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Multivariate Analysis

Tests Concerning Equicorrelation Matrices with Grouped Normal Data

PARAMJIT S. GILL,[1] SARATH G. BANNEHEKA,[2] AND TIM B. SWARTZ[3]

[1]Department of Mathematics and Statistics, Okanagan University College, Kelowna, BC, Canada
[2]Department of Statistics and Computer Science, University of Sri Jayewardenepura, Gangodawila, Nugegoda, Sri Lanka
[3]Department of Statistics and Actuarial Science, Simon Fraser University, Burnaby, BC, Canada

This article considers three practical hypotheses involving the equicorrelation matrix for grouped normal data. We obtain statistics and computing formulae for common test procedures such as the score test and the likelihood ratio test. In addition, statistics and computing formulae are obtained for various small sample procedures as proposed in Skovgaard (2001). The properties of the tests for each of the three hypotheses are compared using Monte Carlo simulations.

Keywords Intraclass correlation; Likelihood ratio test; Maximum likelihood estimation; Score test; Small-sample inference.

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1. Introduction

A common data structure in applied statistics occurs when a small number of multivariate measurements are collected from a finite number of groups or classes. For example, instruments known as computer-analyzed corneal topographers (CACT) are used to obtain multivariate observations on curvature points where the sample size is typically around 10 (Viana et al., 1993). In such situations, a parsimonious model for the covariance matrix is desired. Also, in order to increase sample size, one may wish to combine data from different sources (e.g., instruments). Therefore, there is a need to test the equality of covariance matrices arising from different groups.

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Address correspondence to Paramjit S. Gill, Department of Mathematics and Statistics, Okanagan University College, Kelowna, BC, Canada; E-mail: pgill@ouc.bc.ca

In this article we consider tests of hypotheses regarding a special form of the covariance matrix called the *equicorrelation* or *intraclass correlation* matrix. Such a covariance matrix is suitable when subjects are related (such as a family or a litter of animals) or when measurements are made repeatedly on the same subject. More formally, suppose that we have samples of p-dimensional measurements from K groups. Let n_i be the number of observations from group $i = 1, 2, \ldots, K$ and let $n = \sum_{i=1}^{K} n_i$. Let

$$X_i = \begin{pmatrix} x_{i11} & x_{i12} & \ldots & x_{i1p} \\ x_{i21} & x_{i22} & \ldots & x_{i2p} \\ \vdots & \vdots & \vdots & \vdots \\ x_{in_i1} & x_{in_i2} & \ldots & x_{in_ip} \end{pmatrix}$$

denote the i-th data matrix with $\underline{x}'_{ij} = (x_{ij1}, x_{ij2}, \ldots, x_{ijp})$ as the vector of observations on the j-th sample from group i. Assume

$$\underline{x}_{ij} \sim \mathrm{N}_p(\mu_i \underline{1}, \Sigma_i), \quad i = 1, 2, \ldots, K, \;\; j = 1, 2, \ldots, n_i$$

where $\underline{1}$ denotes the unit vector of dimension p and the equicorrelation or intraclass correlation matrix is given by

$$\Sigma_i = \sigma_i^2 \begin{pmatrix} 1 & \rho_i & \ldots & \rho_i \\ \rho_i & 1 & \ldots & \rho_i \\ \vdots & \vdots & \vdots & \vdots \\ \rho_i & \rho_i & \ldots & 1 \end{pmatrix}, \quad (1-p)^{-1} < \rho_i < 1.$$

Note that for each group, the measurements share a common mean and correlation. Although this is a restrictive structure, it is still appropriate in many contexts. The intraclass correlation coefficient ρ_i measures the degree of resemblance between the members in the i-th group. This covariance pattern also arises in the one-way random effects (within each group i) linear model

$$x_{ijk} = \mu_i + \alpha_{ij} + \varepsilon_{ijk}$$

where α_{ij} is the random effect due to the j-th unit from group i with $E(\alpha_{ij}) = 0$, $Var(\alpha_{ij}) = \sigma^2_{i\alpha}$, ε_{ijk} is the random error with $E(\varepsilon_{ijk}) = 0$, $Var(\varepsilon_{ijk}) = \sigma^2_{i\varepsilon}$, and $Cov(\alpha_{ij}, \varepsilon_{ijk}) = 0$. It is easily seen that the covariance matrix of $\underline{x}_{ij}$ is Σ_i with $\sigma_i^2 = \sigma^2_{i\alpha} + \sigma^2_{i\varepsilon}$ and $\rho_i = \sigma^2_{i\alpha}/(\sigma^2_{i\alpha} + \sigma^2_{i\varepsilon})$ (see, for example, Donner and Koval, 1980).

We are interested in the following three practical hypotheses concerning the equicorrelation matrix:

Case 1. $H_{01} : \rho_1 = \cdots = \rho_K$.
Case 2. $H_{02} : \sigma_1 = \cdots = \sigma_K$.
Case 3. $H_{03} : \rho_1 = \cdots = \rho_K, \; \sigma_1 = \cdots = \sigma_K$.

The Case 1 problem was first addressed by Konishi and Gupta (1989) using a maximum likelihood approach with combined moment estimators for the common ρ. Paul and Barnwal (1990) derived a score-type test based on Neyman's $C(\alpha)$. While estimating the intraclass correlation coefficient, Ahmed et al. (2001) also derived a score-type test statistic for the Case 1 problem. The Case 2 problem does not seem to have been studied. For the Case 3 problem, Han (1975) obtained a modified likelihood ratio test.

Our main concern regarding the Case 1, Case 2, and Case 3 problems involves data sets where the sample sizes are small. For small sample sizes, asymptotic tests such as those described above may be inappropriate. For example, the actual Type 1 error of a statistic may differ considerably from its nominal level. In this regard, we have derived various small sample tests based on the general theory of Skovgaard (2001) and compared these tests to the standard tests. Some of these standard tests have not previously appeared in the literature and are derived here. This article serves as a summary of the variety of tests that one might consider in connection with the Case 1, Case 2, and Case 3 problems.

Small sample tests are typically derived by considering additional terms in asymptotic expansions. These "corrections" attempt to hasten the convergence of statistics to their asymptotic distributions. The downside of such corrections is that they sometimes involve difficult calculations. Well-known examples of correction terms include Bartlett and Edgeworth series type corrections, saddlepoint corrections, and the signed loglikelihood ratio correction. Reviews of asymptotic methods in statistics are given by Barndorff-Nielsen and Cox (1994), Skovgaard (2001), and Reid (2003). Notably, Skovgaard (2001) generalized the Barndorff-Nielsen (1991) correction to the multi-parameter case; we follow Skovgaard's approach to derive small sample corrections for likelihood ratio tests involving equicorrelation matrices.

In Sec. 2, we provide a general description of the modified likelihood ratio test due to Skovgaard (2001) and the score test. We also discuss unrestricted maximum likelihood estimation for the problem considered in this article. In Secs. 3, 4, and 5, Cases 1, 2, and 3 are discussed in detail with an emphasis on deriving asymptotic test procedures. In Sec. 6, simulation results are provided to compare the various tests under each of the three hypotheses.

