SMALL OCTAHEDRAL SYSTEMS

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Abstract. We consider set systems that satisfy a certain octahedral parity property. Such systems arise when studying the colourful simplices formed by configurations of points of in $\mathbb{R}^d$; configurations of low colourful simplicial depth correspond to systems with small cardinality. This construction can be used to find lower bounds computationally for the minimum colourful simplicial depth of a configuration, and, for a relaxed version of colourful depth, provide a simple proof of minimality.

1. Introduction

We are interested in set systems of the following type: the base set $\mathbf{S}$ is partitioned into *colours* $\mathbf{S}_1, \mathbf{S}_2, \ldots \mathbf{S}_m$ for some m, and the sets consist of one element from each $\mathbf{S}_i$. In other words, these are m-uniform hypergraphs where each hyperedge has a unique intersection with each colour $\mathbf{S}_i$, we will sometimes refer to the sets that belong to a given system as *edges*. We call a subset of $\mathbf{S}$ *colourful* if it contains at most one point from each $\mathbf{S}_i$. Thus the edges of any system are colourful. When a colourful set has a point from $\mathbf{S}_i$, we will call this point the ith *coordinate* of the set.

We call a colourful set of $m-1$ points which misses $\mathbf{S}_i$ an $\widehat{i}$-*transversal*, and call any pair of disjoint $\widehat{i}$-*transversal* an *octahedron*. We say that an m-uniform collection of colourful edges forms an *octahedral system* if it satisfies the following property:

Property 1.1. *For any octahedron $\mathbf{\Omega}$, the parity of the set of edges using points from $\mathbf{\Omega}$ and a fixed point s_i for the ith coordinate is the same for all choices of s_i.*

The term octahedron comes from the following geometric motivation. A point $p \in \mathbb{R}^d$ has *simplicial depth* k relative to a set S if it is contained in k closed simplices generated by $(d+1)$ sets of S. This was introduced by Liu [Liu90] as a statistical measure of how representative p is of S, and is a source of challenging problems in computational geometry – see for instance [Alo06], [FR05] and [RS04]. More generally, we consider *colourful simplicial depth*, where the single set S is replaced by $(d+1)$

sets, or colours, $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$, and the *colourful* simplices containing p are generated by taking one point from each set.

From any such colourful configuration, we can form a system of vectors $\mathcal{V}$ where $\mathbf{v} = (s_1, \ldots, s_{d+1})$ is in $\mathcal{V}$ if and only if the colourful simplex described by $\mathbf{v}$ contains $\mathbf{0}$. In this context, $\widehat{i}$-*transversals* are simply vectors with the ith coordinate removed, and *octahedra* are pairs of disjoint $\widehat{i}$-transversals. It is a topological fact that such a system satisfies Property 1.1, see for instance the *Octahedron Lemma* of [BM07] for a proof. Thus $\mathcal{V}$ is an octahedral system with $m = d+1$. When the points of an octahedron $\mathbf{\Omega}$ from $\mathcal{V}$ considered as points in $\mathbb{R}^d$ form a cross-polytope, i.e. a d-dimensional octahedron, in the geometric sense that $\text{conv}(\mathbf{\Omega})$ is a cross-polytope and same coloured points are not adjacent in the skeleton of the polytope, then the even and odd case correspond to $\mathbf{0}$ lying inside and outside $\mathbf{\Omega}$ respectively. Figure 1 illustrates this in a two dimensional case where $\mathbf{0}$ is at the centre of a circle that contains points of the three colours.

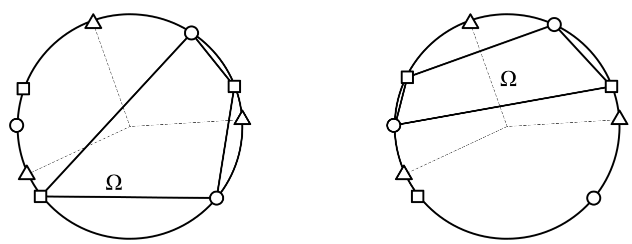

FIGURE 1. Two-dimensional cross-polytopes $\mathbf{\Omega}$ containing $\mathbf{0}$ and not.

