



PAPER

Global energy minimizers for a diffusion-aggregation model on sphere

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citation and DOI.Razvan C Fetecau¹ , Hansol Park^{2,*} and Vishnu Vaidya³ ¹ Department of Mathematics, Simon Fraser University, 8888 University Dr, Burnaby, BC V5A 1S6, Canada² Department of Mathematics, National Tsing Hua University, Section 2, Kuang-Fu Road, Hsinchu 30013, Taiwan³ Department of Mathematics, Indian Institute of Science Education and Research, Dr Homi Bhabha Road, Pune 411008, India

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E-mail: hansolpark@math.nthu.edu.tw, van@math.sfu.ca and vishnu.vaidya@students.iiserpune.ac.in**Keywords:** free energy, global minimizers, nonlinear diffusion, phase transitions, dipolar potential, polymer orientation**Mathematics Subject Classification numbers:** 35A15, 35B38, 58K05, 82D60

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Abstract

We investigate the ground states of a free energy functional on sphere. The energy consists of an entropy and a nonlocal interaction term that are in competition with each other, as they favour spreading and aggregation, respectively. Specifically, the entropy corresponds to slow nonlinear diffusion and the interaction term is modelled by a quadratic interaction potential. We investigate the transitions that occur in the equilibria and the global minimizers of the energy, in terms of the strength of the nonlocal attractive interactions. We consider separately various ranges of the diffusion exponent, which give qualitatively different behaviours of equilibria and ground states. In terms of applications, we note that the energy we consider here is a generalization to nonlinear diffusion of the Onsager free energy with dipolar potential, used to study phase transitions in polymer orientation.

1. Introduction

In this paper we investigate the minimizers of the free energy functional

$$E[\rho] = \frac{1}{m-1} \int_{\mathbb{S}^d} \rho(x)^m dS(x) + \frac{\kappa}{4} \iint_{\mathbb{S}^d \times \mathbb{S}^d} \|x-y\|^2 \rho(x) \rho(y) dS(x) dS(y), \quad (1.1)$$

defined on the space $\mathcal{P}_{ac}(\mathbb{S}^d)$ of absolutely continuous probability measures⁴ on the unit sphere \mathbb{S}^d . Here, $m > 0$ is the diffusion exponent ($m = 1$ requires special consideration), $\kappa > 0$ represents the interaction strength, and $\|\cdot\|$ denotes the Euclidean distance in \mathbb{R}^{d+1} . Also, the integration is with respect to the surface area measure dS of the sphere, and the notion of absolute continuity is with respect to the measure dS .

The functional (1.1) falls in a class of aggregation-diffusion energies extensively studied in various contexts, in particular in self-organizing phenomena such as swarming or flocking of biological organisms [36, 45, 47], emergent behaviour in robotic swarms [41, 43], self-assembly of nanoparticles [33], and opinion formation [42]. In its general form, the energy (1.1) is defined on probability measures on a generic Riemannian manifold M :

$$E[\rho] = \frac{1}{m-1} \int_M \rho(x)^m dx + \frac{1}{2} \iint_{M \times M} W(x,y) \rho(x) \rho(y) dx dy, \quad (1.2)$$

⁴ Note that throughout the paper we refer to an absolutely continuous measure directly by its density, and by an abuse of notation we write $\rho \in \mathcal{P}_{ac}(\mathbb{S}^d)$ instead of $d\rho = \rho dS \in \mathcal{P}_{ac}(\mathbb{S}^d)$.

where $W : M \times M \rightarrow \mathbb{R}$ is an interaction potential, and integration is with respect to the Riemannian volume measure dx . The energy (1.2) relates to the following nonlinear nonlocal evolution equation for the population density ρ :

$$\partial_t \rho(x) - \nabla_M \cdot (\rho(x) \nabla_M W * \rho(x)) = \Delta_M \rho^m(x), \quad (1.3)$$

where

$$W * \rho(x) = \int_M W(x, y) \rho(y) dy,$$

and $\nabla_M \cdot$ and ∇_M represent the Riemannian divergence and gradient, respectively [1]. Specifically, critical points of the energy functional correspond to steady states of (1.3). We also note that for $m = 1$, which corresponds to linear diffusion, the entropy term in (1.2) is replaced by $\int_M \rho(x) \log \rho(x) dx$.

Both the free energy functional (1.2) and its corresponding gradient flow (1.3) have been extensively studied on the Euclidean space $M = \mathbb{R}^d$. A partial list of issues addressed by analysts includes the existence of global energy minimizers for interaction potentials of Riesz type [4, 6, 11], uniqueness and qualitative properties (such as monotonicity and radial symmetry) of energy minimizers or steady states of (1.3) [3, 10, 16, 35], and well-posedness and long time behaviour of solutions to the evolution equation (1.3) [5, 7, 10]. We also refer here to the review papers [5, 12] and the influential monograph [1]. The literature on general manifolds is much less developed however. The case when M is a Cartan–Hadamard manifold was studied recently in [21, 22] for linear diffusion and in [8, 9] for nonlinear diffusion. As far as the manifold setup is concerned, most of the research has focused in fact on the model without the diffusion term (when only nonlocal interactions are considered). The well-posedness of the model without diffusion on manifolds was studied in [23, 24, 48], and emergent behaviours were studied on a variety of specific manifolds such as sphere [13, 23, 25], unitary matrices [32, 38], hyperbolic space [19, 25, 30], the special orthogonal group [18] and Stiefel manifolds [31], as well as on general Riemannian manifolds of bounded curvature [20].

