Appendix A. Example in dimension 4 where **A6** cycles

This example consists of 5 points in each of the 5 colours in $\mathbb{R}^4$. The points are presented in Table 1. They are grouped by colour, with the rows representing x, y, z and w coordinates, respectively.

Red points

7/52	1/89	-1/60	-1/28	4/127
1/176	-8/65	5/49	6/35	9/118
4/29	1/961	-8/191	1/40	-1/75
$-\frac{\sqrt{4238906047}}{66352}$	$\frac{\sqrt{30434652805951}}{5559385}$	$\frac{\sqrt{11360296502737439}}{107254140}$	$-\frac{\sqrt{69789743}}{31640}$	$-7\frac{\sqrt{25600756871}}{1123950}$

Green points

3/85	-5/71	8/45	3/88	-1/114
-1/67	1/10	-38/155	-2/131	-24/185
1/173	-2/101	1/95	3/53	7/85
$\frac{\sqrt{29008089867051134}}{170445655}$	$-\frac{\sqrt{5063381959}}{71710}$	$2\frac{\sqrt{159502559}}{26505}$	$5\frac{\sqrt{14863381455}}{610984}$	$\frac{\sqrt{125498719055}}{358530}$

Blue points

-3/77	4/141	3/22	16/111	-3/46
-3/20	-4/63	-3/17	5/29	3/47
-2/71	-3/173	-5/79	-1/210	1/33
$-\frac{\sqrt{470161115387}}{694309}$	$-8\frac{\sqrt{122080034994545}}{88619769}$	$-\frac{\sqrt{826050579}}{29546}$	$-\frac{\sqrt{48208184671}}{225330}$	$\frac{\sqrt{5043188147}}{71346}$

Tan points

1/59	6/151	8/45	-3/29	11/76
1/29	-1/122	-7/32	4/43	-1/8
3/56	1/536	8/97	-1/14	9/59
$25\frac{\sqrt{14625287}}{95816}$	$\frac{\sqrt{554855708771634695}}{745501496}$	$\frac{\sqrt{17827555751}}{139680}$	$-\frac{\sqrt{297327743}}{17458}$	$\frac{\sqrt{75612155}}{8968}$

White points

1/167	3/43	11/52	-19/65	-3/100
1/241	-1/244	-5/134	2/129	1/62
1/53	2/9	13/142	1/4386	-4/73
$-5\frac{\sqrt{1201121068645021462891}}{173320847963}$	$-\frac{\sqrt{8432767415}}{94428}$	$-\frac{\sqrt{57852799351}}{247364}$	$-\frac{\sqrt{74312211919}}{285090}$	$\frac{\sqrt{50998516979}}{226300}$

Table 1. Coordinates of points of an example where **A6** cycles in dimension 4.

The initial simplex is taken to be (1,1,1,1,1), i.e., the first point of each colour. The algorithm proceeds to visit simplices (1,1,4,1,1), (3,1,4,1,1), (3,1,4,3,1), (3,1,1,3,1) and (1,1,1,3,1) before returning to the original simplex and repeating. At steps one, three and five, there are two candidate colours for pivoting, the candidates that are not chosen for pivoting are 1, 3 and 4 respectively. In the even numbered steps there is a single candidate colour for pivoting.

Appendix B. Example in dimension 3 where **A2** takes 40,847 iterations

This example consists of 4 unnormalized points in each of the 4 colours in $\mathbb{R}^3$. The points are presented in Table 2. They are grouped by colour, with the rows representing x, y and z coordinates, respectively.

The initial simplex is taken to be (1,1,1,1), i.e., the first point of each colour. It then updates to (1,3,1,1), (1,3,2,1), (1,3,2,3), (1,3,2,2) and reaches (3,3,2,2) on the fifth iteration. At this point, it begins to flip between (3,3,2,2) and (2,3,2,2) with y initially alternating

Red points

1.00000320775369	-0.01000436049274	-0.01000129525998	1.00000089660284
0.00000340785030	0.99999739350954	-1.00000497855619	0.00000051797159
0.00999859615603	0.00000371775824	0.00000030149139	-0.01999639732055

Green points

1.00000363763560	-0.00999644886160	-0.00999943004295	1.00000335962280
-0.00000325123594	1.00000064545156	-1.00000169806216	-0.00000080450760
0.01000493174811	-0.00000024088601	0.00000009099437	-0.01999811804365

Blue points

0.99999949817337	-0.00999587145461	-0.00999627213896	0.99999551963712
-0.00000260397964	1.00000485455718	-1.00000419710665	-0.00000024626161
0.00999854691703	0.00000123671997	-0.00000381812529	-0.01999801526314

Tan points

0.99999980645233	0.10000000280522	-0.60000327600988	0.99999642880542
0.00000024487465	-0.98999719313413	0.79999695643245	-0.00000429109491
0.01000455311709	-0.00000405877812	0.00000372117690	-0.01000272055280

Table 2. Coordinates of points of an example taking 40,847 iterations of **A2** in dimension 3.

between values close to (0.2,±0.00200,0.00285). The values of all these coordinates decrease very slowly as the algorithm continues. At iteration 40,847 it chooses fourth point of colour 1 instead of the third. This makes the current simplex (4,3,2,2) which contains **0**.