2. Preliminaries

2.1. *The Skovgaard Modifications*

Suppose that a random variable X belongs to the exponential family having a density of the form $f(x;\theta) = b(x)\exp\{\theta' t(x) - \kappa(\theta)\}$. Then θ is called the canonical parameter and t is the corresponding canonical sufficient statistic. Let $\omega' = (\nu', \psi')$ be a (possibly nonlinear) function of θ where $\psi' = (\psi_1, \ldots, \psi_q)$ is the vector of parameters of interest and $\nu' = (\nu_1, \ldots, \nu_{r-q})$ is a vector of nuisance parameters. Suppose one is interested in testing the null hypothesis $H_0 : \psi_1 = \cdots = \psi_q = \psi_0$ (unspecified). Let $\hat{\omega}' = (\hat{\nu}', \hat{\psi}')$ denote the maximum likelihood (ML) estimate of the full parameter vector and $\tilde{\omega}_0' = (\tilde{\nu}', \tilde{\psi}_0')$ be the ML estimate of the free parameters under the null hypothesis. Let $\ell(\hat{\omega})$ and $\ell(\tilde{\omega}_0)$ be the corresponding maximum values of the loglikelihood. Denote $\mathcal{J}_{\nu\nu}$ and $\mathcal{I}_{\nu\nu}$ as the observed and expected Fisher information matrices for the ν-part of the parameter vector and let $\widetilde{\mathcal{J}}_{\nu\nu}$ and $\widetilde{\mathcal{I}}_{\nu\nu}$ denote the corresponding estimates obtained by replacing the unknown parameter values by their restricted ML estimates. Furthermore, let τ and Ω denote the mean and variance matrix of t; $\tilde{\tau}$ and $\tilde{\Omega}$ are the corresponding estimates when the unknown parameters are replaced by the restricted ML estimates and $\widehat{\Omega}$ is the estimate under unrestricted ML estimation.

Let $w = 2(\ell(\hat{\omega}) - \ell(\tilde{\omega}_0))$ denote the likelihood ratio test statistics for testing H_0. Following Skovgaard (2001), we define

$$\delta = \frac{\{(t-\tilde{\tau})'\tilde{\Omega}^{-1}(t-\tilde{\tau})\}^{(q-1)/2}}{w^{(q-1)/2-1}(\hat{\theta}-\tilde{\theta})'(t-\tilde{\tau})}\left(\frac{|\tilde{\Omega}||\widetilde{\mathcal{J}}_{vv}|}{|\widehat{\Omega}||\widetilde{\mathcal{J}}_{vv}|}\right)^{1/2}. \tag{1}$$

The corrected likelihood ratio is then given by

$$w^* = w(1 - w^{-1}\ln\delta)^2 \tag{2}$$

where it is hoped that $w^* \sim \chi^2_{q-1}$ is a better approximation under H_0 than the likelihood ratio test $w \sim \chi^2_{q-1}$. Another corrected χ^2_{q-1} test statistic is (Skovgaard, 2001)

$$w^{**} = w - 2\ln\delta \tag{3}$$

which may assume a negative value, especially when w is small.

2.2. *The Score Test*

The score test is attractive for its relative simplicity of computations as we need ML estimation only under the null hypothesis. For the r-dimensional parameter vector $\omega' = (v', \psi')$, let $U' = (U'_v, U'_\psi)$ be the score vector where

$$U_v = \frac{\partial \ell}{\partial v}, \quad U_\psi = \frac{\partial \ell}{\partial \psi}.$$

Then under the null hypothesis $H_0 : \psi_1 = \cdots = \psi_q = \psi_0$,

$$U \sim \mathrm{N}_r\left[\underline{0}, \begin{pmatrix} T & S' \\ S & R \end{pmatrix}\right],$$

where T is a $(r-q) \times (r-q)$ matrix with

$$T(i,j) = -E\left(\frac{\partial^2 \ell}{\partial v_i \partial v_j}\right)\Bigg|_{H_0},$$

S is a $q \times (r-q)$ matrix with

$$S(i,j) = -E\left(\frac{\partial^2 \ell}{\partial \psi_i \partial v_j}\right)\Bigg|_{H_0}$$

and R is a $q \times q$ matrix with

$$R(i,j) = -E\left(\frac{\partial^2 \ell}{\partial \psi_i \partial \psi_j}\right)\Bigg|_{H_0}.$$

The score test statistic is then given by

$$\xi = U'_\psi(R - ST^{-1}S')^{-1}U_\psi$$

and under the null hypothesis, ξ is asymptotically distributed as χ^2_{q-1}. Note that all the terms involved in ξ are evaluated at the ML estimates of ω under H_0. In this article, we compare the performance of the score test with other tests using Monte Carlo simulations.

2.3. *Unrestricted Maximum Likelihood Estimation*

In Cases 1, 2, and 3, tests in this article require ML estimation under the alternative hypotheses. We refer to this as unrestricted ML estimation. Let A be the $p \times p$ matrix

$$A = \begin{pmatrix} \frac{1}{p} & \frac{1}{p} & \frac{1}{p} & \cdots & \frac{1}{p} & \frac{1}{p} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{p(1-p)}} & \frac{1}{\sqrt{p(1-p)}} & \frac{1}{\sqrt{p(1-p)}} & \cdots & \frac{1}{\sqrt{p(1-p)}} & -\frac{(p-1)}{\sqrt{p(1-p)}} \end{pmatrix}$$

Define $Z_i = X_i A'$, $i = 1, 2, \ldots, K$ or equivalently, $\underline{z}_{ij} = A\underline{x}_{ij}, i = 1, 2, \ldots, K,\ j = 1, 2, \ldots, n_i$. Then $\underline{z}_{ij} \sim \mathrm{N}_p(\underline{\mu}_i^*, \Sigma_i^*)$, where

$$\underline{\mu}_i^* = \begin{pmatrix} \mu_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \Sigma_i^* = A\Sigma_i A' = \sigma_i^2 \begin{pmatrix} \rho_i + p^{-1}(1-\rho_i) & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1-\rho_i & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1-\rho_i & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1-\rho_i \end{pmatrix}.$$

Note that $z_{ij1} \sim \mathrm{N}(\mu_i, \sigma_i^2\{\rho_i + p^{-1}(1-\rho_i)\})$ and $z_{ij2}, \ldots, z_{ijp} \overset{\text{iid}}{\sim} \mathrm{N}(0, \sigma_i^2\{1-\rho_i\})$, $i = 1, 2, \ldots, K$; $j = 1, 2, \ldots, n_i$. Also, z_{ij1} is independent of $z_{ij2}, \ldots, z_{ijp}$.

The log-likelihood in terms of μ_i, σ_i, and ρ_i is

$$\begin{aligned} \ell &= \ell(\mu_1, \ldots, \mu_K, \sigma_1^2, \ldots, \sigma_K^2, \rho_1, \ldots, \rho_K) \\ &= -\frac{1}{2}p\sum_{i=1}^{K} n_i \ln \sigma_i^2 - \frac{1}{2}\sum_{i=1}^{K} n_i \ln\{\rho_i + p^{-1}(1-\rho_i)\} - \frac{1}{2}(p-1)\sum_{i=1}^{K} n_i \ln(1-\rho_i) \\ &\quad - \sum_{i=1}^{K}\sum_{j=1}^{n_i} \frac{(z_{ij1}-\mu_i)^2}{2\sigma_i^2\{\rho_i + p^{-1}(1-\rho_i)\}} - \sum_{i=1}^{K}\sum_{j=1}^{n_i}\sum_{k=2}^{p} \frac{z_{ijk}^2}{2\sigma_i^2(1-\rho_i)} \end{aligned}$$

It follows that the ML estimates $\hat{\mu}_i$, $\hat{\sigma}_i^2$, and $\hat{\rho}_i$ of μ_i, σ_i^2, and ρ_i are given by

$$\hat{\mu}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} z_{ij1}$$

$$\hat{\sigma}_i^2 = \frac{\sum_{j=1}^{n_i}(z_{ij1} - \hat{\mu}_i)^2}{n_i p\{\hat{\rho}_i + p^{-1}(1-\hat{\rho}_i)\}} + \frac{\sum_{j=1}^{n_i}\sum_{k=2}^{p} z_{ijk}^2}{n_i p(1-\hat{\rho}_i)}$$

and

$$\hat{\rho}_i = \frac{p\sum_{j=1}^{n_i}(z_{ij1}-\hat{\mu}_i)^2 - (p-1)^{-1}\sum_{j=1}^{n_i}\sum_{k=2}^{p} z_{ijk}^2}{p\sum_{j=1}^{n_i}(z_{ij1}-\hat{\mu}_i)^2 + \sum_{j=1}^{n_i}\sum_{k=2}^{p} z_{ijk}^2}. \tag{4}$$

3. Case 1: $H_{01} : \rho_1 = \cdots = \rho_K$

The Case 1 problem with grouped normal data has been well studied in the literature. We fill in some gaps concerning the likelihood ratio test and derive the Skovgaard (2001) test. For comparison purposes, we present the score test (Paul and Barnwal, 1990).

