Empirical Banzhaf Indices

Andrew Heard and Tim Swartz *

Abstract

This paper considers a generalization of the Banzhaf index which is used to quantify voting power in constituent bodies. The generalization allows unequal probabilities in voting outcomes and relies on historical voting data to highlight natural coalitions that exist between various constituent bodies. The approach is based on Bayesian modelling where estimates of voting probabilities are given by the posterior means of Dirichlet distributions. An example is provided concerning the voting power of provinces according to the amending formula to Canada's Constitution.

Keywords : Bayesian modelling, Dirichlet distributions, voting power.

*A. Heard is Assistant Professor, Department of Political Science, Simon Fraser University, Burnaby BC, Canada V5A1S6. T. Swartz is Associate Professor, Department of Mathematics and Statistics, Simon Fraser University, Burnaby BC, Canada V5A1S6. T. Swartz was partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

1. INTRODUCTION

The Banzhaf index (Banzhaf (1965, 1968)) is a numerical measure of power for members of a voting party. For a given voting member, it is defined as the proportion of all voting outcomes in which a reversal of vote by the voting member would result in an opposite decision. For example, with m voting members and a positive decision requiring consensus, each voting member would have a Banzhaf index of $2/2^m = 2^{-(m-1)}$. Alternatively, the Banzhaf index for a voting member can be expressed as the probability that its vote determines the decision given that each combination of "Yes" and "No" votes by the voting members is equally likely. Sometimes the Banzhaf index is scaled so that the indices of all voting members sum to 100. For example, see Kilgour (1983), Levesque and Moore (1984) and Mintz (1985a). However, with such scaling, the probability interpretation is lost.

In an attempt to distinguish between a voting member's ability to veto and to enable a positive decision, the Banzhaf index is sometimes modified. For example, in Mintz (1985b), the power to initiate measure for a given voting member is defined as the proportion of losing coalitions which the member can change to winning by reversing its vote from "No" to "Yes". Again, with m voting members and a positive decision requiring consensus, each voting member would have a power to initiate measure of $1/(2^m - 1)$. The power to initiate measure also admits a probabilistic interpretation. For a given voting member, it is the conditional probability that reversing its vote from "No" to "Yes" causes a positive decision given that each combination of "Yes" and "No" votes by the voting members is equally likely and that a negative decision has been reached.

The power to prevent measure for a given voting member is defined similarly as the proportion of winning coalitions which the member can change to losing by reversing its vote from "Yes" to "No". The power to prevent measure also admits a probabilistic interpretation.

Whether one uses the straight Banzhaf index, the power to initiate measure or the power to prevent measure, all of these indices share the common and sometimes unrealistic assumption of equal probability (ie. 1/2) of a "Yes" and "No" vote for each voting member. In this paper we relax the equal probability assumption allowing the possibility that certain voting members tend to align in their voting patterns. Historical voting records provide this information as modelling is carried out in the Bayesian framework based on the Dirichlet distribution. Merrill (1978) considers an empirical measure of citizen voting power using a model based on the normal distribution. Unlike our model, Merrill requires historical data consisting of repeated votes on the same issue. For example, Merrill looks at state voting records of voting Democrat and Republican in past presidential elections.

In section 2 we describe the model, obtain the posterior distribution and provide estimates of voting probabilities for the various voting member combinations. We then use these probabilities to construct empirical Banzhaf indices. The approach is illustrated using a simple example based on 2 constituent voting members. In section 3, we apply the methodology to a real example where we obtain empirical Banzhaf indices for the Canadian provinces based on the current amending formula to Canada's Constitution.

2. THE MODEL AND ESTIMATES

Consider a voting system based on m constituent voting members. Let the subscript $i_j = 1(0)$ denote that voting member j votes yes(no) and let $p_{i_1 \cdots i_m}$ be the corresponding probability in a random election. Our modelling constraints require that

$$0 \le p_{i_1 \cdots i_m} \le 1 \qquad \forall\ i_1, \ldots i_m \tag{1}$$

$$\text{and} \qquad \sum_{\substack{i_j = 0,1 \\ j = 1, \ldots, m}} p_{i_1 \cdots i_m} = 1. \tag{2}$$

Now rather than assuming $p_{i_1 \cdots i_m} = 1/2^m$ as is done in the Banzhaf calculations, we retain some flexibility by requiring only that

$$p_{i_1 \cdots i_m} = p_{i'_1 \cdots i'_m} \quad \text{where } i'_j = \begin{cases} 1 & i_j = 0 \\ 0 & i_j = 1 \end{cases} \qquad j = 1, \ldots, m. \tag{3}$$

In assumption (3), note that $i_1 \cdots i_m$ and $i'_1 \cdots i'_m$ have the same alignment (ie. a specified subset of voting members agreee and the complementary subset disagree). This is a reasonable assumption from the point of view that the election is random. For if election A is random, then so is $\overline{A}$, and we would require that $p_{i_1 \cdots i_m}(A) = p_{i_1 \cdots i_m}(\overline{A})$ for all $i_1 \cdots i_m$ which in turn implies (3). Therefore, apart from assuming indifference between a "Yes" versus "No" vote, assumptions (1), (2) and (3) are completely flexible in allowing stochastic coalitions between voting members.