It is interesting to get lower bounds for the number of colourful simplices containing p for given configurations, for instance satisfying convexity properties as described in Section 1.1 below. Besides the intrinsic appeal of the problem, its solution is a bound on the number of solutions to a colourful linear program in the sense of [BO97] and [DHST08]. One strategy for establishing this bound is to show that certain small octahedral systems cannot exist. In particular, it leads to two nice combinatorial questions: what is the smallest non-empty octahedral system in terms of the number of edges on m (i.e. $d+1$) sets of m points, and what is the smallest such system where every point is contained in some edge. In Section 2 we show that the answer to the first question is m and use this to prove a conjecture about point configurations. The second question suggests a method of computationally attacking the colourful simplicial depth problem, see below, at least for small dimension. Some progress on this is described in Section 3. Finally, in Section 4 we consider some further questions about octahedral systems.

1.1. **Colourful Simplicial Depth Problems.** Consider the colourful configurations described above. Without loss of generality we assume that $p = \mathbf{0}$ and that the points in $\mathbf{S} \cup \{\mathbf{0}\}$ are in general position. If the convex hulls of the $\mathbf{S}_i$'s contain $\mathbf{0}$ in their interior, we say that the configuration satisfies the *core* condition. Bárány's Colourful Carathéodory Theorem [Bár82] shows that the core conditions imply that $\mathbf{0}$ must be contained in some colourful simplex. In other words, we have $\mu(d) \geq 1$ where $\mu(d)$ denotes the minimum number of colourful simplices drawn from $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ that contain $\mathbf{0}$ for all configurations with the core condition. The sets $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ must each contain at least $(d+1)$ points for $\mathbf{0}$ to be in the interior of their convex hulls, and since we are minimizing we can assume they contain no additional points, i.e. that $|\mathbf{S}_i| = d + 1$ for each i.

The quantity $\mu(d)$ was investigated in [DHST06], where it is shown that $2d \leq \mu(d) \leq d^2+1$, that $\mu(d)$ is even for odd d, and that $\mu(2) = 5$. This paper also conjectures that $\mu(d) = d^2 + 1$ for all $d \geq 1$. Subsequently, [BM07] verified the conjecture for $d = 3$ and provided a lower bound of $\mu(d) \geq \max(3d, \left\lceil \frac{d(d+1)}{5} \right\rceil)$ for $d \geq 3$, while [ST08] independently provided a lower bound of $\mu(d) \geq \left\lfloor \frac{(d+2)^2}{4} \right\rfloor$, before [DSX11] showed that $\mu(d) \geq \lceil \frac{(d+1)^2}{2} \rceil$.

A recent generalization of the Colourful Carathéodory Theorem in [ABB$^+$09] and [HPT08] relaxes the condition of $\mathbf{0}$ being in the convex hull of each $\mathbf{S}_i$ to require only that $\mathbf{0}$ is in the convex hull of $\mathbf{S}_i \cup \mathbf{S}_j$ for all i and j, and $\mathbf{S}_i$ not empty for all i. The analogous quantity $\mu^\diamond(d)$, which denotes the minimum number of colourful simplices drawn from $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ that contain $\mathbf{0} \in \mathbb{R}^d$ given that $|\mathbf{S}_i| = d + 1$ for all i and $\mathbf{0} \in \mathbf{S}_i \cup \mathbf{S}_j$ for each $i \neq j$, has been investigated in [DSX11] where it is shown that $\mu^\diamond(d) \leq d + 1$, $\mu^\diamond(2) = 3$, and $\mu^\diamond(3) = 4$. The associated octahedral system of $(d+1)$ points in $(d+1)$ colours satisfies Property 1.1.

