V.8 Renormalization of the Φ^4 Model

The starting point of the RG calculation is the Ginzburg-Landau functional (5.11)

$$\mathcal{L} = \int d^d r \left[\frac{1}{2}(\nabla\Phi)^2 + \frac{1}{2}r_0\Phi^2 + \frac{1}{4}u\Phi^4\right] ,$$

where the magnetic field $h = 0$ was set to zero for simplicity. Since the RG calculation will be performed in momentum space, the functional is expressed in terms of the Fourier transforms $\widehat{\Phi}(\mathbf{k})$ of the fields Φ. Using the abbreviation $\int_{\mathbf{k}} \equiv \int_{|\mathbf{k}|<\Lambda} \frac{d^d k}{(2\pi)^2}$ we find

$$\mathcal{L} = \frac{1}{2}\int_{\mathbf{k}}(r_0 + k^2)\widehat{\Phi}(\mathbf{k})\widehat{\Phi}(-\mathbf{k}) + \frac{1}{4}u\int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}(2\pi)^d\delta(\mathbf{k}_1 + \ldots + \mathbf{k}_4)\widehat{\Phi}(\mathbf{k}_1)\widehat{\Phi}(\mathbf{k}_2)\widehat{\Phi}(\mathbf{k}_3)\widehat{\Phi}(\mathbf{k}_4) . \tag{5.62}$$

In the first step (i) of the RG procedure we want to integrate out the $\mathbf{k}$-modes with $\Lambda/b < k < \Lambda$. Whereas this was easily done for the Gaussian model, because the quadratic terms do not couple different $\mathbf{k}$-modes, this is no longer the case for the Φ^4-model. In order to deal with this complication, we first split up the fields Φ into two parts:

$$\begin{aligned}&\widehat{\Phi}(\mathbf{k}) = \widehat{\Phi}^<(\mathbf{k}) + \widehat{\Phi}^>(\mathbf{k})\\ &\text{with}\quad \widehat{\Phi}^<(\mathbf{k}) = \begin{cases}\widehat{\Phi}(\mathbf{k}) & \text{for } k < \Lambda/b\\ 0 & \text{otherwise}\end{cases} \quad\text{and}\quad \widehat{\Phi}^>(\mathbf{k}) = \begin{cases}\widehat{\Phi}(\mathbf{k}) & \text{for } \Lambda/b < k < \Lambda\\ 0 & \text{otherwise}\end{cases} .\end{aligned} \tag{5.63}$$

Then the GL functional can be formally written as

$$\begin{aligned}&\mathcal{L}[\widehat{\Phi}] = \mathcal{L}_G[\widehat{\Phi}^<] + \mathcal{L}_G[\widehat{\Phi}^>] + V[\widehat{\Phi}^<,\widehat{\Phi}^>]\\ &\text{with}\quad \mathcal{L}_G[\widehat{\Phi}^<] = \int_{\mathbf{k}}^{<}\frac{1}{2}(r_0 + k^2)\widehat{\Phi}^<(\mathbf{k})\widehat{\Phi}^<(-\mathbf{k}) ,\\ &\qquad\quad \mathcal{L}_G[\widehat{\Phi}^>] = \int_{\mathbf{k}}^{>}\frac{1}{2}(r_0 + k^2)\widehat{\Phi}^>(\mathbf{k})\widehat{\Phi}^>(-\mathbf{k})\\ &\text{and}\quad \int_{\mathbf{k}}^{<} \equiv \int_{k<\Lambda/b}\frac{d^d k}{(2\pi)^d} ,\quad \int_{\mathbf{k}}^{>} \equiv \int_{\Lambda/b<k<\Lambda}\frac{d^d k}{(2\pi)^d} .\end{aligned} \tag{5.64}$$

so that $V[\widehat{\Phi}^<,\widehat{\Phi}^>]$ contains the quartic terms that couple different $\mathbf{k}$-modes. Hence, the partition function is

$$Z = \int \mathcal{D}\widehat{\Phi}\, e^{-\mathcal{L}[\widehat{\Phi}]} = \int \mathcal{D}\widehat{\Phi}^<\mathcal{D}\widehat{\Phi}^>\, e^{-\mathcal{L}_G[\widehat{\Phi}^<]-\mathcal{L}_G[\widehat{\Phi}^>]-V[\widehat{\Phi}^<,\widehat{\Phi}^>]} .$$

In the first step of the RG procedure we integrate over $\widehat{\Phi}^>$. This leads to a renormalized functional

$$e^{-\mathcal{L}_b[\widehat{\Phi}^<]} = e^{-\mathcal{L}_G[\widehat{\Phi}^<]}\int \mathcal{D}\widehat{\Phi}^>\, e^{-\mathcal{L}_G[\widehat{\Phi}^>]}e^{-V[\widehat{\Phi}^<,\widehat{\Phi}^>]} . \tag{5.65}$$

Eq. (5.65) can be rewritten as an average with respect to the Hamiltonian $\mathcal{L}_G[\widehat{\Phi}^>]$. Defining such an average through

$$\langle A\rangle_0 = \frac{1}{Z_0}\int \mathcal{D}\widehat{\Phi}^>\, A e^{-\mathcal{L}_G[\widehat{\Phi}^>]} \quad\text{with}\quad Z_0 = \int \mathcal{D}\widehat{\Phi}^>\, e^{-\mathcal{L}_G[\widehat{\Phi}^>]} \tag{5.66}$$

we find

$$e^{-\mathcal{L}_b[\widehat{\Phi}^<]} = Z_0 e^{-\mathcal{L}_G[\widehat{\Phi}^<]} \left\langle e^{-V[\widehat{\Phi}^<,\widehat{\Phi}^>]} \right\rangle_0 \; . \tag{5.67}$$

To calculate $\langle e^{-V}\rangle_0$ we use a cumulant expansion

$$\begin{aligned} \left\langle e^{-V} \right\rangle &= \left\langle 1 - V + \frac{1}{2}V^2 \pm \ldots \right\rangle = \exp\left\{ \log\left(1 - \langle V\rangle + \frac{1}{2}\langle V^2\rangle \pm \ldots\right)\right\} \\ &= \exp\left\{ -\langle V\rangle + \frac{1}{2}\left[\langle V^2\rangle - \langle V\rangle^2\right] \pm \ldots \right\} \end{aligned} \tag{5.68}$$

Such an expansion requires that the quartic terms V can be treated as a perturbation, i.e., the parameter u or, more precisely, its fixed point value u^* must be small. For $d > 4$ we know that $u^* = 0$. We now make the nontrivial assumption that u^* is a continuous function of the dimension d and define

$$\epsilon = 4 - d \; . \tag{5.69}$$

For small $\epsilon > 0$ we expect the fixed point value u^* to lie close to zero, i.e., $u^* = \mathcal{O}(\epsilon)$ (this must be checked afterwards). Thus, (5.68) is an expansion in powers of ϵ and commonly called ϵ-expansion.