Appendix C. Iteration counts from our experiments

In this Appendix we present the raw data from our computational experiments. Each table presents results for a single random generator. The entries give the average number of iterations to solution for each algorithm at the given dimension. For each generator at $d = 3$ we sampled 100,000 problems, at $d = 6$ and $d = 12$ we sampled 10,000 problems, at $d = 24$ and $d = 48$ we sampled 1,000 problems and finally for $d \geq 96$ we sampled 100 problems.

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**
$d = 3$	1.31	2.96	1.15	1.15	1.15	1.31	7.15
$d = 6$	2.56	6.87	1.77	1.67	1.67	2.90	63.48
$d = 12$	4.84	13.93	2.42	2.16	2.16	7.01	4133.15
$d = 24$	8.84	27.70	3.07	2.87	2.87	19.07	Large
$d = 48$	16.14	54.88	3.77	4.14	4.14	56.12	Large
$d = 96$	28.80	108.71	4.26	6.39	6.39	185.57	Large
$d = 192$	51.96	217.59	4.99	11.68	11.68	808.78	Large
$d = 384$	Unstable	425.26	Unstable	21.63	Unstable	Large	Large

Table 3. Average iteration counts in **G1** generator tests.

	A1	A2	A3	A4	A5	A6	A7
$d = 3$	5	136	4	4	4	5	102
$d = 6$	7	21	5	5	5	12	579
$d = 12$	10	30	6	6	6	20	47362
$d = 24$	15	37	6	8	8	43	Large
$d = 48$	22	67	6	9	9	105	Large
$d = 96$	39	120	6	10	10	269	Large
$d = 192$	63	241	7	19	19	1574	Large
$d = 384$	Unstable	472	Unstable	30	Unstable	Large	Large

TABLE 4. Maximum iteration counts found in **G1** generator tests.

	A1	A2	A3	A4	A5	A6	A7
$d = 3$	1.39	5.62	1.25	1.43	1.43	1.38	7.30
$d = 6$	2.92	17.00	2.17	3.14	2.89	3.54	66.02
$d = 12$	5.83	33.48	3.23	6.65	5.64	10.26	4296.66
$d = 24$	11.18	64.30	4.29	13.86	10.86	31.75	Large
$d = 48$	20.24	123.02	5.51	27.91	21.11	106.11	Large
$d = 96$	37.12	240.49	6.54	56.70	40.91	406.10	Large
$d = 192$	Unstable	468.52	Unstable	111.84	Unstable	3367.60	Large
$d = 384$	Unstable	909.82	Unstable	220.50	Unstable	Large	Large

TABLE 5. Average iteration counts in **G2** generator tests.

	A1	A2	A3	A4	A5	A6	A7
$d = 3$	5	4783	4	5	5	6	109
$d = 6$	8	2880	6	44	10	14	1079
$d = 12$	13	842	8	60	14	33	78418
$d = 24$	21	217	9	36	23	78	Large
$d = 48$	31	249	9	55	41	258	Large
$d = 96$	47	323	9	77	76	840	Large
$d = 192$	Unstable	561	Unstable	140	Unstable	11784	Large
$d = 384$	Unstable	1013	Unstable	260	Unstable	Large	Large

TABLE 6. Maximum iteration counts found in **G2** generator tests.

	A1	A2	A3	A4	A5	A6	A7
$d = 3$	1.51	5.93	1.31	1.51	1.51	1.48	9.16
$d = 6$	3.48	17.26	2.35	3.31	3.01	4.10	150.31
$d = 12$	7.64	37.22	3.62	8.06	6.43	13.61	Large
$d = 24$	16.59	75.73	5.11	19.11	13.92	48.51	Large
$d = 48$	33.51	155.48	6.57	42.81	28.70	159.29	Large
$d = 96$	61.97	306.64	8.32	90.98	58.44	602.07	Large
$d = 192$	Unstable	619.55	Unstable	186.86	Unstable	9607.73	Large
$d = 384$	Unstable	1221.43	Unstable	382.10	Unstable	Large	Large

TABLE 7. Average iteration counts in **G3** generator tests.

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**
$d = 3$	6	2756	5	6	6	6	127
$d = 6$	9	3704	7	38	9	14	1709
$d = 12$	16	689	8	55	16	46	Large
$d = 24$	28	195	9	52	27	124	Large
$d = 48$	50	257	10	83	47	505	Large
$d = 96$	78	374	11	133	83	2023	Large
$d = 192$	Unstable	736	Unstable	226	Unstable	72317	Large
$d = 384$	Unstable	1399	Unstable	454	Unstable	Large	Large

Table 8. Maximum iteration counts found in **G3** generator tests.

