σ_i and σ_j (i.e., the number of factorizations of $(1\,2\,3)$). Thus, the total number of factorizations in this case is

$$3\binom{k-1}{2}\left|\mathcal{Q}_{n-2,k-2}\right|. \tag{6.3}$$

Note that k may take the values from 3 to $n-1$.

Case 2. Exactly, one of 1, 2 or 3 is a fixed point. There are two subcases here.

Case 2a. If the fixed point is 1 then one of $\sigma_{k+1}, \ldots, \sigma_n$, say σ_j, is the join $(2\,3)$ (since 2 and 3 are not fixed points after the cut). Further, one of $\sigma_1, \ldots, \sigma_{k-1}$, say σ_i is the join $(1\,2)$ (to join 1 to 2 and 3 after the cut). Removing σ_i and σ_j from the factorization gives us a factorization of $(1)(4\cdots n)$, with $(1\,4)$ as the cut in the $(k-1)$-th position (since σ_i is after the cut and σ_j is before the cut). Hence, from [7, Theorem 5] the total number of these factorizations is $|\mathcal{Q}_{n-2,k-1}|$. There are $k-1$ choices for the place of σ_i (it can occur anywhere after the cut) and $n-k$ choices for σ_j (it can occur anywhere before the cut). Thus, the total number of factorizations of this kind is

$$(k-1)(n-k)|\mathcal{Q}_{n-2,k-1}|. \tag{6.4}$$

Note that k can assume the values from 2 to $n-2$.

Case 2b. The other case is when 2 or 3 is the fixed point after the cut. Since there is symmetry between these two, we may assume that 3 is the fixed point after the cut.

The join $(1\,3)$ is the join in $\sigma_1,\ldots,\sigma_{k-1}$ that joins 3 to 1 and 2, and this is the only transposition in which 3 appears. Removing this transposition from the factorization we have a minimal, transitive factorization of $(1\,2)(4\cdots n)$, i.e., a factorization of type $\mathcal{G}^2_{n,k}$, which by Proposition 3.3 is $\widehat{\mathcal{Q}}_{n-1,k-1}$. Since there are $k-1$ places for the join $(1\,3)$ after the cut, we see that there are

$$(k-1)\left|\widehat{\mathcal{Q}}_{n-1,k-1}\right|$$

factorizations of this type and accounting for the symmetry between 2 and 3, then there are

$$2(k-1)\left|\widehat{\mathcal{Q}}_{n-1,k-1}\right| \tag{6.5}$$

total number of factorizations in the case. Note that k may take any value from 2 to $n-2$.

Case 3. The final case is when 1, 2 and 3 are in the same cycle after the cut. If $\mathcal{B}_{n,k}$ is defined to be the set of minimal factorizations of $(1\,2\cdots n)$ such that

(1) $\sigma_1,\sigma_2,\ldots,\sigma_{k-1}$ does not contain a 2, 3 or 4;
(2) $\sigma_k,\sigma_{k+1},\ldots,\sigma_{n-1}$ does contain a 1.

Then using a proof similar to the proof of Lemma 3.2, we can show that there is a bijection from factorizations of the type in Case 3 and $\mathcal{B}_{n,k}$. Further, with a result similar to Proposition 3.3, we can show that $\mathcal{B}_{n,k}$ is in bijection with prime parking functions of length n, where the first 1 appears in the k-th position and no 2 or 3 appears to the left of the left-most 1. We call these prime parking functions $\widehat{\widehat{\mathcal{Q}}}_{n,k}$.

We have grouped the factorizations above in an intuitive way; the grouping is based on which of 1, 2 and 3 are fixed points after the cut. However, from above, we see that the number of factorizations in Cases 1 and 2a (given in (6.3) and (6.4)) have similar expressions. We will, therefore, call this set of factorizations $\mathcal{D}^1_{n,k}$, that is, the set of factorizations in Cases 1 and 2a above will be denoted by $\mathcal{D}^1_{n,k}$. Similarly, we call the set of factorizations in Case 2b $\mathcal{D}^2_{n,k}$ and the set of factorizations in Case 3 $\mathcal{D}^3_{n,k}$. We enumerate these sets in the next section.

7. Enumerating the Classes for the Case $(3, n-3)$

We now enumerate all the cases given in the last section.

7.1. The Number of Elements in $\mathcal{D}^1_{n,k}$ (Case 1 and Case 2a)

In Case 1 and Case 2a, we have similar expressions given in (6.3) and (6.4). Combining and summing over the appropriate k we get

$$\sum_{k=2}^{n-1}\left(3\binom{k-1}{2}+(k-2)(n-k+1)\right)\left|Q_{n-2,k-2}\right|$$
$$=\sum_{k=2}^{n-1}\left(3\binom{k-1}{2}+(k-2)(n-k+1)\right)(n-k)(n-2)^{n-k-1}(n-3)^{k-4}, \tag{7.1}$$

the equality follows from Lemma 4.1.