V.8.1 First-Order Calculation: Determination of $\langle V\rangle_0$ for $n = 1$.

We begin with step (i): integrating over $\widehat{\Phi}(\mathbf{k})$ with $\Lambda/b < k < \Lambda$. Since $\widehat{\Phi} = \widehat{\Phi}^< + \widehat{\Phi}^>$ we have

$$\langle V\rangle_0 = \frac{u}{4}\int d^d r \left\{ \Phi^{<4}(\mathbf{r}) + 6\Phi^{<2}(\mathbf{r})\langle\Phi^{>2}(\mathbf{r})\rangle_0 + \langle\Phi^{>4}(\mathbf{r})\rangle_0 \right\} \; .$$

The average $\langle\Phi^{>2}(\mathbf{r})\rangle_0 = \int_{\mathbf{k}_1,\mathbf{k}_2}\langle\widehat{\Phi}^>(\mathbf{k}_1)\widehat{\Phi}^>(\mathbf{k}_2)\rangle_0 e^{i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}}$ is determined using the result (5.18), i.e.,

$$\begin{aligned} \langle\widehat{\Phi}^>(\mathbf{k}_1)\widehat{\Phi}^>(\mathbf{k}_2)\rangle_0 &= \int d\mathrm{Re}\widehat{\Phi}^>(\mathbf{k}_1) d\mathrm{Im}\widehat{\Phi}^>(\mathbf{k}_1) d\mathrm{Re}\widehat{\Phi}^>(\mathbf{k}_2) d\mathrm{Im}\widehat{\Phi}^>(\mathbf{k}_2)\, \widehat{\Phi}^>(\mathbf{k}_1)\widehat{\Phi}^>(\mathbf{k}_2) e^{-\mathcal{L}_G[\widehat{\Phi}^>]} \\ &= 0 \quad \text{for} \quad \mathbf{k}_1 \neq -\mathbf{k}_2 \end{aligned}$$

and

$$\langle\widehat{\Phi}^>(\mathbf{k})\widehat{\Phi}^>(-\mathbf{k})\rangle_0 = \int d\mathrm{Re}\widehat{\Phi}^>(\mathbf{k}) d\mathrm{Im}\widehat{\Phi}^>(\mathbf{k}) \left(\mathrm{Re}^2\widehat{\Phi}^>(\mathbf{k}) + \mathrm{Im}^2\widehat{\Phi}^>(\mathbf{k})\right) e^{-\mathcal{L}_G[\widehat{\Phi}^>]} = \frac{L^d}{r_0 + k^2} \; .$$

Hence, for $\Lambda/b < k_1, k_2 < \Lambda$

$$\langle\widehat{\Phi}^>(\mathbf{k}_1)\widehat{\Phi}^>(\mathbf{k}_2)\rangle_0 = L^d \delta_{\mathbf{k}_1,-\mathbf{k}_2}\frac{1}{r_0 + \mathbf{k}_1^2} = (2\pi)^d \delta(\mathbf{k}_1 + \mathbf{k}_2)\frac{1}{r_0 + \mathbf{k}_1^2} \tag{5.70}$$

and

$$\langle\Phi^{>2}(\mathbf{r})\rangle_0 = \int_{\mathbf{k}}^{>} \frac{1}{r_0 + \mathbf{k}_1^2} = \int_{\mathbf{k}}^{>} G_0(\mathbf{k}) \; , \tag{5.71}$$

where we have introduced the *propagator* $G_0(\mathbf{k})$.

Analogously, $\langle \Phi^{>4}(\mathbf{r})\rangle_0$ is determined by $\langle \widehat{\Phi}^{>}(\mathbf{k}_1)\widehat{\Phi}^{>}(\mathbf{k}_2)\widehat{\Phi}^{>}(\mathbf{k}_3)\widehat{\Phi}^{>}(\mathbf{k}_4)\rangle_0$. Again, we obtain a nonzero result if and only if pairs of wave vectors $\mathbf{k}$add up to zero. Writing down all possible combinations we find $(\Lambda/b < k_1, k_2, k_3, k_4 < \Lambda)$

$$\begin{aligned}\langle \widehat{\Phi}^{>}(\mathbf{k}_1)\widehat{\Phi}^{>}(\mathbf{k}_2)\widehat{\Phi}^{>}(\mathbf{k}_3)\widehat{\Phi}^{>}(\mathbf{k}_4) = {} & (2\pi)^d\delta(\mathbf{k}_1+\mathbf{k}_2)G_0(\mathbf{k}_1)(2\pi)^d\delta(\mathbf{k}_3+\mathbf{k}_4)G_0(\mathbf{k}_3) \\ & + (2\pi)^d\delta(\mathbf{k}_1+\mathbf{k}_3)G_0(\mathbf{k}_1)(2\pi)^d\delta(\mathbf{k}_2+\mathbf{k}_4)G_0(\mathbf{k}_2) \\ & + (2\pi)^d\delta(\mathbf{k}_1+\mathbf{k}_4)G_0(\mathbf{k}_1)(2\pi)^d\delta(\mathbf{k}_2+\mathbf{k}_3)G_0(\mathbf{k}_2)\ .\end{aligned} \tag{5.72}$$

These results for the correlation functions of $\widehat{\Phi}(\mathbf{k})$ are known as Wick's theorem. In first order the term $\int d^dr\langle \Phi^{>4}(\mathbf{r})\rangle_0$ just gives a constant (independent of $\widehat{\Phi}^{<}$) that can be ignored.