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**
$d = 3$	0.89	2.07	0.82	0.71	0.71	0.89	2.12
$d = 6$	0.99	3.96	0.68	0.66	0.66	0.99	1.94
$d = 12$	0.97	7.61	0.63	0.63	0.63	0.97	1.78
$d = 24$	0.99	15.46	0.64	0.64	0.64	0.99	1.83
$d = 48$	1.01	31.15	0.65	0.65	0.65	1.01	1.87
$d = 96$	1.06	61.44	0.64	0.64	0.64	1.06	1.81
$d = 192$	0.90	122.88	0.64	0.64	0.64	0.90	1.77
$d = 384$	0.77	211.20	0.55	0.55	0.55	0.77	1.50

Table 9. Average iteration counts in **G4** generator tests.

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**
$d = 3$	2	5	2	3	3	2	38
$d = 6$	3	7	2	2	2	3	17
$d = 12$	6	12	1	1	1	6	30
$d = 24$	6	24	1	1	1	6	19
$d = 48$	5	48	1	1	1	5	16
$d = 96$	5	96	1	1	1	5	14
$d = 192$	3	192	1	1	1	4	15
$d = 384$	4	384	1	1	1	4	9

Table 10. Maximum iteration counts found in **G4** generator tests.

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**
$d = 3$	1.26	2.72	0.99	0.91	0.91	1.26	9.67
$d = 6$	2.39	5.97	1.09	0.99	0.99	2.39	161.93
$d = 12$	4.61	12.00	1.12	1.00	1.00	4.61	Large
$d = 24$	8.94	24.00	1.13	1.00	1.00	8.94	Large
$d = 48$	17.82	48.00	1.15	1.00	1.00	17.82	Large
$d = 96$	35.58	96.00	1.19	1.00	1.00	35.58	Large
$d = 192$	71.15	192.00	1.47	1.00	1.00	71.15	Large

Table 11. Average iteration counts in **G5** generator tests.

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**
$d = 3$	3	5	3	2	2	3	128
$d = 6$	5	6	3	1	1	5	1371
$d = 12$	9	12	3	1	1	9	Large
$d = 24$	14	24	2	1	1	14	Large
$d = 48$	24	48	2	1	1	24	Large
$d = 96$	41	96	2	1	1	41	Large
$d = 192$	81	192	3	1	1	81	Large

Table 12. Maximum iteration counts found in **G5** generator tests.

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**
$d = 3$	2.19	3.54	1.96	1.59	1.59	2.26	24.39
$d = 6$	6.27	7.67	3.24	2.23	2.23	6.65	21041.05
$d = 12$	14.64	15.23	3.63	2.92	2.92	16.03	Large
$d = 24$	30.55	30.42	3.40	3.71	3.71	34.25	Large
$d = 48$	61.96	60.95	3.27	4.89	4.89	69.65	Large
$d = 96$	125.31	121.73	3.45	6.26	6.26	140.79	Large
$d = 192$	Unstable	242.06	Unstable	9.31	Unstable	Unstable	Large

Table 13. Average iteration counts in **G6** generator tests.

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**
$d = 3$	5	7	5	4	4	6	242
$d = 6$	12	15	7	6	6	12	173941
$d = 12$	25	25	8	9	9	25	Large
$d = 24$	47	49	9	13	13	51	Large
$d = 48$	101	94	13	22	22	95	Large
$d = 96$	154	174	6	35	35	183	Large
$d = 192$	Unstable	331	Unstable	69	Unstable	Unstable	Large

Table 14. Maximum iteration counts found in **G6** generator tests.

Appendix D. Average time per iteration

In Table 15 we give the average CPU time per iteration for our **G1** experiments. This was computed using the MATLAB `cputime` function.

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**
$d = 3$	0.0075	0.0009	0.0078	0.0019	0.0021	0.0012	0.0002
$d = 6$	0.0087	0.0010	0.0095	0.0033	0.0035	0.0012	0.0002
$d = 12$	0.0124	0.0013	0.0141	0.0073	0.0074	0.0016	0.0004
$d = 24$	0.0229	0.0022	0.0267	0.0182	0.0184	0.0030	0.0007
$d = 48$	0.0625	0.0043	0.0702	0.0474	0.0477	0.0085	0.0014
$d = 96$	0.2510	0.0099	0.2608	0.1318	0.1324	0.0495	0.0035
$d = 192$	1.5592	0.0277	1.2623	0.3275	0.3268	0.7843	0.0121
$d = 384$	Unstable	0.1144	Unstable	1.1381	Unstable	Unstable	0.0619

Table 15. Average iteration times on **G1** generator tests.

The time per iteration is fairly constant across problem types so we do not include data from the other generators. One difference that will occur is that **A5** will have a higher average iteration time as that **A4** for ill-conditioned problems. In random problems, we rarely see slow convergence of **A4** so it is unnecessary to use the slower steps from **A3**. With ill-conditioned problems the **A3** steps become more frequent and increase the average time per iteration.

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