3.1. *Likelihood Ratio Test*

Under the hypothesis $H_0 : \rho_1 = \rho_2 = \cdots = \rho_K = \rho$ (an unspecified value), the log-likelihood is

$$\begin{aligned}\ell_0 &= \ell_0(\mu_1, \dots, \mu_K, \sigma_1^2, \dots, \sigma_k^2, \rho)\\ &= -\frac{1}{2}p\sum_{i=1}^{K} n_i \ln\sigma_i^2 - \frac{1}{2}n\ln\{\rho + p^{-1}(1-\rho)\} - \frac{1}{2}n(p-1)\ln(1-\rho)\\ &\quad - \sum_{i=1}^{K}\sum_{j=1}^{n_i}\frac{(z_{ij1}-\mu_i)^2}{2\sigma_i^2\{\rho + p^{-1}(1-\rho)\}} - \sum_{i=1}^{K}\sum_{j=1}^{n_i}\sum_{k=2}^{p}\frac{z_{ijk}^2}{2\sigma_i^2(1-\rho)}.\end{aligned}$$

The ML estimates for μ_i and σ_i^2 are given by

$$\tilde{\mu}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} z_{ij1} = \hat{\mu}_i$$

and

$$\tilde{\sigma}_i^2 = \frac{\sum_{j=1}^{n_i}(z_{ij1}-\tilde{\mu}_i)^2}{n_i p\{\tilde{\rho} + p^{-1}(1-\tilde{\rho})\}} + \frac{\sum_{j=1}^{n_i}\sum_{k=2}^{p} z_{ijk}^2}{n_i p(1-\tilde{\rho})}. \tag{5}$$

The score equation for ρ

$$\begin{aligned}U(\rho) = \frac{\partial\ell_0}{\partial\rho} &= -\frac{n(p-1)}{2p\{\rho + p^{-1}(1-\rho)\}} + \frac{n(p-1)}{2(1-\rho)} + \frac{(p-1)\sum_{i=1}^{K}\sum_{j=1}^{n_i}(z_{ij1}-\mu_i)^2/\sigma_i^2}{2p\{\rho + p^{-1}(1-\rho)\}^2}\\ &\quad - \frac{1}{2(1-\rho)^2}\sum_{i=1}^{K}\sum_{j=1}^{n_i}\sum_{k=2}^{p} z_{ijk}^2/\sigma_i^2 = 0\end{aligned}$$

does not admit an explicit solution. Estimates are obtained by the scoring method

$$\rho^{(m+1)} = \rho^{(m)} + \frac{U(\rho^{(m)})}{\mathcal{I}(\rho^{(m)})}, \quad m = 0, 1, \dots \tag{6}$$

where $\mathcal{I}(\rho) = -E(\frac{\partial^2 \ell_0}{\partial \rho^2})$ is the Fisher information. To obtain the Fisher information, we have

$$\frac{\partial^2 \ell_0}{\partial \rho^2} = \frac{n(p-1)^2}{2p^2\{\rho + p^{-1}(1-\rho)\}^2} + \frac{n(p-1)}{2(1-\rho)^2} - \frac{(p-1)^2 \sum_{i=1}^{K} \sum_{j=1}^{n_i} (z_{ij1} - \mu_i)^2/\sigma_i^2}{p^2\{\rho + p^{-1}(1-\rho)\}^3} - \frac{\sum_{i=1}^{K} \sum_{j=1}^{n_i} \sum_{k=2}^{p} z_{ijk}^2/\sigma_i^2}{(1-\rho)^3}.$$

Since $E(z_{ij1} - \mu_i)^2 = \sigma_i^2\{\rho + p^{-1}(1-\rho)\}$ and $E(z_{ijk}^2) = \sigma_i^2(1-\rho)$, it follows that

$$\mathcal{I}(\rho) = -E\left(\frac{\partial^2 \ell_0}{\partial \rho^2}\right) = \frac{n(p-1)^2}{2p^2\{\rho + p^{-1}(1-\rho)\}^2} + \frac{n(p-1)}{2(1-\rho)^2}.$$

We summarize the estimation procedure for $\mu_1, \ldots, \mu_K, \sigma_1^2, \ldots, \sigma_K^2$ and ρ in the following algorithm:

1. Let $m = 0$ and $\rho^{(0)} = \sum_{i=1}^{K} \hat{\rho}_i/K$ where $\hat{\rho}_i$ is given by (4).
2. Let $(\sigma_i^2)^{(m)}$ be the m-th approximation to $\tilde{\sigma}_i^2$ obtained by substituting $\rho^{(m)}$ for ρ in (5).
3. Calculate $\rho^{(m+1)}$ using (6) where $(\sigma_i^2)^{(m)}$ and $\rho^{(m)}$ replace σ_i^2 and ρ, respectively.
4. Let $m = m + 1$.
5. If $|U(\rho^{(m)})| < 10^{-6}$ then stop. Otherwise, go to Step 2 and continue.

In all of the simulations that we have considered, we have not experienced any difficulty with the above algorithm.

Let $\hat{\ell} = \ell(\hat{\omega}) = \ell(\hat{\mu}_1, \ldots, \hat{\mu}_K, \hat{\sigma}_1^2, \ldots, \hat{\sigma}_K^2, \hat{\rho}_1, \ldots, \hat{\rho}_K)$ and $\hat{\ell}_0 = \ell_0(\tilde{\omega}_0) = \ell_0(\tilde{\mu}_1, \ldots, \tilde{\mu}_K, \tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_K^2, \tilde{\rho})$ be the maximized values of the loglikelihood. Then, under the null hypothesis H_0, the likelihood ratio test statistic $w = 2(\hat{\ell} - \hat{\ell}_0)$ follows the chi-square distribution with $(K-1)$ degrees of freedom for large sample sizes.

3.2. *Modified Likelihood Ratio Tests*

In the following, we derive the Skovgaard (2001) modifications to the likelihood ratio test statistic which is intended to provide more accurate inference in small-sample situations. For our problem, the canonical parameter $\theta = (\theta_1, \ldots, \theta_K, \theta_{K+1}, \ldots, \theta_{2K}, \theta_{2K+1}, \ldots, \theta_{3K})$ and canonical sufficient statistic $t = (t_1, \ldots, t_K, t_{K+1}, \ldots, t_{2K}, t_{2K+1}, \ldots, t_{3K})$ are as follows

$$\theta_i = \frac{1}{\lambda_i^2}, \quad t_i = -\sum_{j=1}^{n_i} z_{ij1}^2/2,$$

$$\theta_{K+i} = \frac{\mu_i}{\lambda_i^2}, \quad t_{K+i} = \sum_{j=1}^{n_i} z_{ij1},$$

$$\theta_{2K+i} = \frac{1}{\beta_i^2}, \quad t_{2K+i} = -\sum_{j=1}^{n_i} \sum_{k=2}^{p} z_{ijk}^2/2$$

where $i = 1, \dots, K$, $\lambda_i^2 = \sigma_i^2\{\rho_i + p^{-1}(1-\rho_i)\}$ and $\beta_i^2 = \sigma_i^2(1-\rho_i)$. Then the terms in $\tau = E(t)$ are

$$\tau_i = -n_i E(z_{ij1}^2)/2 = -n_i(\mu_i^2 + \lambda_i^2)/2,$$
$$\tau_{K+i} = n_i E(z_{ij1}) = n_i \mu_i,$$
$$\tau_{2K+i} = -n_i(p-1)E(z_{ij2}^2)/2 = -n_i(p-1)\beta_i^2/2$$

and the elements in $\Omega = Var(t)$ are

$$\Omega(i,i) = n_i\, Var(z_{ij1}^2)/4 = n_i(\lambda_i^2 + 2\mu_i^2)\lambda_i^2/2,$$
$$\Omega(K+i, K+i) = n_i Var(Z_{ij1}) = n_i\lambda_i^2,$$
$$\Omega(2K+i, 2K+i) = n_i(p-1)\, Var(z_{ij2}^2)/4 = n_i(p-1)\beta_i^4/2,$$
$$\Omega(K+i, i) = \Omega(K+i, i) = Cov\left(-\sum_{j=1}^{n_i} z_{ij1}^2, \sum_{j=1}^{n_i} z_{ij1}\right)\Big/2 = -n_i\mu_i\lambda_i^2$$

where $i = 1, \dots, K$. All the other covariances are zero.