At the end of step (i) we now have

$$\begin{aligned}\mathcal{L}_b[\widehat{\Phi}^{<}] = {} & \frac{1}{2}\int_{\mathbf{k}}^{<}\left(k^2 + r_0 + \frac{3}{2}u\int_{\mathbf{q}} G_0(\mathbf{q})\right)\widehat{\Phi}^{<}(\mathbf{k})\widehat{\Phi}^{<}(-\mathbf{k}) \\ & + \frac{u}{4}\int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}^{<}(2\pi)^d\delta(\mathbf{k}_1+\ldots\mathbf{k}_4)\widehat{\Phi}^{<}(\mathbf{k}_1)\widehat{\Phi}^{<}(\mathbf{k}_2)\widehat{\Phi}^{<}(\mathbf{k}_3)\widehat{\Phi}^{<}(\mathbf{k}_4)\ .\end{aligned}$$

Since we did not find a contribution to the $k^2\widehat{\Phi}^{<}(\mathbf{k})\widehat{\Phi}^{<}(-\mathbf{k})$ term the rescaling of the $\widehat{\Phi}$ fields (in order to keep the coefficient of the k^2 term equal to 1/2) remains the same as in (5.58),

$$\widehat{\Phi}(\mathbf{k}/b) = \zeta(b)\widehat{\Phi}(\mathbf{k}) \quad \text{with} \quad \zeta(b) = b^{1+d/2}\ . \tag{5.73}$$

Thus after steps (ii) and (iii) we obtain a renormalized functional

$$\mathcal{L}_b[\widehat{\Phi}] = \frac{1}{2}\int_{\mathbf{k}}(r_0' + \mathbf{k}^2)\widehat{\Phi}(\mathbf{k})\widehat{\Phi}(-\mathbf{k}) + \frac{u'}{4}\int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}(2\pi)^d\delta(\mathbf{k}_1+\ldots+\mathbf{k}_4)\widehat{\Phi}(\mathbf{k}_1)\widehat{\Phi}(\mathbf{k}_2)\widehat{\Phi}(\mathbf{k}_3)\widehat{\Phi}(\mathbf{k}_4)$$

$$\begin{aligned}&\text{with} \quad r_0' = b^{-d}\zeta^2(b)\left(r_0 + 3u\int_{\mathbf{q}}^{>} G_0(\mathbf{q})\right) = b^2\left(r_0 + 3u\int_{\mathbf{q}}^{>} G_0(\mathbf{q})\right) && \text{a)} \\ &\text{and} \quad u' = b^{-3d}\zeta^4(b)u = b^{\epsilon}u\ . && \text{b)}\end{aligned} \tag{5.74}$$

Hence, in first order of the ϵ-expansion we do not find a fixed point u^*. Instead of (5.74b) we need something like

$$u' = c_1 u - c_2 u^2 \qquad \Rightarrow \qquad u^* = (c_1 - 1)/c_2\ .$$

Therefore, we must calculate the second-order contributions $\langle V^2\rangle - \langle V\rangle^2$. However, (5.74b) does confirm the result from Ch. V.7 that u is a relevant parameter at the Gaussian fixed point.

Assuming that $u^* = \mathcal{O}(\epsilon)$ we see that we do not need to calculate contributions to r_0' of order u^2, because r_0' is already determined to $\mathcal{O}(\epsilon)$; u^2 terms would give contributions of order ϵ^2, which would be inconsistent, since we can determine u^* only to order ϵ.

V.8.2 Feynman Diagrams

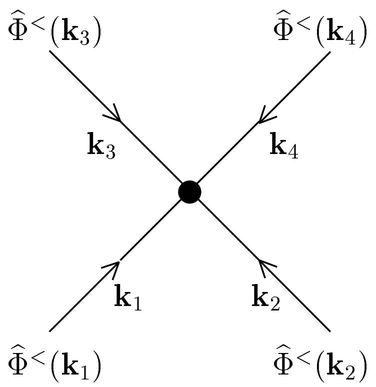
Already in second order of the perturbation expansion (5.68) the number of terms is enormous. Feynman developed a graphical technique to handle the terms generated in the

ϵ-expansion. These so-called Feynman diagrams or Feynman graphs and the corresponding rules will be explained first by repeating the first-order calculations.

In first order there were three contributions:

(1) $\displaystyle \frac{u}{4}\int^{<}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}(2\pi)^d\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\widehat{\Phi}^<(\mathbf{k}_1)\widehat{\Phi}^<(\mathbf{k}_2)\widehat{\Phi}^<(\mathbf{k}_3)\widehat{\Phi}^<(\mathbf{k}_4)$

The corresponding Feynman diagram has the following form:



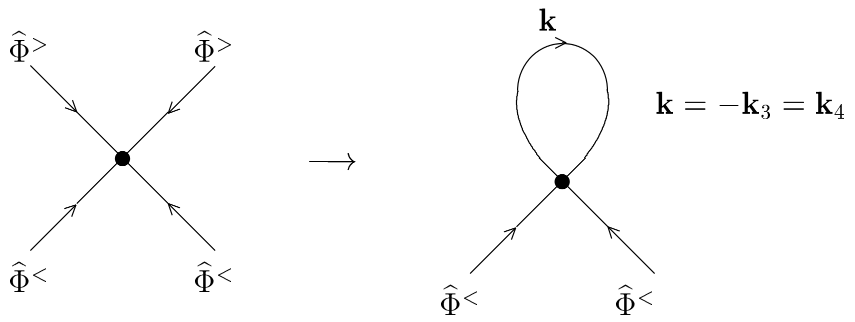
Rules:

- Each external leg represents a $\widehat{\Phi}^<(\mathbf{k})$ field. An external leg has one dangling end that is not connected to any other part of the diagram. The associated momentum $\mathbf{k}$ is drawn as flowing into the leg from the outside.
- The central vertex represents the factor

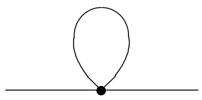
$$\bullet = \frac{u}{4}(2\pi)^d\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\ .$$

  Each vertex is connected to 4 legs, a consequence of the Φ^4 interactions.
- An integral $\int^<_{\mathbf{k}}$ over each $\mathbf{k}$ of the external legs is implied.