Based on historical records, let $x_{i_1 \cdots i_m}$ be the number of votes cast in the direction of $i_1 \cdots i_m$ and let $n = \sum x_{i_1 \cdots i_m}$ be the total number of past elections involving all m voting parties. It follows that the conditional probability of $\underline{x}$ given $\underline{p}$ (known as the likelihood) is given by

$$
\begin{aligned}
(\underline{x} \mid \underline{p}) \quad &\propto \quad \prod_{\substack{i_j=0,1 \\ j=1,\ldots,m}} p_{i_1 \cdots i_m}^{x_{i_1 \cdots i_m}} \\
&\propto \quad \prod_{\substack{i_j=0,1 \\ j=2,\ldots,m}} p_{0 i_2 \cdots i_m}^{(x_{0 i_2 \cdots i_m} + x_{1 i'_2 \cdots i'_m})}.
\end{aligned}
$$

We now introduce the prior density

$$
(\underline{p}) \propto 1 \quad \text{where} \sum_{\substack{i_j=0,1 \\ j=2,\ldots,m}} p_{0 i_2 \cdots i_m} = 1/2.
$$

This flat prior density which describes our apriori (ie. before data) view of the distribution of voting probabilities is a widely accepted reference prior and is often used to express ignorance (Berger (1985)). The Bayesian paradigm then states that the posterior (ie. after data) density of the voting probabilities is proportional to the product of the likelihood and the prior density. The posterior density is therefore given by

$$
(\underline{p} \mid \underline{x}) \propto \prod_{\substack{i_j=0,1 \\ j=2,\ldots,m}} p_{0 i_2 \cdots i_m}^{(x_{0 i_2 \cdots i_m} + x_{1 i'_2 \cdots i'_m})}
$$

where $\sum p_{0 i_2 \cdots i_m} = 1/2$ and has the form of a scaled Dirichlet distribution. Using the posterior means of the Dirichlet distribution, it follows that voting probability estimates are given by

$$\hat{p}_{i_1 \cdots i_m} = \frac{1}{2} \left(\frac{x_{i_1 \cdots i_m} + x_{i'_1 \cdots i'_m} + 1}{n + 2^{m-1}} \right). \tag{4}$$

Note that in the absence of historical data, we have $n = 0$ and $\underline{x} = \underline{0}$. In this case, the voting probability estimates (4) reduce to $\hat{p}_{i_1 \cdots i_m} = 1/2^m$ which corresponds to the values traditionally used in the Banzhaf calculations.

To obtain the various empirical Banzhaf probabilities we simply sum the $\hat{p}_{i_1 \cdots i_m}$ values over the relevant sets. For example, to obtain the power to prevent probability (which is a conditional probability) for a given voting member, we calculate $\sum_{S_1} p_{i_1 \cdots i_m} / \sum_{S_2} p_{i_1 \cdots i_m}$ where S_2 is the set of indices which lead to a positive decision and $S_1 \subseteq S_2$ consists of those indices for which reversing the member's vote results in a negative decision. Whereas the traditional Banzhaf probabilities are useful in quantifying voting power based on the actual voting rules, the empirical Banzhaf probabilities highlight effective voting power due to coalitions that may exist between various constituent bodies.

To fix ideas, consider a simple example based on $m = 2$ constituent voting members where a decision is passed only when a consensus is reached. Suppose further than in $n = 5$ previous votes, $x_{11} = 3$, $x_{00} = 1$, $x_{01} = 1$ and $x_{10} = 0$. We then obtain $\hat{p}_{11} = \hat{p}_{00} = 5/14$ and $\hat{p}_{01} = \hat{p}_{10} = 1/7$. The traditional calculations for the Banzhaf probability, the power to initiate probability and the power to prevent probability give values of $1/2$, $1/3$ and 1 for each voting member. On the other hand, the new empirical calculations for the Banzhaf probability, the power to initiate probability and the power to prevent probability for the first voting member give values $\hat{p}_{01} + \hat{p}_{11} = 1/2$, $\hat{p}_{01}/(\hat{p}_{00} + \hat{p}_{01} + \hat{p}_{10}) = 2/9$ and $\hat{p}_{11}/\hat{p}_{11} = 1$. The same values are obtained for the second voting member.

3. CANADA'S CONSTITUTIONAL AMENDING FORMULA

We now turn to an example that has been much discussed in the literature. See, for example, Heard and Swartz (1996), Kilgour (1983), Levesque and Moore (1984), Mintz (1985a,b) and Straffin (1977). It concerns the calculation of power indices for the $m = 10$ Canadian provinces with respect to the amending formula to Canada's Constitution Act. The current amending formula is a somewhat complicated rule that depends on provincial populations. Based on 1996 population estimates, it essentially provides that an amendment passes if and only if it has the approval of each of Ontario, Quebec, British Columbia and Alberta, at least one of Manitoba and Saskatchewan and at least two of Nova Scotia, New Brunswick and Newfoundland. Note that Prince Edward Island is rendered powerless in the amending formula.

