Due: Monday, April 8th (by e-mail)

Reminders: Project presentations begin April 5th. The final exam will be on Monday, April 22nd at noon in SRYC 2750.

1. Problem 4-4 from the `AMPL` book. Please group all the submission files for this assignment into a single e-mail.

2. This problem considers the value of the optimal solution to a linear program when an entry of the matrix A is perturbed by a small constant. Here the linear program is taken to be in equality standard form and A is assumed to have independent rows. Consider a non-degenerate optimal solution $\vec{x}^*$ with corresponding to optimal basis $\mathcal{B}$. Assume also that the corresponding dual solution is non-degenerate. Without loss of generality, take the first column of A to be part of the basis.

Now consider what happens when we change the first entry of the first row of A by a small amount. That is, replace a_{11} by $a_{11} + \delta$ for some small $\delta > 0$. Let E be the $m \times m$ matrix whose only non-zero element is $E_{11} = 1$.

a. Show that if δ is small enough, $A_{\mathcal{B}} + \delta E$ is a basis matrix for the new problem.

b. Show that under the basis $A_{\mathcal{B}} + \delta E$, the vector $\vec{x}_{\mathcal{B}}$ of basic variables in the new problem is $(I + \delta A_{\mathcal{B}}^{-1} E)^{-1} A_{\mathcal{B}}{}^{-1} b$.

c. Show that if δ is sufficiently small, $A_{\mathcal{B}} + \delta E$ is an optimal basis for the new problem.

d. Explain roughly how the objective value changes under this perturbation.

3. Consider the linear program:

$$
\begin{array}{rlllllll}
\text{Maximize} & & x_{12} & & & + x_{22} & + x_{23} & \\
\text{subject to} & x_{11} & & & & & + x_{23} & \leq 15 \\
& x_{11} & + x_{12} & + x_{13} & & & & = 20 \\
& & & & x_{21} & + x_{22} & + x_{23} & = 20 \\
& x_{11} & & & + x_{21} & & & = 20 \\
& & x_{12} & & & + x_{22} & & = 10 \\
& & & x_{13} & & & + x_{23} & = 10 \\
& & x_{ij} \geq 0 & \text{for all} & i, j. & & &
\end{array}
$$

Consider a Dantzig-Wolfe decomposition to this problem where the inequality constraint is the lone linking constraint, and the remaining constraints form a single subproblem.

a. Consider the solutions to the subproblem given by:

$$\vec{x}^1 = (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}) = (20, 0, 0, 0, 10, 10)$$

and

$$\vec{x}^2 = (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}) = (0, 10, 10, 20, 0, 0).$$

Construct a restricted master problem where $\vec{x}$ is constrained to be a convex combination of $\vec{x^1}$ and $\vec{x}^2$. Find the optimal solution to this restricted problem, along with the corresponding dual solution (q, r) corresponding to the inequality and convexity constraints, respectively.

b. Now consider the subproblem formed by removing the inequality constraint and adjusting the objective function by subtracting $q(x_{11} + x_{23})$. Solve the resulting subproblem by inspection.

c. Part (b) should give an extreme point for the subproblem, which thus corresponds to a variable in the master problem. What is the reduced cost of this variable? Is the solution from part (a) optimal?

4. Consider the linear program $\min \vec{c}^t\vec{x}$ subject to $A\vec{x} = \vec{b}$ and $\vec{0} \leq \vec{x} \leq \vec{e}$.

a. Suppose you apply a Dantzig-Wolfe decomposition with a single subproblem $Q = \{\vec{0} \leq \vec{x} \leq \vec{e}\}$. Describe the corresponding master problem.

b. Suppose you apply a Dantzig-Wolfe decomposition where each variable is a subproblem $Q_i = \{0 \leq x_i \leq 1\}$. Describe the corresponding master problem.

5. Consider the problem of finding a feasible point in the set $S = \{1 \leq x + y \leq 2 \mid x, y \geq 0\}$. Suppose that we try to find a feasible point by using the ellipsoid algorithm starting with a centre at the origin.

a. What is the smallest ellipsoid centred at zero containing S? What is its (two-dimensional) volume?

b. Using that ellipsoid, run one iteration of the ellipsoid method to find a second ellipsoid. What is it centre and volume?

c. Of course to compute the smallest bounding ellipsoid in part (a), we relied on detailed knowledge of the feasible set. Suppose that we did not observe anything about the feasible set at the start, but computed an initial ellipsoid based on the input. What would that starting ellipsoid be?

6. Suppose we want to run the affine scaling algorithm described in class for the problem:

$$\text{Minimize } x + y + z \text{ subject to } x + 2y + 3z = 6 \text{ and } x, y, z, \geq 0.$$

Compute the first step using the direction of steepest descent with step length $\frac{1}{2}$.

The research paper presentations will take place on April 5th, 10th and 12th, with two presentations on April 10th.