In order to obtain the Skovgaard (2001) term δ in (1), we need to evaluate $\widetilde{\mathcal{J}}_{vv}$ and $\widetilde{\mathcal{I}}_{vv}$ for the nuisance parameter $v' = (\mu_1, \dots, \mu_K, \sigma_1^2, \dots, \sigma_K^2)$. We have

$$\widetilde{\mathcal{J}}(\mu_i, \mu_i) = -\frac{\partial^2 \ell}{\partial \mu_i^2}\bigg|_{\omega=\tilde{\omega}} = \frac{n_i}{\tilde{\sigma}_i^2\{\tilde{\rho} + p^{-1}(1-\tilde{\rho})\}},$$
$$\widetilde{\mathcal{J}}(\mu_i, \sigma_i^2) = -\frac{\partial^2 \ell}{\partial \mu_i \partial \sigma_i^2}\bigg|_{\omega=\tilde{\omega}} = \frac{\sum_{j=1}^{n_i}(z_{ij1} - \tilde{\mu}_i)}{\tilde{\sigma}_i^4\{\tilde{\rho} + p^{-1}(1-\tilde{\rho})\}} = 0,$$
$$\widetilde{\mathcal{J}}(\sigma_i^2, \sigma_i^2) = -\frac{\partial^2 \ell}{\partial \sigma_i^4}\bigg|_{\omega=\tilde{\omega}} = -\frac{n_i p}{2\tilde{\sigma}_i^4} + \frac{\sum_{j=1}^{n_i}(z_{ij1} - \tilde{\mu}_i)^2}{\tilde{\sigma}_i^6\{\tilde{\rho} + p^{-1}(1-\tilde{\rho})\}}$$
$$+ \frac{\sum_{j=1}^{n_i}\sum_{k=2}^{p} z_{ijk}^2}{\tilde{\sigma}_i^6(1-\tilde{\rho})} = \frac{n_i p}{2\tilde{\sigma}_i^4} \text{ using equation (5).}$$

In addition, $\widetilde{\mathcal{J}}(\mu_i, \mu_j) = \widetilde{\mathcal{J}}(\mu_i, \sigma_j^2) = \widetilde{\mathcal{J}}(\sigma_i^2, \sigma_j^2) = 0, i \neq j$ since

$$\frac{\partial^2 \ell}{\partial \mu_i \partial \mu_j} = \frac{\partial^2 \ell}{\partial \sigma_i^2 \partial \sigma_j^2} = \frac{\partial^2 \ell}{\partial \mu_i \partial \sigma_j^2} = 0, i \neq j.$$

Furthermore,

$$\widetilde{\mathcal{I}}(\mu_i, \mu_i) = -E\left(\frac{\partial^2 \ell}{\partial \mu_i^2}\right)\bigg|_{\omega=\tilde{\omega}} = \frac{n_i}{\tilde{\sigma}_i^2\{\tilde{\rho} + p^{-1}(1-\tilde{\rho})\}},$$
$$\widetilde{\mathcal{I}}(\mu_i, \sigma_i^2) = -E\left(\frac{\partial^2 \ell}{\partial \mu_i \partial \sigma_i^2}\right)\bigg|_{\omega=\tilde{\omega}} = 0,$$
$$\widetilde{\mathcal{I}}(\sigma_i^2, \sigma_i^2) = -E\left(\frac{\partial^2 \ell}{\partial \sigma_i^4}\right)\bigg|_{\omega=\tilde{\omega}} = \frac{n_i p}{2\tilde{\sigma}_i^4},$$
$$\widetilde{\mathcal{I}}(\mu_i, \mu_j) = \widetilde{\mathcal{I}}(\mu_i, \sigma_j^2) = \widetilde{\mathcal{I}}(\sigma_i^2, \sigma_j^2) = 0, \quad i \neq j.$$

Note that as $\widetilde{\mathcal{J}}_{vv} = \widetilde{\mathcal{I}}_{vv}$, it is not necessary to evaluate $|\widetilde{\mathcal{I}}_{vv}|$ and $|\widetilde{\mathcal{J}}_{vv}|$ as they cancel out in δ.

3.3. *Score Test*

In order to evaluate the score test statistic

$$\xi = U_\psi'(R - ST^{-1}S')^{-1}U_\psi$$

we require the vector of scores $U_\psi' = (U_1, \ldots, U_K)$ where

$$U_i = \frac{\partial \ell}{\partial \rho_i}\bigg|_{\omega=\tilde{\omega}}$$

and the $K \times K$ matrix R, the $K \times 2K$ matrix S, and the $2K \times 2K$ matrix T where

$$R(i,j) = -E\left(\frac{\partial^2 \ell}{\partial \rho_i \partial \rho_j}\right)\bigg|_{\omega=\tilde{\omega}},$$

$$S(i,j) = -E\left(\frac{\partial^2 \ell}{\partial \rho_i \partial \mu_j}\right)\bigg|_{\omega=\tilde{\omega}},$$

$$S(i,K+j) = -E\left(\frac{\partial^2 \ell}{\partial \rho_i \partial \sigma_j^2}\right)\bigg|_{\omega=\tilde{\omega}},$$

$$T(i,j) = -E\left(\frac{\partial^2 \ell}{\partial \mu_i \partial \mu_j}\right)\bigg|_{\omega=\tilde{\omega}},$$

$$T(i,K+j) = T_{(K+i,j)} = -E\left(\frac{\partial^2 \ell}{\partial \mu_i \partial \sigma_j^2}\right)\bigg|_{\omega=\tilde{\omega}},$$

$$T(K+i,K+j) = -E\left(\frac{\partial^2 \ell}{\partial \sigma_i^2 \partial \sigma_j^2}\right)\bigg|_{\omega=\tilde{\omega}}.$$

Explicit expressions for the components of ξ are given by

$$U_i = -\frac{n_i}{2\{\rho + p^{-1}(1-\rho)\}} - \frac{n_i(p-1)}{2(1-\rho)} + \frac{(p-1)\sum_{j=1}^{n_i}(\bar{x}_{ij} - \bar{x}_i)^2}{2p\sigma_i^2\{\rho + p^{-1}(1-\rho)\}^2} - \frac{\sum_{j=1}^{n_i}\sum_{k=2}^{p} z_{ijk}^2}{2\sigma_i^2(1-\rho)^2},$$

$$R = \operatorname{diag}\left(\frac{3n_i(p-1)}{2(1-\rho)^2} - \frac{n_i(3p-2)(p-1)}{2p^2\{\rho + p^{-1}(1-\rho)\}^2}\right)_{K\times K},$$

$$S = \left[0_{K\times K}\ \operatorname{diag}\left(-\frac{n_i(p-1)\rho}{2\sigma_i^2(1-\rho)\{\rho + p^{-1}(1-\rho)\}}\right)_{K\times K}\right],$$

$$T = \begin{bmatrix} \operatorname{diag}\left(-\frac{n_i}{\sigma_i^2\{\rho + p^{-1}(1-\rho)\}}\right)_{K\times K} & 0_{K\times K} \\ 0_{K\times K} & \operatorname{diag}\left(\frac{n_i p}{2\sigma_i^4}\right)_{K\times K} \end{bmatrix}.$$

As the matrices R, S, and T have a special structure, the inverses required in the score test statistic ξ can be easily computed.