In Table 1, we present the results of $n = 17$ amending votes during the period 1907-1992. The rule used in defining a vote is that a draft resolution must have been presented for approval at a formal federal-provincial conference. Note that the 1992 referendum vote is included as no provincial legislature would have proceeded with the amendment after it's failure in the referendum. Note also that the 1983 vote is included where Quebec's boycott is interpreted as a "No" vote. Finally, the 1981b vote is included; it was not tabled in a conference but was passed by Parliament and subsequently halted by court actions from the "No" provinces.

In Table 2 we give the traditional Banzhaf probabilities and the empirical Banzhaf probabilities which were calculated using a Fortran program. We note that the differences between the Banzhaf probabilities and the empirical Banzhaf probabilities are not large. This is due to the small number (17) of historical votes compared with the large number (1024) of voting combinations. Keeping this comment in mind, we observe that Quebec gained the most power in comparing its empirical Banzhaf probability (.059) to its traditional Banzhaf probability (.047). This is explained by noting that Quebec has often taken a negative stand on amendment votes. In these cases, using the current amending formula, the amendment result would have changed from negative to positive had Quebec instead voted "Yes"'.

In summary, Bayesian modelling based on the Dirichlet distribution provides a useful adaptation of the Banzhaf index. A variety of applications may benefit from using historical voting data in order to determine power values that account for observed patterns in voting combinations. More effective analyses may be done by moving away from the assumption that all voting combinations are equally likely.

REFERENCES

Banzhaf, J.R. (1965), "Weighted voting doesn't work: a mathematical analysis", *Rutgers Law Review*, 317-343.

Banzhaf, J.R. (1968), "One man, 3.312 votes: a mathematical analysis of the electoral college", *Villanova Law Review*, 304-332.

Berger, J.O. (1985), Statistical Decision Theory and Bayesian Analysis, Second Edition, Springer-Verlag.

Heard, A.D. and Swartz, T.B. (1996), "Measuring the effects of the regional veto amending formula", *Manuscript.*

Kilgour, D.M. (1983), "A formal analysis of the amending formula of Canada's Constitution Act", *Canadian Journal of Political Science*, 16:14, 771-777.

Levesque, T.J. and Moore, J.W. (1984), "Citizen and provincial power under alternative amending formulae: an extension of Kilgour's analysis", *Canadian Journal of Political Science*, 17:1, 157-166.

Merrill, S. (1978), "Citizen voting power under the electoral college: a stochastic model based on state voting patterns", *SIAM Journal of Applied Mathematics*, 34:2, 376-390.

Mintz, E. (1985a), "Banzhaf's power index and Canada's constitutional amending formula: a comment on Kilgour's analysis", *Canadian Journal of Political Science*, 18:2, 385-387.

Mintz, E. (1985b), "Changing Canada's constitutional amending formula: a comment", *Canadian Public Policy*, 11:3, 623-624.

Straffin, P.D. (1977), "Homogeneity, independence and power indices", *Public Choice*, 30, 107-118.

Table 1: Results of $n = 17$ amending votes involving the $m = 10$ Canadian provinces where a 1(0) denotes a yes(no) vote.

Date - Issue	Ont	Que	BC	Alta	Man	Sask	NS	NB	Nfld	PEI
1992 - Charlottetown Accord	1	0	0	0	0	0	0	1	1	1
1990 - Meech Lake Accord	1	1	1	1	0	1	1	1	0	1
1983 - Aboriginal rights	1	0	1	1	1	1	1	1	1	1
1981a - Patriation package	1	0	1	1	1	1	1	1	1	1
1981b - Unilateral package	1	0	0	0	0	0	0	1	0	0
1980 - Best Efforts package	1	1	1	1	1	1	1	1	1	1
1971 - Victoria Charter	1	0	1	1	1	1	1	1	1	1
1964a - Pension amendment	1	1	1	1	1	1	1	1	1	1
1964b - Fulton-Favreau formula	1	0	1	1	1	1	1	1	1	1
1960a - Judges' mandatory retirement	1	1	1	1	1	1	1	1	1	1
1960b - Fulton formula	1	0	1	1	1	1	1	1	1	1
1951 - Indirect taxation	0	0	1	1	1	1	1	1	1	1
1950 - Old Age Pension	1	1	1	1	1	1	1	1	1	1
1940 - Unemployment Insurance	1	1	1	1	1	1	1	1	1	1
1935 - Amending formula proposal	1	1	1	1	1	1	1	0	1	1
1931 - Statute of Westminster	1	1	1	1	1	1	1	1	1	1
1907 - Provincial subsidy payments	1	1	1	1	1	1	1	1	1	1

Table 2: Banzhaf probabilities and empirical Banzhaf probabilities for the 10 Canadian provinces.

Province	Banzhaf	Empirical Banzhaf
Ont	.047	.055
Que	.047	.059
BC	.047	.054
Alta	.047	.054
Man	.016	.015
Sask	.016	.016
NS	.023	.025
NB	.023	.024
Nfld	.023	.024
PEI	.000	.000