Remark 1.2. Colourful simplicial depth was introduced in the context of lower bounds for ordinary simplicial depth. This problem is quite challenging even in two dimensions: it has been studied at least since Kártesi [Kár55]; the bound of $\frac{1}{27}n^3 + O(n^2)$ was established in [BF84], but the the construction in that paper of a set of points meeting this bound needed to be revised, see [BMN10]. For general d, finding a tight bound remains a challenging problem. Recently Gromov [Gro10] introduced a topological method which among other things improves the lower bound. See also [Kar11].

1.2. **Octahedral Problems.** The strong version of Bárány's Colourful Carathéodory Theorem says that when a colourful configuration satisfies the core condition that every point in $\mathbf{S}$ is part of some colourful simplex. Thus the octahedral system generated by such a colourful configuration must satisfy:

Property 1.3. *Every element of* $\{1, 2, \ldots, d+1\}$ *appears as the ith coordinate of some* $\mathbf{v} \in \mathcal{V}$ *for each* $i \in \{1, 2, \ldots, d+1\}$.

In particular, any colourful configuration satisfying the core condition must generate a system $\mathcal{V}$ satisfying Property 1.1 and Property 1.3. For example, the low colourful simplicial depth configurations of [DHST06] generate such a system with $(d+1)$ sets of $(d+1)$ points, containing (d^2+1) vectors. We define $\nu(d)$ to be the minimum number of vectors in an octahedral system of $(d+1)$ points in $(d+1)$ colours satisfying Properties 1.1 and 1.3, and $\nu^\diamond(d)$ to be the minimum number of vectors of a similar system satisfying Property 1.1 only. Then we have $\nu(d) \leq \mu(d) \leq d^2+1$ and $\nu^\diamond(d) \leq \mu^\diamond(d) \leq d+1$. In Section 2 we show that $\nu^\diamond(d) = \mu^\diamond(d) = d+1$. In Section 3 we show that $\nu(d) = d^2+1$ for $d = 2, 3$, and conjecture that it holds for all d. In particular, computation of $\nu(d)$ for small d gives us a finite procedure that can prove lower bounds for $\mu(d)$.

Remark 1.4. In [DHST06] it was observed that $\mu(d)$ is even for odd d. Similarly it is easy to see that when $m = d+1$ is even, all octahedral systems have an even number of vectors. In particular, both $\nu(d)$ and $\nu^\diamond(d)$ are even for odd d.

2. Proof that $\mu^\diamond(d) = d+1$

A construction in [DSX11] shows that $\nu^\diamond(d) \leq \mu^\diamond(d) \leq d+1$ for $d \geq 2$. In fact, in this section we show that any non-empty octahedral system of $(d+1)$ sets of $(d+1)$ points has at least $(d+1)$ vectors, and hence that $\nu^\diamond(d) = \mu^\diamond(d) = d+1$.

Proposition 2.1. *For any* $d \geq 2$*, we have* $\nu^\diamond(d) = \mu^\diamond(d) = d+1$.

Proof. Assume that there is an octahedron $\mathbf{\Omega}$ consisting of two $\widehat{i}$-transversals and a point $s \in \mathbf{S}_i$ such that there are an odd number of edges using points from $\mathbf{\Omega}$ and s. Then it follows immediately from Property 1.1 that there is at least one edge that uses points from $\mathbf{\Omega}$ and any point in $\mathbf{S}_i$. Therefore $\nu^\diamond(d) \geq d+1$ as $|\mathbf{S}_i| = d+1$.

Assume then that there exists no such octahedron with odd parity, but that the system contains some edge E. We view E as being formed by a $\widehat{i}$-transversal T and a point $s \in \mathbf{S}_i$ and generate edges in the following way. Consider the d disjoint $\widehat{i}$-transversals T_j for $j = 1, 2, \ldots, d$ generated from the remaining points, and the d octahedra $\mathbf{\Omega}_1, \mathbf{\Omega}_2, \ldots, \mathbf{\Omega}_d$ given by pairing T_j with T for $j = 1, 2, \ldots, d$. For each j, besides E, there is at least one other edge that uses s and the points from $\mathbf{\Omega}_j$ due to the even parity. Therefore $\nu^\diamond(d) \geq d + 1$.