4. Case 2: $H_{02}: \sigma_1 = \cdots = \sigma_K$

As mentioned previously, there has been considerable amount of work directed at the Case 1 problem. The Case 2 problem does not seem to have been studied. It is conceivable that group correlations may differ yet group variances may be the same. For example, if the same type of instruments are used across groups and the source of variability is due to measurement error, then group variances are comparable. However, if additional sources of variability are at play, then H_{02} may be false. For the three Case 2 statistics that follow, the asymptotic distributions follow a χ^2_{K-1}distribution.

4.1. *Likelihood Ratio Test*

Under $H_{02}: \sigma_1 = \cdots = \sigma_K = \sigma$ (an unspecified value), the log-likelihood is

$$\begin{aligned}\ell_0 &= \ell_0(\mu_1, \ldots, \mu_K, \sigma^2, \rho_1, \ldots, \rho_K)\\ &= -\frac{1}{2}np\ln\sigma^2 - \frac{1}{2}\sum_{i=1}^{K} n_i \ln\{\rho_i + p^{-1}(1-\rho_i)\} - \frac{1}{2}(p-1)\sum_{i=1}^{K} n_i \ln(1-\rho_i)\\ &\quad - \sigma^{-2}\sum_{i=1}^{K}\sum_{j=1}^{n_i}\frac{(z_{ij1}-\mu_i)^2}{2\{\rho_i + p^{-1}(1-\rho_i)\}} - \sigma^{-2}\sum_{i=1}^{K}\sum_{j=1}^{n_i}\sum_{k=2}^{p}\frac{z_{ijk}^2}{2(1-\rho_i)}.\end{aligned}$$

The ML estimates for $\mu_1, \ldots, \mu_K$ and σ^2 are given by

$$\tilde{\mu}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} z_{ij1} = \hat{\mu}_i$$

and

$$\tilde{\sigma}^2 = \frac{1}{np}\sum_{i=1}^{K}\sum_{j=1}^{n_i}\left\{\frac{(z_{ij1}-\tilde{\mu}_i)^2}{\{\tilde{\rho}_i + p^{-1}(1-\tilde{\rho}_i)\}} + \sum_{k=2}^{p}\frac{z_{ijk}^2}{(1-\tilde{\rho}_i)}\right\}. \tag{7}$$

The equations

$$\begin{aligned}\frac{\partial\ell_0}{\partial\rho_i} &= -\frac{n_i(p-1)}{2p\{\rho_i + p^{-1}(1-\rho_i)\}} + \frac{n_i(p-1)}{2(1-\rho_i)} + \frac{(p-1)\sum_{j=1}^{n_i}(z_{ij1}-\mu_i)^2}{2p\sigma^2\{\rho_i + p^{-1}(1-\rho_i)\}^2}\\ &\quad - \frac{\sum_{j=1}^{n_i}\sum_{k=2}^{p} z_{ijk}^2}{2\sigma^2(1-\rho_i)^2} = 0\end{aligned} \tag{8}$$

do not lead to an explicit solution for ρ_i, $i = 1, \ldots, K$. To obtain the ML estimate $\tilde{\rho}_i$ of ρ_i under H_{02}, our first instinct was to use the scoring method. However, we experienced some convergence problems. Instead, the method of bisection was successfully used on Eq. (8). Because of the interdependence between $\tilde{\sigma}^2$ and $\tilde{\rho}_i$, the following algorithm was used for estimation.

1. Let $m = 0$ and $\rho_i^{(0)} = \hat{\rho}_i$ where $\hat{\rho}_i$ is given by (4), $i = 1, \ldots, K$.
2. Let $(\sigma^2)^{(m)}$ be the m-th approximation to $\tilde{\sigma}^2$ obtained by substituting $\rho_i^{(m)}$ for ρ_i in (7).

3. Calculate $\rho_i^{(m+1)}$ using bisection on (8) where $(\sigma^2)^{(m)}$ replaces σ^2, $i = 1, \ldots, K$.
4. Let $m = m + 1$.
5. If $|\frac{\partial \ell_0}{\partial \rho_i}| < 10^{-6}$ for $i = 1, \ldots, K$ then stop. Otherwise, go to Step 2 and continue.

The likelihood ratio test statistic is given by $w = 2(\widehat{\ell} - \widehat{\ell}_0)$ where $\widehat{\ell}$ is the maximized unrestricted loglikelihood and $\widehat{\ell}_0$ is the maximized restricted loglikelihood.

4.2. *Modified Likelihood Ratio Tests*

Everything except the terms $\widetilde{\mathcal{I}}_{vv}$ and $\widetilde{\mathcal{J}}_{vv}$ in the δ formula (1) remains the same as in Case 1. The matrices $\widetilde{\mathcal{J}}_{vv}$ and $\widetilde{\mathcal{I}}_{vv}$ for the nuisance parameter $v' = (\mu_1, \ldots, \mu_K, \rho_1, \ldots, \rho_K)$ are as follows. For $i = 1, \ldots, K$,

$$\widetilde{\mathcal{J}}_{vv}(i, i) = -\frac{\partial^2 \ell}{\partial \mu_i^2}\bigg|_{\omega=\tilde{\omega}} = \frac{n_i}{\tilde{\sigma}^2\{\tilde{\rho}_i + p^{-1}(1 - \tilde{\rho}_i)\}}$$

$$\widetilde{\mathcal{J}}_{vv}(i, K+i) = -\frac{\partial^2 \ell}{\partial \mu_i \rho_i}\bigg|_{\omega=\tilde{\omega}} = \frac{(p-1)\sum_{j=1}^{n_i}(z_{ij1} - \tilde{\mu}_i)}{p\tilde{\sigma}^2\{\tilde{\rho}_i + p^{-1}(1 - \tilde{\rho}_i)\}^2} = 0$$

$$\widetilde{\mathcal{J}}_{vv}(K+i, K+i) = -\frac{\partial^2 \ell}{\partial \rho_i^2}\bigg|_{\omega=\tilde{\omega}} = -\frac{n_i(p-1)^2}{2p^2\{\tilde{\rho}_i - p^{-1}(1 - \tilde{\rho}_i)\}^2} - \frac{n_i(p-1)}{2(1 - \tilde{\rho}_i)^2} + \frac{(p-1)^2\sum_{j=1}^{n_i}(z_{ij1} - \tilde{\mu}_i)^2}{p^2\tilde{\sigma}^2\{\tilde{\rho}_i + p^{-1}(1 - \tilde{\rho}_i)\}^3} + \frac{\sum_{j=1}^{n_i}\sum_{k=2}^{p} z_{ijk}^2}{\tilde{\sigma}^2(1 - \tilde{\rho}_i)^3}.$$

All other terms are zero since

$$\frac{\partial^2 \ell}{\partial \mu_i \partial \mu_j} = \frac{\partial^2 \ell}{\partial \rho_i \partial \rho_j} = \frac{\partial^2 \ell}{\partial \mu_i \partial \rho_j} = 0, \quad i \neq j.$$

For $i = 1, \ldots, K$,

$$\widetilde{\mathcal{I}}_{vv}(i, i) = -E\left(\frac{\partial^2 \ell}{\partial \mu_i^2}\right)\bigg|_{\omega=\tilde{\omega}} = \frac{n_i}{\tilde{\sigma}^2\{\tilde{\rho}_i + p^{-1}(1 - \tilde{\rho}_i)\}}$$

$$\widetilde{\mathcal{I}}_{vv}(i, K+i) = -E\left(\frac{\partial^2 \ell}{\partial \mu_i \partial \rho_i}\right)\bigg|_{\omega=\tilde{\omega}} = 0$$

$$\widetilde{\mathcal{I}}_{vv}(K+i, K+i) = -E\left(\frac{\partial^2 \ell}{\partial \rho_i^2}\right)\bigg|_{\omega=\tilde{\omega}} = \frac{n_i(p-1)^2}{2p^2\{\tilde{\rho}_i + p^{-1}(1 - \rho_i)\}^2} + \frac{n_i(p-1)}{2(1 - \tilde{\rho}_i)^2}$$

and all other terms are zero.

