Balanced Generalized Weighing Matrices and Optimal Codes

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Summary

- Constant weight error-correcting codes.
- Balanced generalized weighing matrices (BGWs).
- Use BGWs to construct optimal constant weight codes.

Codes

- A finite collection of "strings" (say $\mathscr{C}$) of given length over a given finite alphabet (say $\mathcal{A}$).
- $\mathcal{A}$ has 0.
- Does not assume that $\mathcal{A}$ is endowed with an arithmetic.
- Usually take $\mathcal{A} = GF(q)$.

Codes

- Take $\mathcal{A} = GF(5) = \{0, 1, \omega, \omega^2, \omega^3\}$, where ω is some primitive element.

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$$\underbrace{\begin{matrix} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ \omega & \omega^3 & 1 & \omega^3 & 0 & 1 \end{matrix}}_{6}$$

- Length 6 ($n = 6$).

Codes

- Take $\mathcal{A} = GF(5) = \{0, 1, \omega, \omega^2, \omega^3\}$, where ω is some primitive element.

$$2 \left\{ \begin{array}{cccccc} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ \omega & \omega^3 & 1 & \omega^3 & 0 & 1 \end{array} \right.$$

- Number of codewords is 2 ($M = 2$).

Codes

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- Take $\mathcal{A} = GF(5)$, where ω is some primitive element.

$$\begin{array}{cccccc} \color{red}{\omega^3} & \color{red}{1} & \color{red}{\omega^3} & 0 & \color{red}{1} & \color{red}{1} \\ \omega & \omega^3 & 1 & \omega^3 & 0 & 1 \end{array}$$

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- Constant weight 5. $(w = 5)$

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- Take $\mathcal{A} = GF(5)$ and ω some primitive element.

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- The codewords are at Hamming distance 5.

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- $d = \min_{\substack{c,c' \in \mathscr{C} \\ c \neq c'}} d(c, c')$.

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- The codewords are at Hamming distance 5.
- $d = \min_{\substack{c,c' \in \mathscr{C} \\ c \neq c'}} d(c, c')$.
- $d = 5$.

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$$\begin{array}{cccccc} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ \omega & \omega^3 & 1 & \omega^3 & 0 & 1 \end{array}$$

- Write $(6, 5, 5)_5$-code.

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$$\begin{matrix} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ \omega & \omega^3 & 1 & \omega^3 & 0 & 1 \end{matrix}$$

- Write $(6, 5, 5)_5$-code.
- More generally, $(n, d, w)_q$-code.

Codes

- Fundamental Question:

$$\text{Max } M$$
$$\text{Given } n, w, d, q.$$

denoted $A_q(n, d, w)$.

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Restricted Johnson Bound

$$A_q(n,d,w) \leq \left\lfloor \frac{nd(q-1)}{qw^2 - 2(q-1)nw + nd(q-1)} \right\rfloor, \tag{1}$$

if $qw^2 - 2(q-1)nw + nd(q-1) > 0$.

Optimal Code

$$\begin{matrix} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ \omega & \omega^3 & 1 & \omega^3 & 0 & 1 \end{matrix}$$

Optimal Code

$$\omega^3 \quad 1 \quad \omega^3 \quad 0 \quad 1 \quad 1$$

Optimal Code

$$\begin{array}{cccccc} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ & \omega^3 & & & & \end{array}$$

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$$\omega^3 \quad 1 \quad \omega^3 \quad 0 \quad 1 \quad 1$$
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$$\begin{array}{cccccc} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ & \omega^3 & 1 & \omega^3 & & \end{array}$$

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Optimal Code

$$\begin{matrix} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ \omega & \omega^3 & 1 & \omega^3 & 0 & 1 \end{matrix}$$

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$$\begin{matrix} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ \omega & \omega^3 & 1 & \omega^3 & 0 & 1 \\ \omega & \omega & \omega^3 & 1 & \omega^3 & 0 \end{matrix}$$

Optimal Code

$$\begin{matrix} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ \omega & \omega^3 & 1 & \omega^3 & 0 & 1 \\ \omega & \omega & \omega^3 & 1 & \omega^3 & 0 \\ 0 & \omega & \omega & \omega^3 & 1 & \omega^3 \end{matrix}$$

Optimal Code

$$
\begin{matrix}
\omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\
\omega & \omega^3 & 1 & \omega^3 & 0 & 1 \\
\omega & \omega & \omega^3 & 1 & \omega^3 & 0 \\
0 & \omega & \omega & \omega^3 & 1 & \omega^3 \\
1 & 0 & \omega & \omega & \omega^3 & 1
\end{matrix}
$$