Proposition 7.1. *The following is a closed form for* (7.1); *that is, the number of elements in* $\mathcal{D}^1_{n,k}$ *is*

$$2(n-2)^{n-2}(n-3)-5(n-2)^{n-1}+\frac{1}{2}(n-3)^{n-3}(17n^2-63n+54).$$

Proof. We use a proof similar to that in Proposition 4.2 by noting that (7.1) is equal to

$$\frac{1}{n-3}\left(\frac{3}{2}\frac{\partial^3}{\partial y^2\partial x}+\frac{\partial^3}{\partial x^2\partial y}\right)\left[z^{n-1}\right]\frac{1}{(1-xz)(1-yz)}\Bigg|_{\substack{x=n-2\\ y=n-3}}.$$

Simplifying the last expression gives the desired result. ∎

7.2. The Number of Elements in $\mathcal{D}^2_{n,k}$ (Case 2b)

In (6.5), the number of factorizations in the Case 2b is given in terms of the prime parking functions $\widehat{Q}_{n,k}$. From the discussion immediately preceding (4.4), we see that these prime parking functions correspond to the subset of the trees $\widehat{\mathcal{T}}_n$ where the root has the label k. Call these trees $\widehat{\mathcal{T}}_{n,k}$. The following lemma finds a closed form for the number of trees in $\widehat{\mathcal{T}}_{n,k}$.

Proposition 7.2. *For* $n=k=1$ *we have* $\left|\widehat{\mathcal{T}}_{n,k}\right|=1$ *and for* $n>k\geq 1$ *we have*

$$\left|\widehat{\mathcal{T}}_{n,k}\right|=(n-2)^{k-2}n^{n-k-1}(n-2k)+(n-2)^{k-2}(n-1)^{n-k-1}(k-1).$$

Proof. The first claim is clear. For a fixed $1\leq k\leq n-1$, let $\widehat{\mathcal{T}}(x,y)$ be the multivariate generating series defined by

$$\widehat{\mathcal{T}}(x,y)=\sum_{n\geq 1}\left|\widehat{\mathcal{T}}_{n,k}\right|\frac{x^{n-k}}{(n-k)!}\frac{y^{k-1}}{(k-1)!},$$

i.e., for a fixed k, $\widehat{\mathcal{T}}(x,y)$ is the generating series for $\widehat{\mathcal{T}}_{n,k}$. In $\widehat{\mathcal{T}}(x,y)$ the root of a tree has no marker, x marks vertices larger than the root and y marks vertices smaller than the root. Recall that $T(t)$ is the generating function for rooted forests. Using Figure 1 and the proof of Theorem 4.3 (in particular, we use (4.5)) we see that

$$\widehat{\mathcal{T}}(x,y) = \sum_{\substack{\ell\geq 1\\ i\geq 0}} \binom{\ell+i}{\ell} \frac{x^{\ell+i}}{(\ell+i)!} \left(\exp(T(x+y))\right)^{\ell+i-1}.$$

Therefore, we obtain

$$\begin{aligned}
\left|\widehat{\mathcal{T}}_{n,k}\right| &= \left[\frac{x^{n-k}}{(n-k)!}\frac{y^{k-1}}{(k-1)!}\right] \sum_{\substack{\ell\geq 1\\ i\geq 0}} \binom{\ell+i}{\ell} \frac{x^{\ell+i}}{(\ell+i)!} \left(\exp(T(x+y))\right)^{\ell+i-1}\\
&= \left[\frac{x^{n-k}}{(n-k)!}\frac{y^{k-1}}{(k-1)!}\right] \exp(-T(x+y))\\
&\qquad (\exp(2x\exp(T(x+y))) - \exp(x\exp(T(x+y))))\\
&= \left[\frac{x^{n-k}}{(n-k)!}\frac{y^{k-1}}{(k-1)!}\right] \sum_{j\geq 0} \frac{x^j}{j!} \exp((j-1)T(x+y))(2^j-1)\\
&= (n-k)!(k-1)! \sum_{j=1}^{n-k} \frac{2^j-1}{j!} \left[x^{n-k-j}y^{k-1}\right] \exp((j-1)T(x+y))\\
&= (n-k)!(k-1)! \sum_{j=1}^{n-k} \frac{2^j-1}{j!} \binom{n-j-1}{k-1} \left[t^{n-j-1}\right] \exp((j-1)T(t))\\
&= (n-k)!(k-1)! \sum_{j=1}^{n-k} \frac{2^j-1}{j!} \binom{n-j-1}{k-1} (j-1) \frac{(n-2)^{n-j-2}}{(n-j-1)!}\\
&= \sum_{j=1}^{n-k} \binom{n-k}{j} (j-1)(2^j-1)(n-2)^{n-j-2}.
\end{aligned}$$