4.3. *Score Test*

The score vector U_ψ and the matrices R, S, and T are given by the following expressions where $i, j = 1, \ldots, K$.

$$U_i = \frac{\partial \ell}{\partial \sigma_i^2}\bigg|_{\omega=\tilde{\omega}} = -\frac{n_i p}{2\tilde{\sigma}^2} + \frac{\sum_{j=1}^{n_i}(z_{ij1}-\mu_i)^2}{2\tilde{\sigma}^4\{\tilde{\rho}_i + p^{-1}(1-\tilde{\rho}_i)\}} + \frac{\sum_{j=1}^{n_i}\sum_{k=2}^{p} z_{ijk}^2}{2\tilde{\sigma}^4(1-\tilde{\rho}_i)},$$

$$R(i,i) = -E\left(\frac{\partial^2 \ell}{\partial \sigma_i^4}\right)\bigg|_{\omega=\tilde{\omega}} = \frac{n_i p}{2\tilde{\sigma}^4},$$

$$R(i,j) = -E\left(\frac{\partial^2 \ell}{\partial \sigma_i^2 \partial \sigma_j^2}\right)\bigg|_{\omega=\tilde{\omega}} = 0, \quad i \neq j,$$

$$S(i,i) = -E\left(\frac{\partial^2 \ell}{\partial \sigma_i^2 \partial \mu_i}\right)\bigg|_{\omega=\tilde{\omega}} = 0,$$

$$S(i,K+i) = -E\left(\frac{\partial^2 \ell}{\partial \sigma_i^2 \partial \rho_i}\right)\bigg|_{\omega=\tilde{\omega}} = \frac{(p-1)n_i}{2p\tilde{\sigma}^2\{\tilde{\rho}_i + p^{-1}(1-\tilde{\rho}_i)\}} - \frac{(p-1)n_i}{2\tilde{\sigma}^2(1-\tilde{\rho}_i)},$$

$$S(i,j) = -E\left(\frac{\partial^2 \ell}{\partial \sigma_i^2 \partial \mu_j}\right)\bigg|_{\omega=\tilde{\omega}} = 0, \quad i \neq j$$

$$S(i,K+j) = -E\left(\frac{\partial^2 \ell}{\partial \sigma_i^2 \partial \rho_j}\right)\bigg|_{\omega=\tilde{\omega}} = 0, \quad i \neq j,$$

$$T(i,j) = -E\left(\frac{\partial^2 \ell}{\partial v_i \partial v_j}\right)\bigg|_{\omega=\tilde{\omega}} = \widetilde{\mathcal{I}}_{vv}(i,j)$$

where $\widetilde{\mathcal{I}}_{vv}$ is given in Subsec. 4.2.

The score statistic for testing $H_{02}: \sigma_1 = \cdots = \sigma_K$ is then given by

$$\xi = U_\psi'(R - ST^{-1}S')^{-1}U_\psi.$$

5. Case 3: $H_{03}: \sigma_1 = \cdots = \sigma_K, \rho_1 = \cdots = \rho_K$

Since H_{01} (Case1) and H_{02} (Case2) are common hypotheses of interest with respect to the grouped normal data problem, it follows that the composite hypothesis H_{03} is also of interest. Although there have been a number of test procedures proposed for H_{03} (Han, 1975) , we make use of the methodologies developed for Cases 1 and 2 to derive the Skovgaard (2001) modifications. The likelihood ratio test and the score test are presented for comparison purposes. For the three Case 3 statistics that follow, the asymptotic distributions follow a χ^2_{2K-2}distribution.

5.1. *Likelihood Ratio Test*

Under $H_{03}: \sigma_1 = \cdots = \sigma_K = \sigma$ and $\rho_1 = \cdots = \rho_K = \rho$, the log-likelihood is

$$\begin{aligned}\ell_0 &= \ell_0(\mu_1, \ldots, \mu_K, \sigma^2, \rho) \\ &= -\frac{1}{2}np\ln\sigma^2 - \frac{1}{2}n\ln\{\rho + p^{-1}(1-\rho)\} - \frac{1}{2}n(p-1)\ln(1-\rho) \\ &\quad - \frac{1}{2\sigma^2\{\rho + p^{-1}(1-\rho)\}}\sum_{i=1}^{K}\sum_{j=1}^{n_i}(z_{ij1}-\mu_i)^2 - \frac{1}{2\sigma^2(1-\rho)}\sum_{i=1}^{K}\sum_{j=1}^{n_i}\sum_{k=2}^{p} z_{ijk}^2.\end{aligned}$$

The ML estimates are given by

$$\tilde{\mu}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} z_{ij1} = \hat{\mu}_i,$$

$$\tilde{\sigma}^2 = \frac{1}{np}\sum_{i=1}^{K}\sum_{j=1}^{n_i}\left\{\frac{(z_{ij1}-\tilde{\mu}_i)^2}{\tilde{\rho}+p^{-1}(1-\tilde{\rho})} + \sum_{k=2}^{p}\frac{z_{ijk}^2}{(1-\tilde{\rho})}\right\},$$

$$\tilde{\rho} = \frac{p\sum_{i=1}^{K}\sum_{j=1}^{n_i}(z_{ij1}-\tilde{\mu}_i)^2 - (p-1)^{-1}\sum_{i=1}^{K}\sum_{j=1}^{n_i}\sum_{k=2}^{p} z_{ijk}^2}{p\sum_{i=1}^{K}\sum_{j=1}^{n_i}(z_{ij1}-\tilde{\mu}_i)^2 + \sum_{i=1}^{K}\sum_{j=1}^{n_i}\sum_{k=2}^{p} z_{ijk}^2}.$$

5.2. *Modified Likelihood Ratio Tests*

For the Skovgaard (2001) modifications, the matrices $\tilde{\mathcal{J}}_{vv}$ and $\tilde{\mathcal{I}}_{vv}$ for the nuisance parameter $v' = (\mu_1, \ldots, \mu_K)$ are as follows

$$\tilde{\mathcal{J}}_{vv}(i,i) = -\frac{\partial^2\ell}{\partial\mu_i^2}\bigg|_{\omega=\tilde{\omega}} = \frac{n}{\tilde{\sigma}^2\{\tilde{\rho}+p^{-1}(1-\tilde{\rho})\}},$$

$$\tilde{\mathcal{J}}_{vv}(i,j) = -\frac{\partial^2\ell}{\partial\mu_i\partial\mu_j}\bigg|_{\omega=\tilde{\omega}} = 0, \quad i \neq j,$$

$$\tilde{\mathcal{I}}_{vv}(i,i) = -E\left(\frac{\partial^2\ell}{\partial\mu_i^2}\right)\bigg|_{\omega=\tilde{\omega}} = \frac{n}{\tilde{\sigma}^2\{\tilde{\rho}+p^{-1}(1-\tilde{\rho})\}},$$

$$\tilde{\mathcal{I}}_{vv}(i,j) = -E\left(\frac{\partial^2\ell}{\partial\mu_i\partial\mu_j}\right)\bigg|_{\omega=\tilde{\omega}} = 0, \quad i \neq j.$$

Note that $\tilde{\mathcal{I}}_{vv} = \tilde{\mathcal{J}}_{vv}$ providing a simplification in the δ formula (1).