(2) $\displaystyle \frac{u}{4}\int^{<}_{\mathbf{k}_1,\mathbf{k}_2}\widehat{\Phi}^<(\mathbf{k}_1)\widehat{\Phi}^<(\mathbf{k}_2)\int^{>}_{\mathbf{k}_3,\mathbf{k}_4}\langle\widehat{\Phi}^>(\mathbf{k}_3)\widehat{\Phi}^>(\mathbf{k}_4)\rangle_0(2\pi)^d\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$

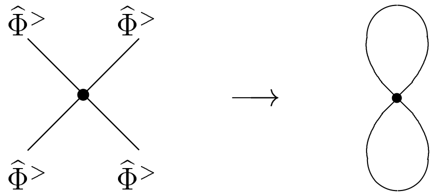


- The average over two $\widehat{\Phi}^>$ fields ties the two legs together: the Gaussian integral is nonzero only if $\mathbf{k}_3 = -\mathbf{k}_4$. This procedure creates an internal line. Each internal line carries momentum $\mathbf{k}$ and represents a factor $G_0(\mathbf{k})$. An integration $\int_{\mathbf{k}}^{>}$ is associated with each internal line.

$\Rightarrow$  $= \dfrac{u}{4}\displaystyle\int_{\mathbf{k}}^{<} \widehat{\Phi}^{<}(\mathbf{k})\widehat{\Phi}^{<}(-\mathbf{k}) \int_{\mathbf{k}}^{>} G_0(\mathbf{k})$

The conservation of momentum at each vertex requires that the sum of the momenta of the external legs is zero. There are several possibilities of drawing the diagram (2) that correspond to the number of ways to choose 2 legs for the $\widehat{\Phi}^{>}$ fields out of a total of 4 legs: the multiplicity of graph (2) is $\binom{4}{2} = 6$.

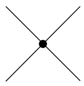
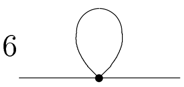
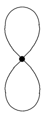
(3) $$\frac{u}{4}\int_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3,\mathbf{k}_4}^{>} \langle \widehat{\Phi}^{>}(\mathbf{k}_1)\widehat{\Phi}^{>}(\mathbf{k}_2)\widehat{\Phi}^{>}(\mathbf{k}_3)\widehat{\Phi}^{>}(\mathbf{k}_4)\rangle_0 (2\pi)^d\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$$



$= \dfrac{u}{4}\left[\displaystyle\int_{\mathbf{k}}^{>} G_0(\mathbf{k})\right]^2$

This diagram has a multiplicity of 3 corresponding to the number of ways of choosing a partner for the first $\widehat{\Phi}^{>}$ leg.

Thus, we finally can write the results of the first-order calculation using Feynman diagrams,

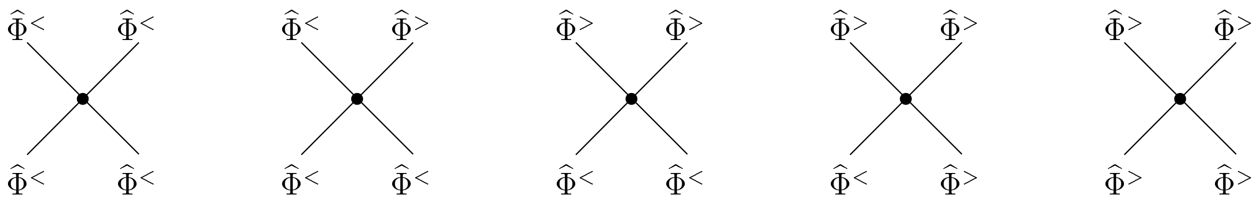
$\langle V\rangle_0 =$  $+\,6$  $+\,3$  . (5.75)

V.8.3 ϵ-Expansion to Second Order

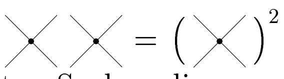
In this section the contributions to the second-order cumulant

$$\langle V^2\rangle_c := \frac{1}{2}[\langle V^2\rangle_0 - \langle V\rangle_0^2] \tag{5.76}$$

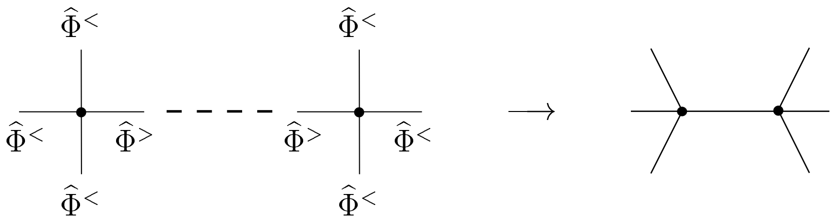
will be evaluated using Feynman graphs. As before the potential is given by $V = \frac{u}{4}\int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}$ $(2\pi)^d\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\widehat{\Phi}(\mathbf{k}_1)\widehat{\Phi}(\mathbf{k}_2)\widehat{\Phi}(\mathbf{k}_3)\widehat{\Phi}(\mathbf{k}_4)$ with $\widehat{\Phi}(\mathbf{k}) = \widehat{\Phi}^{<}(\mathbf{k}) + \widehat{\Phi}^{>}(\mathbf{k})$. Since $V^2 \sim u^2$ we now have to take into account diagrams with two vertices. *Before* averaging each vertex may be one of the following 5:



a b c d e

Each of these can be paired with all others resulting in 25 diagrams. However, not all of these combinations give a nonzero contribution to the cumulant $\frac{1}{2}[\langle V^2\rangle_0 - \langle V\rangle_0^2]$. E.g., the diagram a-a is simply . This is a *disconnected* diagram that is simply the product of its parts. Such a diagram also appears in $\langle V\rangle_0^2$, hence it drops out when calculating $\langle V^2\rangle_c$. This does not happen by chance: the fact that disconnected diagrams do not contribute to $\langle e^V\rangle_0$, i.e., give no contribution to any order in the cumulant expansion, is called the *linked cluster theorem.*