The second last equality follows from Lagrange's Implicit Function Theorem. Finally, using the technique used in Propositions 4.2 and 7.1, the result follows from the last equation. ∎

Of course, Proposition 7.2 leaves us with the task of having to compute the quantity in (6.5) summed over $k = 2$ to $n-2$, that is, we must compute

$$\sum_{k=2}^{n-2} 2(k-1) \left|\widehat{\mathcal{Q}}_{n-1,k-1}\right|.$$

However, in order to obtain a closed form for this quantity using Proposition 7.2, we can again use the technique of Propositions 4.2 and 7.1 to obtain

$$\sum_{k=2}^{n-2} 2(k-1) \left|\widehat{\mathcal{Q}}_{n-1,k-1}\right| = -\frac{1}{2}(n-3)^{n-2}(13n-27) + 4(n-2)^{n-1} - \frac{1}{2}(n-1)^{n-1}.$$

Therefore, the last quantity is the number of elements in $\mathcal{D}^2_{n,k}$. We write this as a proposition for later reference.

Proposition 7.3. *The number of elements in $\mathcal{D}^2_{n,k}$ is*

$$-\frac{1}{2}(n-3)^{n-2}(13n-27)+4(n-2)^{n-1}-\frac{1}{2}(n-1)^{n-1}.$$

7.3. The Number of Elements in $\mathcal{D}^3_{n,k}$ (Case 3)

At the end of Section 6, we saw that the number of factorizations in $\mathcal{D}^3_{n,k}$ is the number of prime parking functions in $\hat{\hat{Q}}_{n,k}$ i.e., the number of prime parking functions of length n with the first 1 in the k-th position and no 2 or 3 occurring to the left of the left-most 1. At this point we can drop the parameter k; that is, we will be interested in prime parking functions such that no 2 or 3 appears to the left of the left-most 1, where the left-most 1 can be in any position (we do this because we can enumerate these objects over all k at once). Using the reasoning immediately preceding (4.4), we see that these prime parking functions correspond to rooted trees where

(1) the root is smaller than all its children,
(2) the root is smaller than all the children of its smallest child,
(3) the root is smaller than all the children of its smallest child's smallest child,
(4) if the root's smallest child has no children, than the root is smaller than the children of its second smallest child.

Call this set of trees on n vertices $\hat{\hat{T}}_n$. Note in the above description that there are two types of trees given; the first are the trees in (3) and the second being the trees in (4). The first of these types of trees is given in Figure 2 and the second type of trees is given in Figure 3.

Let $\hat{\hat{T}}(x)$ be the generating series

$$\hat{\hat{T}}(x)=\sum_{n\geq 2}\left|\hat{\hat{T}}_n\right|\frac{x^n}{n!}.$$

Using the reasoning used in Theorem 4.3, we see that the contribution of the first type of trees to $\hat{\hat{T}}(x)$ is

$$\sum_{\substack{i\geq 0\\ j\geq 1\\ \ell\geq 1}}\binom{i+j+\ell}{i,\, j,\, \ell}\frac{x^{i+j+\ell+1}}{(i+j+\ell)!}T^iT^{j-1}T^{\ell-1},$$

and the contribution of the second type of trees is

$$\sum_{\substack{i\geq 0\\ j\geq 2}}\binom{i+j}{i}\frac{x^{i+j+1}}{(i+j+1)!}T^iT^{j-2},$$