Optimal Code

$$
\begin{matrix}
\omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\
\omega & \omega^3 & 1 & \omega^3 & 0 & 1 \\
\omega & \omega & \omega^3 & 1 & \omega^3 & 0 \\
0 & \omega & \omega & \omega^3 & 1 & \omega^3 \\
1 & 0 & \omega & \omega & \omega^3 & 1 \\
\omega & 1 & 0 & \omega & \omega & \omega^3
\end{matrix}
$$

Optimal Code

$$
\begin{array}{cccccc}
\omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\
\omega & \omega^3 & 1 & \omega^3 & 0 & 1 \\
\omega & \omega & \omega^3 & 1 & \omega^3 & 0 \\
0 & \omega & \omega & \omega^3 & 1 & \omega^3 \\
1 & 0 & \omega & \omega & \omega^3 & 1 \\
\omega & 1 & 0 & \omega & \omega & \omega^3 \\
\hline
1 & \omega & 1 & 0 & \omega & \omega \\
\omega^2 & 1 & \omega & 1 & 0 & \omega \\
\omega^2 & \omega^2 & 1 & \omega & 1 & 0 \\
0 & \omega^2 & \omega^2 & 1 & \omega & 1 \\
\omega & 0 & \omega^2 & \omega^2 & 1 & \omega \\
\omega^2 & \omega & 0 & \omega^2 & \omega^2 & 1
\end{array}
$$

Optimal Code

$$
\left.\begin{array}{cccccc|cccccc}
\omega^3 & 1 & \omega^3 & 0 & 1 & 1 & \omega & \omega^2 & \omega & 0 & \omega^2 & \omega^2 \\
\omega & \omega^3 & 1 & \omega^3 & 0 & 1 & \omega^3 & \omega & \omega^2 & \omega & 0 & \omega^2 \\
\omega & \omega & \omega^3 & 1 & \omega^3 & 0 & \omega^3 & \omega^3 & \omega & \omega^2 & \omega & 0 \\
0 & \omega & \omega & \omega^3 & 1 & \omega^3 & 0 & \omega^3 & \omega^3 & \omega & \omega^2 & \omega \\
1 & 0 & \omega & \omega & \omega^3 & 1 & \omega^2 & 0 & \omega^3 & \omega^3 & \omega & \omega^2 \\
\omega & 1 & 0 & \omega & \omega & \omega^3 & \omega^3 & \omega^2 & 0 & \omega^3 & \omega^3 & \omega \\
\hline
1 & \omega & 1 & 0 & \omega & \omega & & & & & & \\
\omega^2 & 1 & \omega & 1 & 0 & \omega & & & & & & \\
\omega^2 & \omega^2 & 1 & \omega & 1 & 0 & & & & & & \\
0 & \omega^2 & \omega^2 & 1 & \omega & 1 & & & & & & \\
\omega & 0 & \omega^2 & \omega^2 & 1 & \omega & & & & & & \\
\omega^2 & \omega & 0 & \omega^2 & \omega^2 & 1 & & & & & &
\end{array}\right.
$$

Optimal Code

$$
\left[\begin{array}{cccccc|cccccc}
\omega^3 & 1 & \omega^3 & 0 & 1 & 1 & \omega & \omega^2 & \omega & 0 & \omega^2 & \omega^2 \\
\omega & \omega^3 & 1 & \omega^3 & 0 & 1 & \omega^3 & \omega & \omega^2 & \omega & 0 & \omega^2 \\
\omega & \omega & \omega^3 & 1 & \omega^3 & 0 & \omega^3 & \omega^3 & \omega & \omega^2 & \omega & 0 \\
0 & \omega & \omega & \omega^3 & 1 & \omega^3 & 0 & \omega^3 & \omega^3 & \omega & \omega^2 & \omega \\
1 & 0 & \omega & \omega & \omega^3 & 1 & \omega^2 & 0 & \omega^3 & \omega^3 & \omega & \omega^2 \\
\omega & 1 & 0 & \omega & \omega & \omega^3 & \omega^3 & \omega^2 & 0 & \omega^3 & \omega^3 & \omega \\
\hline
1 & \omega & 1 & 0 & \omega & \omega & \omega^2 & \omega^3 & \omega^2 & 0 & \omega^3 & \omega^3 \\
\omega^2 & 1 & \omega & 1 & 0 & \omega & 1 & \omega^2 & \omega^3 & \omega^2 & 0 & \omega^3 \\
\omega^2 & \omega^2 & 1 & \omega & 1 & 0 & 1 & 1 & \omega^2 & \omega^3 & \omega^2 & 0 \\
0 & \omega^2 & \omega^2 & 1 & \omega & 1 & 0 & 1 & 1 & \omega^2 & \omega^3 & \omega^2 \\
\omega & 0 & \omega^2 & \omega^2 & 1 & \omega & \omega^3 & 0 & 1 & 1 & \omega^2 & \omega^3 \\
\omega^2 & \omega & 0 & \omega^2 & \omega^2 & 1 & 1 & \omega^3 & 0 & 1 & 1 & \omega^2
\end{array}\right]
$$