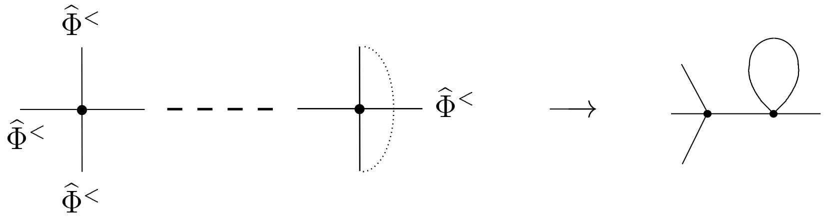
It remains to determine the *connected* diagrams. Vertex a cannot form any connected diagram because it has no free $\widehat{\Phi}^>$ fields that could tie the vertices together ("tying together" means a Gaussian average over the corresponding $\widehat{\Phi}^>$ fields). Vertex b can pair with itself:



b-b

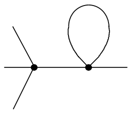
The vertex b-b has six external legs, i.e., it contributes to a term $u_6\Phi^6$ in the GL functional. However, u_6 is an irrelevant parameter as can be shown by power counting. Therefore, b-b can be ignored.

The vertex b can also be paired with d:



b-d

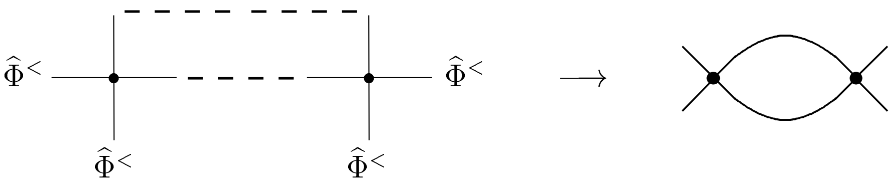
This graph has 4 external legs and therefore gives a contribution to the renormalized u:



$$
\begin{aligned}
&= \frac{u^2}{16}\int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}^{<} \widehat{\Phi}^<(\mathbf{k}_1)\widehat{\Phi}^<(\mathbf{k}_2)\widehat{\Phi}^<(\mathbf{k}_3)\int_{\mathbf{k}_4}^{>} G_0(\mathbf{k}_4)\int_{\mathbf{k}}^{>} G_0(\mathbf{k})\int_{\mathbf{k}_5}^{<}\widehat{\Phi}^<(\mathbf{k}_5) \\
&\qquad\times(2\pi)^d\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)(2\pi)^d\delta(\mathbf{k}_4+\mathbf{k}_5) \\
&= 0\ .
\end{aligned}
$$

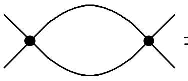
The result is obtained because of the last δ-function: since $\Lambda/b < k_4 < \Lambda$ and $0 < k_5 < \Lambda/b$ it follows that $\mathbf{k}_4 \neq -\mathbf{k}_5$. The graph b-d is an example of so-called *one-particle reducible* diagrams that can be divided into two disconnected parts by cutting a single internal line. These graphs give no contribution.

Vertex c can be paired with itself:



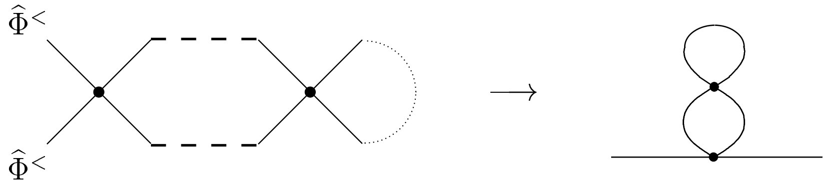
c-c

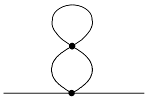
This also contributes to u':



$$
\begin{aligned}
&= \frac{u}{16}\int^{<}_{\mathbf{k}_1,\mathbf{k}_2} \widehat{\Phi}^{<}(\mathbf{k}_1)\widehat{\Phi}^{<}(\mathbf{k}_2)\int^{>}_{\mathbf{k}_3} G_0(\mathbf{k}_3)\int^{>}_{\mathbf{k}_4} G_0(\mathbf{k}_4)\int_{\mathbf{k}_5,\mathbf{k}_6} \widehat{\Phi}^{<}(\mathbf{k}_5)\widehat{\Phi}^{<}(\mathbf{k}_6)\\
&\qquad\qquad \times(2\pi)^d\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)(2\pi)^d\delta(\mathbf{k}_3+\mathbf{k}_4+\mathbf{k}_5+\mathbf{k}_6)\\
&= \frac{u}{16}\int^{<}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} \widehat{\Phi}^{<}(\mathbf{k}_1)\widehat{\Phi}^{<}(\mathbf{k}_2)\widehat{\Phi}^{<}(\mathbf{k}_3)\widehat{\Phi}^{<}(\mathbf{k}_4)(2\pi)^d\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\\
&\qquad\qquad \times\int^{>}_{\mathbf{k}} G_0(\mathbf{k})G_0(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\ .
\end{aligned} \tag{5.77}
$$

c can also be paired with e:

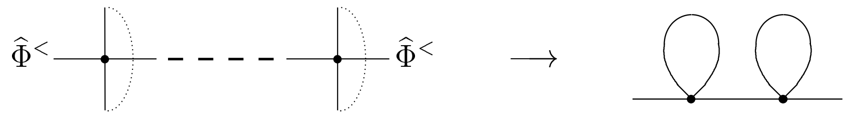




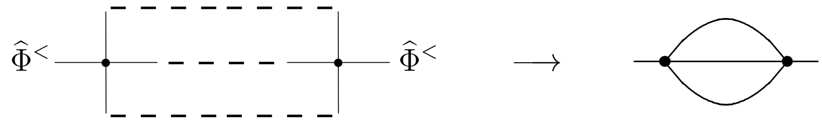
$$
= \frac{u}{16}\int^{<}_{\mathbf{k}} \widehat{\Phi}^{<}(\mathbf{k})\widehat{\Phi}^{<}(-\mathbf{k})\int^{>}_{\mathbf{k}_1} G_0(\mathbf{k}_1)\int^{>}_{\mathbf{k}_2} G_0(\mathbf{k}_2)
$$