5.3. *Score Test*

The score vector $U'_\psi = (U_1, \ldots, U_{2K})$ and the matrices R and S are given by

$$U_i = \frac{\partial\ell}{\partial\sigma_i^2}\bigg|_{\omega=\tilde{\omega}} = -\frac{n_i p}{2\tilde{\sigma}^2} + \frac{\sum_{j=1}^{n_i}(z_{ij1}-\mu_i)^2}{2\tilde{\sigma}^4\{\tilde{\rho}+p^{-1}(1-\tilde{\rho})\}} + \frac{\sum_{j=1}^{n_i}\sum_{k=2}^{p} z_{ijk}^2}{2\tilde{\sigma}^4(1-\tilde{\rho})},$$

$$U_{K+i} = \frac{\partial\ell}{\partial\rho_i}\bigg|_{\omega=\tilde{\omega}} = -\frac{n_i(p-1)}{2p\{\tilde{\rho}+p^{-1}(1-\tilde{\rho})\}} + \frac{n_i(p-1)}{2(1-\tilde{\rho})} + \frac{(p-1)\sum_{j=1}^{n_i}(z_{ij1}-\mu_i)^2}{2p\tilde{\sigma}^2\{\tilde{\rho}+p^{-1}(1-\tilde{\rho})\}^2} - \frac{\sum_{j=1}^{n_i}\sum_{k=2}^{p} z_{ijk}^2}{2\tilde{\sigma}^2(1-\tilde{\rho})^2},$$

$$R(i,i) = -E\left(\frac{\partial^2\ell}{\partial\sigma_i^4}\right)\bigg|_{\omega=\tilde{\omega}} = \frac{n_i p}{2\tilde{\sigma}^4},$$

$$R(i,K+i) = R(K+i,i) = -E\left(\frac{\partial^2\ell}{\partial\sigma_i^2\partial\rho_i}\right)\bigg|_{\omega=\tilde{\omega}} = \frac{(p-1)n_i}{2p\tilde{\sigma}^2\{\tilde{\rho}+p^{-1}(1-\tilde{\rho})\}} - \frac{(p-1)n_i}{2\tilde{\sigma}^2(1-\tilde{\rho})},$$

$$R(K+i,K+i) = -E\left(\frac{\partial^2\ell}{\partial\rho_i^2}\right)\bigg|_{\omega=\tilde{\omega}} = \frac{n(p-1)^2}{2p^2\{\tilde{\rho}+p^{-1}(1-\tilde{\rho})\}^2} + \frac{n(p-1)}{2(1-\tilde{\rho})^2}$$

and all other terms are zero since

$$E\left(\frac{\partial^2 \ell}{\partial\sigma_i^2 \partial\sigma_j^2}\right)\Big|_{\omega=\tilde{\omega}} = E\left(\frac{\partial^2 \ell}{\partial\rho_i \partial\rho_j}\right)\Big|_{\omega=\tilde{\omega}} = E\left(\frac{\partial^2 \ell}{\partial\sigma_i^2 \partial\rho_j}\right)\Big|_{\omega=\tilde{\omega}} = 0, \quad i \neq j.$$

Furthermore,

$$S = -E\left(\frac{\partial^2 \ell}{\partial\psi_i \partial v_j}\right)\Big|_{\omega=\tilde{\omega}} = 0$$

since

$$E\left(\frac{\partial^2 \ell}{\partial\sigma_i^2 \partial\mu_i}\right)\Big|_{\omega=\tilde{\omega}} = E\left(\frac{\partial^2 \ell}{\partial\rho_i \partial\mu_i}\right)\Big|_{\omega=\tilde{\omega}} = 0,$$

$$E\left(\frac{\partial^2 \ell}{\partial\sigma_i^2 \partial\mu_j}\right)\Big|_{\omega=\tilde{\omega}} = E\left(\frac{\partial^2 \ell}{\partial\rho_i \partial\mu_j}\right)\Big|_{\omega=\tilde{\omega}} = 0, \quad i \neq j.$$

Table 1
Case 1 ($H_{01} : \rho_1 = \cdots = \rho_K$ where $\sigma_1^2 = \cdots = \sigma_K^2 = 1.0$ and $K = 5$)

		Type I error rates				Cramer-von Mises statistic			
	ρ_i	w	w^*	w^{**}	ξ	w	w^*	w^{**}	ξ
$p = 5$									
($n_i = 5$)	0.00	0.126	0.046	0.042	0.048	1201.8	20.7	91.5	240.8
	0.25	0.125	0.048	0.043	0.049	1207.5	19.7	89.8	246.0
	0.50	0.125	0.046	0.041	0.049	1249.7	15.6	80.8	258.8
	0.75	0.127	0.047	0.043	0.050	1249.2	15.3	79.9	259.6
	0.95	0.125	0.048	0.043	0.049	1213.5	20.6	91.8	247.6
($n_i = 20$)	0.00	0.062	0.049	0.049	0.049	63.6	1.6	3.0	14.4
	0.25	0.063	0.049	0.049	0.049	56.1	3.1	5.0	12.2
	0.50	0.062	0.048	0.048	0.048	60.3	2.4	4.1	13.6
	0.75	0.064	0.049	0.048	0.048	59.8	2.4	4.1	13.4
	0.95	0.063	0.048	0.048	0.049	65.3	1.6	3.1	16.3
$p = 10$									
($n_i = 5$)	0.00	0.134	0.045	0.040	0.065	1453.3	60.0	254.3	302.2
	0.25	0.135	0.046	0.041	0.066	1485.9	55.7	247.6	318.2
	0.50	0.135	0.047	0.041	0.066	1454.7	64.9	265.3	309.9
	0.75	0.135	0.045	0.040	0.066	1467.0	56.5	247.8	308.4
	0.95	0.136	0.048	0.042	0.065	1511.3	53.7	243.8	329.7
($n_i = 20$)	0.00	0.065	0.049	0.048	0.052	71.9	4.6	9.3	15.5
	0.25	0.064	0.048	0.048	0.052	77.0	3.8	8.1	18.3
	0.50	0.064	0.049	0.048	0.051	76.1	3.6	7.8	17.6
	0.75	0.064	0.049	0.048	0.053	71.7	5.5	10.5	16.3
	0.95	0.065	0.048	0.048	0.052	67.3	7.2	12.8	14.0

Then, the score statistic for testing H_{03} reduces to

$$\xi = U'_\psi(R - ST^{-1}S')^{-1}U_\psi = U'_\psi R^{-1}U_\psi.$$

6. Simulation Results and Discussion

We present the results of some Monte Carlo simulations to investigate the adequacy of the test statistics discussed in this article. The following tables show the Type I error probabilities of the likelihood ratio statistic (w), the modified likelihood ratio statistics (w^* and w^{**}), and the score statistic (ξ) based on $N = 100,000$ simulations. In each case, we used the nominal value 0.05 and note that the results are comparable for the nominal value 0.01.