Optimal Code

- Parameters: $n = 6, q = 5, d = 5, w = 5, M = 24$.

$$\left\lfloor \frac{nd(q-1)}{qw^2 - 2(q-1)nw + nd(q-1)} \right\rfloor = \left\lfloor \frac{6 \cdot 5 \cdot 4}{5^3 - 2 \cdot 4 \cdot 6 \cdot 5 + 6 \cdot 5 \cdot 4} \right\rfloor$$
$$= 24$$

- The code is optimal.
- $A_5(6, 5, 5) = 24$.

BGWs

- G some finite group.
- $W = [w_{ij}]$ a $(0, G)$-matrix of order v.
- k non-zero entries in every row.
- The multisets

$$\{w_{ih}w_{jh}^{-1} : w_{ih} \neq 0 \neq w_{jh}, 0 \leq h < v\}, \text{ for } i \neq j.$$

  contain each group element a constant $\lambda/|G|$ times.
- W is a balanced generalized weighing matrix.
- Write $BGW(v, k, \lambda; G)$.

BGWs

ω^3	1	ω^3	0	1	1	ω	ω^2	ω	0	ω^2	ω^2
ω	ω^3	1	ω^3	0	1	ω^3	ω	ω^2	ω	0	ω^2
ω	ω	ω^3	1	ω^3	0	ω^3	ω^3	ω	ω^2	ω	0
0	ω	ω	ω^3	1	ω^3	0	ω^3	ω^3	ω	ω^2	ω
1	0	ω	ω	ω^3	1	ω^2	0	ω^3	ω^3	ω	ω^2
ω	1	0	ω	ω	ω^3	ω^3	ω^2	0	ω^3	ω^3	ω
1	ω	1	0	ω	ω	ω^2	ω^3	ω^2	0	ω^3	ω^3
ω^2	1	ω	1	0	ω	1	ω^2	ω^3	ω^2	0	ω^3
ω^2	ω^2	1	ω	1	0	1	1	ω^2	ω^3	ω^2	0
0	ω^2	ω^2	1	ω	1	0	1	1	ω^2	ω^3	ω^2
ω	0	ω^2	ω^2	1	ω	ω^3	0	1	1	ω^2	ω^3
ω^2	ω	0	ω^2	ω^2	1	1	ω^3	0	1	1	ω^2

BGWs

$$\begin{matrix} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ \omega & \omega & \omega^3 & 1 & \omega^3 & 0 \end{matrix}$$

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$$\begin{matrix} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ \omega & \omega & \omega^3 & 1 & \omega^3 & 0 \end{matrix}$$

BGWs

$$\begin{matrix} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ \omega^{-1} & \omega^{-1} & \omega^{-3} & 1 & \omega^{-3} & 0 \end{matrix}$$

BGWs

$$\begin{array}{cccccc} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ \omega^{-1} & \omega^{-1} & \omega^{-3} & 1 & \omega^{-3} & 0 \\ \hline \omega^2 & \omega^3 & 1 & & \omega & \end{array}$$

BGWs

$$\begin{array}{cccccc}
\omega^{3} & 1 & \omega^{3} & 0 & 1 & 1 \\
\omega^{-1} & \omega^{-1} & \omega^{-3} & 1 & \omega^{-3} & 0 \\
\hline
\{\omega^{2}, & \omega^{3}, & 1, & & \omega\} &
\end{array}$$

Trace Construction

- q a prime power, $m > 1$.
- $K = GF(q)$, $F = GF(q^m)$.
- Relative trace $F \to K$:

$$\mathrm{Tr}_{F/K}(\alpha) = \alpha + \alpha^q + \cdots + \alpha^{q^{m-1}}, \qquad \alpha \in F.$$

- $\beta \in F$ a primitive element.
- $\omega = \beta^{-\ell}$, where $\ell = \frac{q^m-1}{q-1}$.