This gives an $\mathcal{O}(\epsilon^2)$ contribution to r_0' that we do not need in this order of the calculation. Besides forming the vertex b-d, vertex d can be paired with itself in two different ways:

(i)

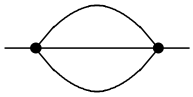


(ii)



d-d

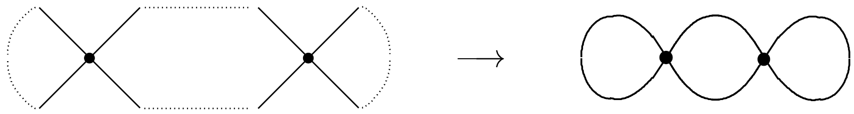
The first of these two is one-particle reducible and zero. The second,



$$
\begin{aligned}
&= \frac{u^2}{16}\int^{<}_{\mathbf{k}_1,\mathbf{k}_2} \widehat{\Phi}^{<}(\mathbf{k}_1)\widehat{\Phi}^{<}(\mathbf{k}_2)\int^{>}_{\mathbf{k}_3,\mathbf{k}_4\mathbf{k}_5} G_0(\mathbf{k}_3)G_0(\mathbf{k}_4)G_0(\mathbf{k}_5)\\
&\qquad\qquad \times(2\pi)^d\delta(\mathbf{k}_1+\mathbf{k}_3+\mathbf{k}_4+\mathbf{k}_5)(2\pi)^d\delta(\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4-\mathbf{k}_5)\\
&= \frac{u^2}{16}\int^{<}_{\mathbf{k}} \widehat{\Phi}^{<}(\mathbf{k})\widehat{\Phi}^{<}(-\mathbf{k})\int^{>}_{\mathbf{k}_3,\mathbf{k}_4} G_0(\mathbf{k}_3)G_0(\mathbf{k}_4)G_0(-\mathbf{k}-\mathbf{k}_3-\mathbf{k}_4)\ ,
\end{aligned} \tag{5.78}
$$

gives an $\mathcal{O}(\epsilon^2)$ contribution to r_0' *and*, since $\int^{>}_{\mathbf{k}_3,\mathbf{k}_4} G_0(\mathbf{k}_3)G_0(\mathbf{k}_4)G_0(-\mathbf{k}-\mathbf{k}_3-\mathbf{k}_4)$ depends on $\mathbf{k}$, it also affects the prefactor of the $k^2\widehat{\Phi}(\mathbf{k})\widehat{\Phi}(-\mathbf{k})$ term and therefore leads to a nonzero result for $\eta = \mathcal{O}(\epsilon^2)$.

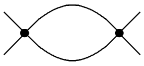
Finally, the vertex e can be paired with itself in two different ways:

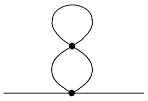
(i) 

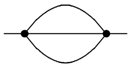
(ii) 

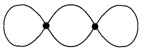
Both contributions do not depend on $\widehat{\Phi}^<$ and therefore represent constants that are unimportant for the RG procedure.

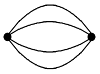
It remains to determine the multiplicities of the various diagrams that give nonzero contributions.

 There are $\binom{4}{2} = 6$ possibilities to select the two $\widehat{\Phi}^>$ legs out of the 4 legs of each vertex. Furthermore, the $\widehat{\Phi}^>$ legs can be paired in two different ways. The resulting factor is $6 \times 6 \times 2 = 72$.

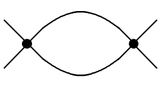
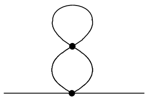
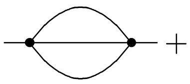
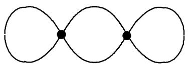
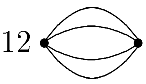
 The counting is almost the same as for the previous graph. However, since the graph is not symmetric, there is an additional factor of 2 for number of ways of choosing the upper and lower vertex: total factor 144.

 There are 4 possibilities to select the external leg at both vertices. Then there are 3×2 ways of pairing the internal lines: factor $4 \times 4 \times 6 = 96$.

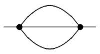
 6×6 possibilities of selecting the legs for the internal lines that connect the two vertices; 2 possibilities to form these two connecting lines $\Rightarrow$ factor $6 \times 6 \times 2 = 72$.

 The first leg can be paired with any of the for legs of the other vertex, the second leg with the remaining 3, etc. $\Rightarrow$ factor $4 \times 3 \times 2 = 24$.

Collecting everything, the result for the second-order cumulant is

$\frac{1}{2}\left\langle V^2 \right\rangle_c = 36$  $+\,72$  $+\,48$  $+\,36$  $+\,12$  . (5.79)

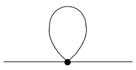
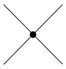
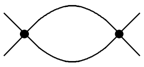
V.8.4 Fixed Points to Order ϵ

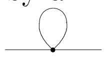
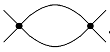
To find the renormalized coupling constants (5.74) we still have to do the steps (ii) and (iii). Step (ii) (rescaling of $\mathbf{k}$ so that the upper bound is again Λ) gives a factor b^{-d} for the quadratic terms and a factor b^{-3d} for the quartic terms. In step (iii) the fields $\widehat{\Phi}$ have to be rescaled so that the coefficient of the $k^2\widehat{\Phi}(\mathbf{k})\widehat{\Phi}(-\mathbf{k})$ term is again $1/2$. Since we are going to determine the fixed point only to order $\epsilon \sim u$ and ignore all terms $\sim u^2$ (in particular ) in the recursion relation for r_0, we do not get any contributions to this coefficient from step (i). Therefore, the scale factor is still [see (5.73)] $\zeta(b) = b^{1+d/2}$ and consequently