In both cases $\nu^\diamond(d) \geq d + 1$. Thus we have $\nu^\diamond(d) = \mu^\diamond(d) = d + 1$ as $\nu^\diamond(d) \leq \mu^\diamond(d) \leq d + 1$. □

In Figure 2 we illustrate the 2-dimensional configuration described in [DSX11] where $\mathbf{0} \in \text{conv}(S_i \cup S_j)$ for all $i \neq j$ and is contained in exactly 3 colourful simplices. In general, the construction is to place

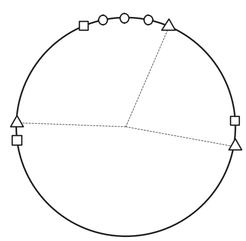

FIGURE 2. Minimal 2-dimensional configuration for the relaxed core condition.

one point of each of the first d colours below the equator in such a way that $\mathbf{0} \in \text{conv}(\mathbf{S}_i)$. Then the conditions are satisfied regardless of the position of the points of $\mathbf{S}_{d+1}$. These points are placed near the north pole in order that each one generates a unique colourful simplex containing $\mathbf{0}$: the simplex is formed using the d points below the equator.

We remark that if we remove the condition that $|\mathbf{S}_i| \geq d+1$ for each i then it is easy to modify the proof to show that $\mathbf{0}$ lies in at least $\min_i |\mathbf{S}_i|$ colourful simplices, and the example can be modified to show that this is tight.

3. Computational Approach

For a given d, the computational approach consists of ruling out a given value k for $\nu(d)$ via an exhaustive computer search showing that no system $\mathcal{V}$ of size k can satisfy Property 1.3 and Property 1.1. This approach was used in [DSX11] on a laptop to show in a few seconds that $\nu(2) > 3$ and in a few hours that $\nu(3) > 8$. In other words, this approach verifies computationally that $\nu(2) = \mu(2) = 5$ and $\nu(3) = \mu(3) = 10$ – using the fact that $\nu(3)$ must be even, see Remark 1.4. Instances of higher dimensions are currently under computation.

In this section we propose ways to normalize the vector system which significantly speed up the enumeration. We also present a constraint programming formulation of the problem.

3.1. **Normalization of vector system.** Recolouring and relabelling of the points does not change the combinatorics of the point configuration. This symmetry will result in many duplicates in enumeration. In order to speed up the enumeration of vector systems for $\nu(d)$ we normalize the vector system in the following ways.

- (i) First, since the vector system $\mathcal{V}$ is not empty, we can assume vector $(0, 0, \ldots, 0) \in \mathcal{V}$.
- (ii) If there is a *covering* octahedron, i.e. one that generates an odd number of vectors for each point of the excluded colour, we can take the excluded colour to be the final one, an octahedron of the system to be $\{(0, \ldots, 0), (1, \ldots, 1)\}$, with the labellings of the points of colours $1, \ldots, d$ chosen so that (i) is satisfied.

A Python routine that searches for small octahedral systems using these normalization is available at [Xie].

3.2. **Pivoting.** We may also be able to take advantage of the following pivoting structure of octahedral systems. Given a particular $\widehat{i}$-transversal T, we can *pivot* from the current octahedral system $\mathbf{\Omega}$ to an *adjacent* one $\mathbf{\Omega}'$ by removing all vectors containing T and replacing them with vectors $T \cup \{s\}$ for each $s \in \mathbf{S}_i$ such that $T \cup \{s\}$ was not in $\mathbf{\Omega}$.

If we have a transversal T which forms vectors with more than half the points of colour i, then pivoting on T will reduce the number of vectors in the system, although it may also break Property 1.3. We remark that pivoting is also seen in the setting of colourful simplicial, it corresponds to moving a point of colour i across a hyperplane defined by and $\widehat{i}$-transversal.

3.3. **Constraint programming approach.** The other computational approach for $\nu(d)$ is to exploit the fact that there is a sphere covering octahedron for each missing colour and model the search for a valid vector system as a constraint programming problem.