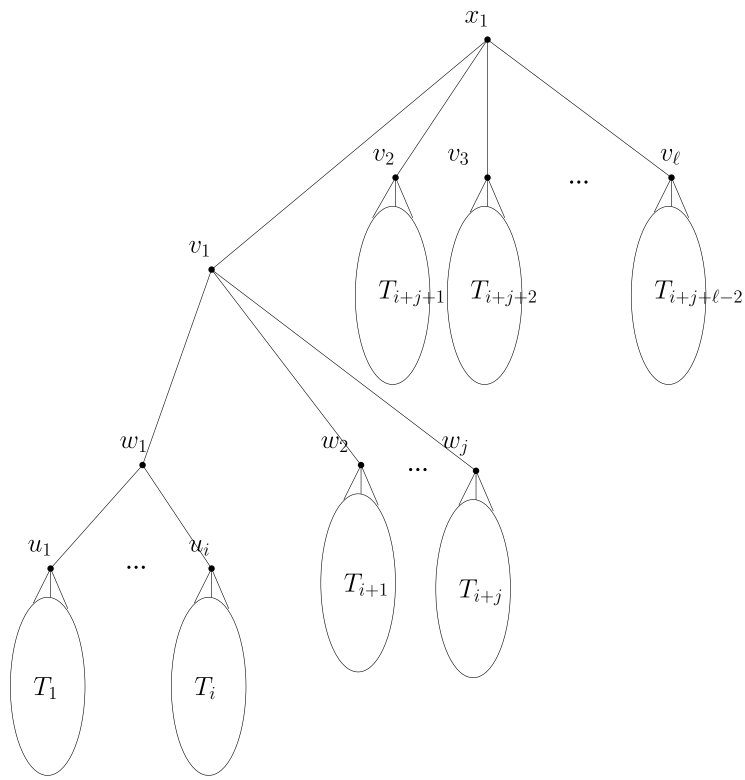

Figure 2: The first type of tree in Case 3. Note the root x_1 satisfies $x_1 < v_m, w_m, u_m$ for all appropriate m.

where, as before, T is the generating series for rooted forests. Combining the last two equations we get

$$\begin{aligned}\widehat{\widehat{T}}(x) &= \frac{1}{T^3}\int_0^{xT}(\exp y - 1)^2 \exp y \, dy + \frac{1}{T^3}\int_0^{xT}(\exp y - y - 1)\exp y \, dy \\ &= \frac{1}{3} + \frac{1}{2T} - \frac{x}{T} + \frac{1}{T^2} - \frac{5}{6T^3}. \qquad (7.2)\end{aligned}$$

Using this last equation, we can find the number of elements in $\mathcal{D}^3_{n,k}$.

Proposition 7.4. *The number of elements in* $\mathcal{D}^3_{n,k}$ *is*

$$\left|\widehat{\widehat{T}}_n\right| = \frac{1}{2}(n-1)^{n-1} + n(n-2)^{n-2} - 2(n-2)^{n-1} + \frac{5}{2}(n-3)^{n-1}.$$

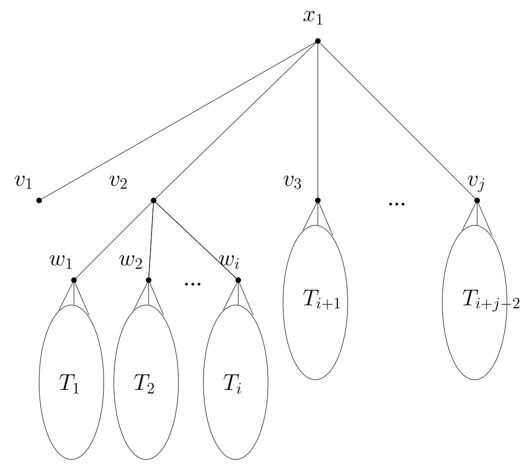

Figure 3: The second type of trees in Case 3.

Proof. From the discussion at the beginning of this subsection, $\left|\mathcal{D}^3_{n,k}\right| = \left|\widehat{\widehat{T}}_{n,k}\right|$. However, applying Lagrange's Implicit Function Theorem on (7.2) gives the result. ∎

8. Main Theorem for the $(3, n-3)$ Case

Theorem 8.1. *The equality in* (6.1) *holds*; *that is,*

$$\left|F_{(3,n-3)}\right| = \frac{27}{2}(n-1)(n-2)(n-3)^{n-2}.$$

Proof. As discussed at the beginning of Section 6, it suffices to show that the number of factorizations with $(1\ 4)$ as the cut is the number given in (6.2); that is, it suffices to show that the number of such factorizations is

$$\frac{9}{2}(n-1)(n-2)(n-3)^{n-3}.$$

However, all such factorizations are in one of our sets $\mathcal{D}^1_{n,k}$, $\mathcal{D}^2_{n,k}$ or $\mathcal{D}^3_{n,k}$. But these sets are enumerated in Propositions 7.1, 7.3 and 7.4 and summing the quantities given in those propositions gives the result. ∎

9. Conclusion

After the proof of (1.1) by Goulden and Jackson in [4], combinatorial proofs of (1.1) have been found for the cases $\lambda = (n)$, $\lambda = (1, n-1)$ and now $\lambda = (2, n-2)$ and $\lambda = (3, n-3)$. The combinatorial form of (1.1) suggests that there should be a combinatorial proof in general. It would be desirable to extend the methods in this paper to general λ. The next step in extending this method seems clear, namely to extend this method to a general $\lambda = (k, n-k)$ partition. Indeed, there are many similarities between the proofs of the cases $(2, n-2)$ and $(3, n-3)$ given here suggesting that a generalization is possible. It was initially regarded by this author that a complete solution for the case $(3, n-3)$ was unattainable using parking functions, but this is demonstrably not so. This gives hope that the solution can be extended to the general case.

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