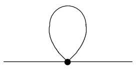
$$\eta = 2 + d - \frac{\log \zeta}{\log b} = 0 \; . \tag{5.80}$$

Hence after steps (ii) and (iii) the renormalized coupling constants are

$$
\begin{aligned}
r_0' &= b^{-d}\zeta^2(b)\left(r_0 + 12 \right. && \text{a)} \\
\frac{u}{4} &= b^{-3d}\zeta^4(b)\left(\quad - 36 \right) && \text{b)}
\end{aligned}
\tag{5.81}
$$

In (5.81) the Feynman graphs must be understood not to contain the external $\widehat{\Phi}^<$ fields and the integration over those fields. Below such an omission will be indicated by a $\sim$ sign instead of an $=$ sign. In order to proceed we have to calculate the diagrams  and .



$$
\sim \frac{u}{4}\int_{\mathbf{k}}^{>} G_0(\mathbf{k}) = \frac{u}{4}\int_{\Lambda/b<k<\Lambda} \frac{d^dk}{(2\pi)^d}\frac{1}{r_0+k^2} \sim \frac{u}{4}\frac{A_d}{(2\pi)^2}\int_{\Lambda/b}^{\Lambda} dk\frac{k^{d-1}}{r_0+k^2}
$$

with $A_d = 2\pi^{d/2}/\Gamma(d/2)$ the surface area of the d-dimensional unit sphere. We now set $b = 1+\delta\ell$ and evaluate the integral for $\delta\ell \ll 1$:

$$
\int_{\Lambda/b}^{\Lambda} dk\frac{k^{d-1}}{r_0+k^2} \simeq \frac{\Lambda^{d-1}}{r_0+\Lambda^2}\left[\Lambda - \frac{\Lambda}{b}\right] = \frac{\Lambda^d}{r_0+\Lambda^2}\left[1-\frac{1}{1+\delta\ell}\right] \simeq \frac{\Lambda^d}{r_0+\Lambda^2}\delta\ell \ .
$$

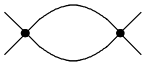
Therefore,

$$
r_0' = b^2\left(r_0 + 3u\frac{A_d}{(2\pi)^d}\frac{\Lambda^d}{r_0+\Lambda^2}\delta\ell\right) \simeq r_0 + \left[2r_0 + 3u\frac{A_d}{(2\pi)^d}\frac{\Lambda^d}{r_0+\Lambda^2}\right]\delta\ell \ .
$$

Using $A_4/(2\pi)^d = 2\pi^2/(2\pi)^4 = \frac{1}{8\pi^2}$ we find

$$
r_0' = r_0 + \left[1r_0 + \frac{3}{8\pi^2}u\frac{\Lambda^4}{r_0+\Lambda^2}\right]\delta\ell + \mathcal{O}(u^2, u\epsilon) \ . \tag{5.82}
$$

For (5.81b) we need to calculate



$$
\sim \frac{u^2}{16}\int^{>}\frac{d^dk}{(2\pi)^d}\frac{1}{r_0+k^2}\frac{1}{r_0+(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})^2} =: I((\mathbf{k}_1+\mathbf{k}_2)^2) \ .
$$

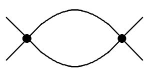
This diagram leads to a contribution

$$
\sim \int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} \widehat{\Phi}(\mathbf{k}_1)\widehat{\Phi}(\mathbf{k}_2)\widehat{\Phi}(\mathbf{k}_3)\widehat{\Phi}(\mathbf{k}_4)(2\pi)^d\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)I((\mathbf{k}_1+\mathbf{k}_2)^2) \ .
$$

Expanding $I(\mathbf{k}^2)$ in powers of $\mathbf{k}^2$ will generate terms of order $\widehat{\Phi}^4$, $\mathbf{k}^2\widehat{\Phi}^4$, $\mathbf{k}^4\widehat{\Phi}^4$, etc. It is easily shown by power counting that all terms, but the $\widehat{\Phi}^4$ term, are irrelevant. Thus, it suffices to find $I(0)$.

$$
I(0) = \frac{u^2}{16}\int_{\mathbf{k}}^{>}\frac{1}{(r_0+\mathbf{k}^2)^2} = \frac{u^2}{16}\frac{A_d}{(2\pi)^d}\int_{\Lambda/b}^{\Lambda} dk\frac{k^{d-1}}{(r_0+\mathbf{k}^2)^2} \simeq \frac{u^2}{16}\frac{A_d}{(2\pi)^d}\frac{\Lambda^d}{(r_0+\Lambda^2)^2}\delta\ell
$$

and therefore



$$\sim \frac{u^2}{128\pi^2}\frac{\Lambda^4}{(r_0+\Lambda^2)^2}\,\delta\ell \ .$$

Thus, the recursion relations (5.81) become

$$\begin{aligned} r_0' &= r_0 + \left[2r_0 + \frac{3}{8\pi^2}u\frac{\Lambda^4}{r_0+\Lambda^2}\right]\delta\ell + \mathcal{O}(u^2, u\epsilon) && \text{a)} \\ u' &= b^\epsilon\left(u - \frac{9}{8\pi^2}u^2\frac{\Lambda^4}{(r_0+\Lambda^2)^2}\delta\ell\right) \simeq u + u\left[\epsilon - \frac{9}{8\pi^2}u\frac{\Lambda^4}{(r_0+\Lambda^2)^2}\right]\delta\ell + \mathcal{O}(u^2\epsilon) \ . && \text{b)} \end{aligned} \tag{5.83}$$