We can start with the following collection of vectors $\mathcal{V}^{\circ}$. Each block of $(d+1)$ vectors represents the simplices derived from a sphere covering octahedron for a missing colour.

$$\begin{gathered}(1, x_{1,1}^2, x_{1,1}^3, \ldots, x_{1,1}^{d+1}),\ (2, x_{1,2}^2, x_{1,2}^3, \ldots, x_{1,2}^{d+1}),\ \ldots,\\ (d+1, x_{1,d+1}^2, x_{1,d+1}^3, \ldots, x_{1,d+1}^{d+1});\\ (x_{2,1}^1, 1, x_{2,1}^3, \ldots, x_{3,1}^{d+1})\ ,\ (x_{2,2}^1, 2, x_{2,2}^3, \ldots, x_{2,2}^{d+1}),\ \ldots,\\ (x_{2,d+1}^1, d+1, x_{2,d+1}^3, \ldots, x_{2,d+1}^{d+1});\\ \ldots\\ (x_{d+1,1}^1, \ldots, x_{d+1,1}^d, 1),\ (x_{d+1,2}^1, \ldots, x_{d+1,2}^d, 2),\ \ldots,\\ (x_{d+1,d+1}^1, \ldots, x_{d+1,d+1}^d, d+1).\end{gathered}$$

The domain of each variable is $\{1, 2, \ldots, d+1\}$. Then we have a constraint programming satisfaction problem: Given a value k, find an assignment of values to the variables such that $|\mathcal{V}^\circ| \leq k$ and the following constraints are satisfied:

(1) $x_{1,1}^i = 1$ for all i and $x_{1,j}^i \in \{1, 2\}$ for all i and $j \geq 2$. These constraints are derived from the normalization of the vector system.
(2) $|\{x_{j,1}^i, x_{j,2}^i, \ldots, x_{j,d+1}^i\}| \leq 2$ for all i and j because they are from an octahedron.
(3) Constraints corresponding to Property 1.1.

If no solution is found, then $\nu(d) \neq k$.

4. Conclusions and remarks

Octahedral systems appear to be interesting combinatorial objects. Using the observation that colourful point configurations generate small octahedral systems, we propose a computational approach to establishing lower bounds for colourful simplicial depth. We can ask several other questions about octahedral systems.

We remark that the maximum cardinality octahedral system is the set of all possible edges; if we have m (i.e. $d+1$) sets of cardinality m it has size m^m. As with the other configurations discussed in this paper, it can be realized as arising from a colourful configuration of points in $\mathbb{R}^d$, in this case the one that places the sets $\mathbf{S}_1, \ldots \mathbf{S}_{d+1}$ close to vertices $v_1, \ldots v_{d+1}$ respectively of a regular simplex containing $\mathbf{0}$.

Question 4.1. Can all octahedral systems of $(d+1)$ sets of $(d+1)$ points be obtained as the vectors of point configurations in $\mathbb{R}^d$, and can all such configurations covering all points be obtained as the vectors of configurations satisfying a core condition?

Question 4.2. How many octahedral systems and covering octahedral systems are there for a given m? We remark that for $m = 1$ we have 2 systems, 1 of which is covering, and for $m = 2$ we have 8 and 3; if we count only up to isomorphism these numbers are 4 and 2 respectively.

Question 4.3. Finally, it would be interesting to explore the pivoting structure of octahedral systems by understand its adjacency graph. For instance, we can ask about connectedness, i.e. can we get to any octahedral system from the empty octahedral system via a sequence of pivots? If so, how long must that sequence be?

We conclude by mentioning that many aspects of colourful simplices are just beginning to be explored. For instance, the combinatorial complexity of a system of colour simplices is anaylsed in [ST10]. As far as we know the algorithmic question of computing colourful simplicial depth is untouched, even for $d = 2$ where several interesting algorithms for computing the monochrome simplicial depth have been developed, see for instance [Alo06], [BRS06], [CO01], [EEM11], [GSW92] and [KM90].

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