Trace Construction

- Construct the ℓ-dimensional vector

$$u = (\mathrm{Tr}_{F/K}(\beta^0), \mathrm{Tr}_{F/K}(\beta^1), \ldots, \mathrm{Tr}_{F/K}(\beta^{\ell-1})).$$

- Take u as the first row of W.
- Remaining rows are the first $\ell - 1$ ω-shifts of u.
- Jungnickel and Tonchev (2002) showed that these structures are

$$BGW\left(\frac{q^m - 1}{q - 1}, q^{m-1}, q^{m-1} - q^{m-2}; GF(q)^*\right) s.$$

Codes From BGWs

- If W is a classical parameter BGW over $GF(q)^*$, then the rows of $W, \omega W, \ldots, \omega^{q-2} W$, form an optimal, constant weight code.
- Can be assumed to be generated by single codeword.
- Parameters:

$$n = \frac{q^m - 1}{q - 1}, d = q^{m-1}, w = q^{m-1}, M = q^m - 1.$$

Theorem

$$A_q\left(\frac{q^m - 1}{q - 1}, q^{m-1}, q^{m-1}\right) = q^m - 1,$$

Projections

$$\begin{bmatrix} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ \omega & \omega^3 & 1 & \omega^3 & 0 & 1 \\ \omega & \omega & \omega^3 & 1 & \omega^3 & 0 \\ 0 & \omega & \omega & \omega^3 & 1 & \omega^3 \\ 1 & 0 & \omega & \omega & \omega^3 & 1 \\ \omega & 1 & 0 & \omega & \omega & \omega^3 \end{bmatrix}$$

- A $BGW(6, 5, 4; GF(5)^*)$.

Projections

$$\begin{bmatrix} \omega^3 & 1 & \omega^3 & 0 & 1 & 1 \\ \omega & \omega^3 & 1 & \omega^3 & 0 & 1 \\ \omega & \omega & \omega^3 & 1 & \omega^3 & 0 \\ 0 & \omega & \omega & \omega^3 & 1 & \omega^3 \\ 1 & 0 & \omega & \omega & \omega^3 & 1 \\ \omega & 1 & 0 & \omega & \omega & \omega^3 \end{bmatrix}$$

- A $BGW(6,5,4;GF(5)^*)$.
- Apply $\omega \mapsto -1$.

Projections

$$\begin{bmatrix} -1 & 1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 1 & -1 \\ 1 & 0 & -1 & -1 & -1 & 1 \\ -1 & 1 & 0 & -1 & -1 & -1 \end{bmatrix}$$

Projections

$$\begin{bmatrix} -1 & 1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 1 & -1 \\ 1 & 0 & -1 & -1 & -1 & 1 \\ -1 & 1 & 0 & -1 & -1 & -1 \end{bmatrix}$$

- A $BGW(6,5,4;\{-1,1\})$.

Ternary Codes

- If q is odd, then apply $\omega \mapsto -1$.
- The result is a BGW over $\{-1, 1\}$.
- The matrix and its negative form an optimal ternary code.

Theorem

$$A_3\left(\frac{q^m-1}{q-1}, q^{m-2}\left(\frac{q+3}{2}\right), q^{m-1}\right) = 2\left(\frac{q^m-1}{q-1}\right),$$

for q odd.

- Östergård and Svanström (2002) considered the case $m = 2$.

- Our optimal constant weight $(6,5,5)_5$-code becomes...

$$\begin{array}{rrrrrr}
-1 & 1 & -1 & 0 & 1 & 1 \\
-1 & -1 & 1 & -1 & 0 & 1 \\
-1 & -1 & -1 & 1 & -1 & 0 \\
0 & -1 & -1 & -1 & 1 & -1 \\
1 & 0 & -1 & -1 & -1 & 1 \\
-1 & 1 & 0 & -1 & -1 & -1 \\
\hline
1 & -1 & 1 & 0 & -1 & -1 \\
1 & 1 & -1 & 1 & 0 & -1 \\
1 & 1 & 1 & -1 & 1 & 0 \\
0 & 1 & 1 & 1 & -1 & 1 \\
-1 & 0 & 1 & 1 & 1 & -1 \\
1 & -1 & 0 & 1 & 1 & 1
\end{array}$$

... an optimal constant weight $(6,3,5)_3$-code.

The End!!

Thank You!

References

Jungnickel, D. and Tonchev, V. D. (2002). Perfect codes and balanced generalized weighing matrices. II. *Finite Fields Appl.*, 8(2):155–165.

Östergård, P. R. J. and Svanström, M. (2002). Ternary constant weight codes. *Electron. J. Combin.*, 9(1):Research Paper 41, 23.

Codes

Unrestricted Johnson Bound

1. If $2w < d$, then $A_q(n, d, w) = 1$; and
2. if $2w \geq d$ and $d \in \{2e - 1, 2e\}$, then

$$A_q(n, d, w) \leq \left\lfloor \frac{n(q-1)}{w} \left\lfloor \frac{(n-1)(q-1)}{w-1} \left\lfloor \cdots \left\lfloor \frac{(n-w+e)(q-1)}{e} \right\rfloor \cdots \right\rfloor \right\rfloor \right\rfloor$$