(5.83b) finally gives a nonzero result for the fixed point value of u: setting $u^* = u' = u$ we find, besides the value $u_G^* = 0$ of the Gaussian fixed point, a nonzero value for the new Ising fixed point:

$$u_I^* = \frac{8\pi^2}{9}\frac{(r_{0I}^*+\Lambda^2)^2}{\Lambda^4}\epsilon \ , \tag{5.84}$$

where r_{0I}^* is to be determined from (5.83a), i.e.,

$$2r_{0I}^* = -\frac{3}{8\pi^2}u_I^*\frac{\Lambda^4}{(r_{0I}^*+\Lambda^2} + \mathcal{O}(u^2, u\epsilon) = -\frac{1}{3}\left(r_{0I}^*+\Lambda^2\right)\epsilon + \mathcal{O}(u^2, u\epsilon) \ . \tag{5.85}$$

The result (5.85) shows that $r_{0I}^* = \mathcal{O}(\epsilon)$ as well. Hence we can neglect r_{0I}^* on the right-hand side and find the following values for the parameters at the Ising fixed point:

$$\begin{aligned} r_{0I}^* &= -\frac{\Lambda^2}{6}\epsilon && \text{a)} \\ u_I^* &= \frac{8\pi^2}{9}\epsilon \ . && \text{b)} \end{aligned} \tag{5.86}$$

V.8.5 Critical Exponents to Order ϵ

In order to find the critical exponents for the Ising fixed point, we need to linearize the recursion relations (5.83) at the fixed point (5.86):

$$\begin{aligned} \left.\frac{\partial r_0'}{\partial r_0}\right|_{\substack{r_0=r_{0I}^* \\ u=u_I^*}} &= 1 + \left[2 - \frac{3}{8\pi^2}u_I^*\frac{\Lambda^4}{(r_{0I}^*+\Lambda^2)^2}\right]\delta\ell = 1 + \left[2 - \frac{\epsilon}{3}\right]\delta\ell + \mathcal{O}(\epsilon^2) \ , \\ \left.\frac{\partial r_0'}{\partial u}\right|_{\substack{r_0=r_{0I}^* \\ u=u_I^*}} &= \frac{3}{8\pi^2}\frac{\Lambda^2}{1-\frac{\epsilon}{6}}\delta\ell \simeq \frac{\Lambda^2}{16\pi^2}\epsilon\delta\ell \ , \\ \frac{\partial u'}{\partial r_0} &= \mathcal{O}(\epsilon^2) \ , \\ \left.\frac{\partial u'}{\partial u}\right|_{\substack{r_0=r_{0I}^* \\ u=u_I^*}} &= 1 + \left[\epsilon - \frac{9}{4\pi^2}u_I^*\right]\delta\ell + \mathcal{O}(\epsilon^2) \simeq 1 - \epsilon\delta\ell \ . \end{aligned}$$

Because terms of order ϵ^2 have been neglected in the derivation of (5.83a) one may argue that r_0 in the term multiplying u in (5.83a) must be set to zero for consistency reasons since r_0 is of order ϵ as well. Literally, that is correct. In that case no corrections to the mean-field exponents would be found in this order of the calculation. The reason that r_0 is kept when doing the derivatives above is the following: In the derivation of (5.83a) terms of order u^2 and $u\epsilon$ have been neglected. These terms give $\mathcal{O}(\epsilon^2)$ contributions to the derivative $\partial r_0'/\partial r_0$. Since no terms of order $r_0 u$ or $r_0\epsilon$ were neglected in the derivation of (5.83a) the derivative $\partial r_0'/\partial r_0$ is still correct to order ϵ. This justifies the expression written down for $\partial r_0'/\partial r_0$ above.

Thus, the renormalization group equations (5.83) linearized about the Ising fixed point have the form $\binom{r_0'-r_{0I}^*}{u'-u_I^*} = \underset{\sim}{L}\binom{r_0-r_{0I}^*}{u-u_I^*}$ with

$$\underset{\sim}{L} = \begin{pmatrix} 1+\left[2-\frac{\epsilon}{3}\right]\delta\ell & \frac{\Lambda^2}{16\pi^2}\epsilon\delta\ell \\ 0 & 1-\epsilon\delta\ell \end{pmatrix} . \tag{5.87}$$

$\underset{\sim}{L}$ has an eigenvector $\mathbf{e}_1 = \binom{1}{0}$ with eigenvalue $\lambda_t = 1+(2-\epsilon/3)\delta\ell \stackrel{!}{=} b^{y_t} \simeq 1+y_t\delta\ell$. Hence,

$$\begin{aligned} y_t &= 2-\frac{\epsilon}{3} && \text{a)} \\ \text{and} \qquad \nu &= \frac{1}{y_t} = \frac{1}{2-\epsilon/3} \simeq \frac{1}{2}+\frac{\epsilon}{6} && \text{b)} \end{aligned} \tag{5.88}$$

The second eigenvalue is found from the relation $\mathrm{Tr}\,\underset{\sim}{L} = \lambda_t+\lambda_u$, i.e., $\lambda_u = 1-\epsilon\delta\ell = b^{y_u} \simeq 1+y_u\delta\ell$ and therefore

$$y_u = -\epsilon \ . \tag{5.89}$$

Since $y_u = -\epsilon = -(4-d) < 0$ for $d<4$, the Ising fixed point is stable for $d<4$! In the following table the critical exponents for the Ising model to order ϵ are listed together with their values for $d=3$, i.e., $\epsilon=1$.

(5.88b)	$\nu = \frac{1}{2}+\frac{\epsilon}{12}$;	$d=3,\ \epsilon=1$: $\nu = \frac{7}{12} \simeq 0.58$
(5.80)	$\eta = \mathcal{O}(\epsilon^2)$;	$\eta \simeq 0$
(5.34)	$\alpha = \frac{\epsilon}{6}$;	$\alpha = \frac{1}{6} \simeq 0.17$
(5.39)	$\gamma = 1+\frac{\epsilon}{6}$;	$\gamma = \frac{7}{6} \simeq 1.17$
(5.36)	$\beta = \frac{1}{2}-\frac{\epsilon}{6}$;	$\beta = \frac{1}{3} \simeq 0.33$
(5.37)	$\delta = 3+\epsilon$;	$\delta = 4$

Comparison with the values gicen in Ch. V.4 shows that this is already a tremendous improvement over mean-field theory!