In addition to investigating the Type I error rate, we are also interested in the adequacy of the χ^2 approximations over the entire range of the test statistics. To do so, we calculate the Cramér-von Mises statistic (Anderson and Darling, 1954)

$$W^2 = 1/(12N) + \sum\{Z_{(i)} - (2i-1)/(2N)\}^2$$

Table 2
Case 2 ($H_{02} : \sigma_1 = \cdots = \sigma_K$ where $K = 5$)

		Type I error rates				Cramer-von Mises statistic			
	ρ_i	w	w^*	w^{**}	ξ	w	w^*	w^{**}	ξ
$p = 5$									
($n_i = 5$)	0.00	0.042	0.023	0.022	0.024	14.9	154.8	203.8	261.9
	0.25	0.049	0.025	0.024	0.031	5.7	107.9	178.5	133.1
	0.50	0.079	0.038	0.036	0.058	400.6	18.0	83.9	58.1
	0.75	0.126	0.046	0.041	0.077	1306.5	60.8	272.7	308.2
	0.95	0.140	0.045	0.039	0.074	1572.1	127.0	483.5	313.1
($n_i = 20$)	0.00	0.047	0.048	0.048	0.044	0.6	1.1	1.8	6.5
	0.25	0.051	0.048	0.048	0.046	1.5	2.3	4.4	2.7
	0.50	0.060	0.050	0.049	0.052	44.4	3.6	7.5	9.1
	0.75	0.064	0.048	0.047	0.054	71.1	9.3	17.6	13.4
	0.95	0.064	0.048	0.047	0.054	84.4	8.1	16.6	18.5
$p = 10$									
($n_i = 5$)	0.00	0.044	0.035	0.035	0.032	2.9	21.5	28.0	60.8
	0.25	0.055	0.038	0.037	0.044	38.4	22.2	56.5	16.6
	0.50	0.103	0.053	0.049	0.080	842.4	18.6	103.7	244.8
	0.75	0.137	0.046	0.041	0.077	1530.7	89.3	374.6	337.7
	0.95	0.142	0.046	0.040	0.074	1659.9	107.6	449.3	346.5
($n_i = 20$)	0.00	0.050	0.050	0.050	0.048	0.4	0.4	0.6	1.4
	0.25	0.054	0.049	0.049	0.050	7.7	2.6	4.2	0.4
	0.50	0.062	0.048	0.048	0.053	63.0	4.4	9.4	13.5
	0.75	0.066	0.049	0.048	0.055	86.8	6.3	13.8	20.9
	0.95	0.066	0.050	0.049	0.054	78.4	10.6	20.2	15.6

where $Z_{(i)}$ is the i-th order statistic corresponding to $Z = F(Y)$, F is the cumulative distribution function (cdf) of the relevant χ^2 distribution, and Y is the test statistic obtained when generating from the equicorrelation model in question. The Cramér-von Mises statistic is one in the wide class of discrepancy measures given by the Cramér-von Mises family

$$Q = n\int_{-\infty}^{\infty}(F_n(y) - F(y))^2\Psi(y)dF(y)$$

where F and $F_n = F_n(Y_1, \ldots, Y_n)$ are the cdf and the empirical distribution function (edf), respectively, and Ψ is a suitable function which gives weights to the squared difference $(F_n(y) - F(y))^2$. Larger values of the statistic Q (and in our case W^2), provide greater evidence of the lack of fit between the data $Y_1, \ldots, Y_n$ and the proposed F. Theory and practical issues associated with the Cramér-von Mises family of statistics are described in detail in Stephens (1986).

The simulation results are presented in Tables 1, 2, and 3 for $K = 5$ groups where $n_1 = \cdots = n_K$. Before discussing the particular simulation results, we make some general comments. It appears that when the sample size n_i reaches 20, all four

Table 3
Case 3 ($H_{03} : \sigma_1 = \cdots = \sigma_K$, $\rho_1 = \cdots = \rho_K$ where $K = 5$)

		Type I error rates				Cramer-von Mises statistic			
	ρ_i	w	w^*	w^{**}	ξ	w	w^*	w^{**}	ξ
$p = 5$									
($n_i = 5$)	0.00	0.115	0.041	0.038	0.063	1114.4	201.3	398.3	106.4
	0.25	0.115	0.040	0.037	0.063	1126.8	193.2	387.1	108.3
	0.50	0.115	0.039	0.036	0.063	1124.4	194.5	388.1	107.0
	0.75	0.116	0.041	0.038	0.063	1083.9	212.8	414.5	96.7
	0.95	0.117	0.042	0.038	0.063	1085.1	205.2	402.3	94.9
($n_i = 20$)	0.00	0.061	0.048	0.047	0.053	58.6	10.0	14.6	6.3
	0.25	0.061	0.048	0.048	0.053	50.9	14.1	19.5	4.5
	0.50	0.060	0.046	0.046	0.051	54.2	12.3	17.4	5.2
	0.75	0.062	0.048	0.047	0.053	54.3	12.0	17.1	4.8
	0.95	0.062	0.048	0.047	0.053	60.6	9.0	13.4	6.7
$p = 10$									
($n_i = 5$)	0.00	0.111	0.041	0.039	0.066	1025.3	129.1	264.2	136.6
	0.25	0.112	0.042	0.039	0.066	1020.4	132.2	269.9	130.2
	0.50	0.111	0.043	0.040	0.066	1020.9	140.4	282.1	141.1
	0.75	0.112	0.043	0.040	0.068	1003.7	140.5	281.5	130.9
	0.95	0.115	0.043	0.040	0.066	1025.3	136.8	276.6	137.8
($n_i = 20$)	0.00	0.061	0.049	0.048	0.053	43.5	10.2	14.0	4.5
	0.25	0.061	0.048	0.048	0.053	49.1	8.2	11.7	7.1
	0.50	0.060	0.048	0.047	0.053	47.8	8.1	11.6	5.9
	0.75	0.061	0.049	0.048	0.054	57.7	5.9	8.9	9.9
	0.95	0.061	0.048	0.048	0.054	47.6	9.2	12.8	6.6

statistics are adequate as the Type I error rates are close to the nominal values. Therefore, our discussion is focused on small sample problems and we choose $n_1 = \cdots = n_K = 5$. Also, except in a few instances, the likelihood ratio statistic w is inferior to w^*, w^{**}, and ξ. Finally, there seems to be little difference in the results when the number of variables changes from $p = 5$ to $p = 10$.

In Table 1, we present the simulation results with respect to the Case 1 scenario. The w^* and ξ statistics perform best with respect to the Type I error rate. However, our recommendation is the modified likelihood ratio statistic w^* based on its superior goodness of fit.

In Table 2, we present the simulation results with respect to the Case 2 scenario. We observe that the performance of all of the test statistics depends on the correlation ρ_i. Notably, w^*, w^{**}, and ξ perform worse at the extreme values of ρ_i. Again, our recommendation is w^* although its performance is not as good as in Case 1.

In Table 3, we present the simulation results with respect to the Case 3 scenario. In Case 3, the score test ξ has a slight edge over the other tests. However, we note that the score test is anti-conservative whereas the two modified likelihood ratio tests are conservative.

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References

Ahmed, S. E., Gupta, A. K., Khan, S. M, Nicol, C. J. (2001). Simultaneous estimation of several intraclass correlation coefficients. *Ann. Instit. Statist. Math.* 53:354–369.

Anderson, T. W., Darling, D. A. (1954). Asymptotic theory of certain goodness-of-fit criteria based on stochastic processes. *Ann. Math. Statist.* 23:193–212.

Barndorff-Nielsen, O. E. (1991). Modified signed loglikelihood ratio. *Biometrika* 78:557–563.

Barndorff-Nielsen, O. E., Cox, D. R. (1994). *Inference and Asymptotics*. London: Chapman and Hall.

Donner, A., Koval, J. J. (1980). The estimation of intraclass correlation in the analysis of family data. *Biometrics* 36:19–25.

Han, C.-P. (1975). Testing the equality of covariance matrices under intraclass correlation models. *Ann. Instit. Statist. Math.* 27:349–356.

Konishi, S., Gupta, A. K. (1989). Testing the equality of several intraclass correlation coefficients. *J. Statist. Plann. Infer.* 21:93–105.

Paul, S. R., Barnwal, R. K. (1990). Maximum likelihood estimation and a $C(\alpha)$ test for a common intraclass correlation. *The Statistician* 39:19–24.

Reid, N. (2003). Asymptotics and the theory of inference. *Ann. Statist.* 31:1695–1731.

Skovgaard, I. M. (2001). Likelihood asymptotics. *Scand. J. Statist.* 28:3–32.

Stephens, M. A. (1986). Tests based on EDF statistics. In *Goodness of Fit Techniques*. D'Agostino, R. B., Stephens, M. A., eds. New York: Marcel Dekker.

Viana, M. A. G., Olkin, I., McMahon, T. T. (1993). Multivariate assessment of computer-analyzed corneal topographers. *J. Optical Soc. Amer. Ser. A* 10:1826–1834.