In the present paper we consider the case

$$M = \mathbb{S}^d, \quad \text{and} \quad W(x, y) = \frac{\kappa}{2} \|x - y\|^2. \quad (1.4)$$

The interactions corresponding to this potential are purely *attractive*, i.e. any two points experience a pairwise attractive interaction. Consequently, the interaction energy favours points to aggregate together. On the other hand, the entropy component favours spreading, so the two components of the energy (1.1) compete with each other. On compact manifolds such as the sphere, diffusion by itself leads to global energy minimizers that are uniform densities on the entire space. Together with the attractive interactions, diffusion may still dominate (and lead to uniform states) if the attraction is sufficiently weak. The main interest in this paper is to study the competition between attraction and diffusion in terms of the size of the interaction strength κ .

Apart from its own intrinsic interest as an aggregation-diffusion energy on the sphere, one of the main motivations for this work is to consider nonlinear diffusion in the Onsager free energy on the sphere [44]. The free energy (1.1) for the special case $m = 1$ —i.e. with the entropic term given by $\int_{\mathbb{S}^d} \rho(x) \log \rho(x) dS(x)$ —has been used to study isotropic-nematic phase transitions in rod-like polymers and has an extensive literature on its own (see [14, 17] and the references therein). In this context, $\rho(x)$ represents the probability distribution function for the orientation of a polymer viewed as a rigid rod of unit vector $x \in \mathbb{S}^2$, and the interaction potential in (1.4) is called the dipolar potential; note that since points on \mathbb{S}^d have unit norm, an equivalent expression of the potential used in this literature is $-\kappa x \cdot y$. It was shown for $d = 2$ [17], and later generalized to arbitrary dimension in [15, 26] that a phase transition occurs at a certain critical value of κ , where the isotropic state (given by the uniform distribution) loses stability to a nematic equilibrium density. In a similar spirit to this research, equilibria and phase transitions with the Maier–Saupe interaction potential of the form $-\kappa(x \cdot y)^2$, have also been studied in the linear diffusion case ($m = 1$); see [17, 29, 37].

In this paper we work with general exponent $m > 1$, which corresponds to slow nonlinear diffusion, and investigate the ground states of the energy functional (1.1) in terms of the interaction strength κ . Depending on the values of m (we distinguish between the cases $1 < m < 2$, $m = 2$ and $m > 2$), we identify various critical values of κ that lead to phase transitions in the ground states, similar to the isotropic-nematic transitions for the Onsager energy. We also note that, different from the case of linear diffusion, we also account here for equilibria that have support strictly included in \mathbb{S}^d . This adds a

new type of transition, where ground states change from being fully supported to being strictly supported on the sphere. For the range $1 < m \leq 2$ we characterize the global energy minimizers and their phase transitions, in general dimension $d \geq 1$. For $m > 2$, we carry out our investigations only in dimension $d = 2$, which in fact is the most important case as far as applications to phase transitions in polymer orientations are concerned.

We also note that on top of the numerous applications to self-collective behaviour in sciences and engineering, there has been recent interest in such models in the context of inverse problems and machine learning [34, 39, 46]. In particular, some of these works include manifold setups as in the present paper [40]. Very recently, an interacting particle system was used in the context of artificial intelligence, more specifically for large language models [27]. In this application, the phenomena of clustering and synchronization is important for learning tasks. We point out that the system used in [27] is set up on the unit sphere of general dimension, and the dependence of solutions on the relative strength of the noise and diffusion is listed as an interesting question that remains to be investigated.

Summary. The summary of this paper is as follows. In section 2 we identify the possible critical points of the energy functional. In section 3 we focus on the uniform density and study its stability. In section 4 we investigate the critical values of κ , the equilibria, and the global energy minimizers when the diffusion exponent is in the range $1 < m < 2$. Sections 5 and 6 are similar in scope, and they cover the cases $m = 2$ and $m > 2$, respectively. Some technical details of the results are deferred to the appendix.

2. Critical points of the energy

In this section we identify the critical points of the energy functional, which are the candidates for ground states. For any $x, y \in \mathbb{S}^d$, we have

$$\|x - y\|^2 = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2 = 2 - 2\langle x, y \rangle,$$

which allows to simplify the energy as

$$E[\rho] = \frac{1}{m-1} \int_{\mathbb{S}^d} \rho(x)^m \, dS(x) - \frac{\kappa}{2} \iint_{\mathbb{S}^d \times \mathbb{S}^d} \langle x, y \rangle \rho(x) \rho(y) \, dS(x) \, dS(y) + \frac{\kappa}{2}. \tag{2.5}$$

The centre of mass $c_\rho \in \mathbb{R}^{d+1}$ of a density $\rho \in \mathcal{P}_{ac}(\mathbb{S}^d)$ is defined as

$$c_\rho = \int_{\mathbb{S}^d} x \rho(x) \, dS(x).$$

It holds that

$$\begin{aligned} \|c_\rho\|^2 &= \left\langle \int_{\mathbb{S}^d} x \rho(x) \, dS(x), \int_{\mathbb{S}^d} y \rho(y) \, dS(y) \right\rangle \\ &= \iint_{\mathbb{S}^d \times \mathbb{S}^d} \langle x, y \rangle \rho(x) \rho(y) \, dS(x) \, dS(y), \end{aligned}$$

and hence, the energy (1.1) can be written as

$$E[\rho] = \frac{1}{m-1} \int_{\mathbb{S}^d} \rho(x)^m \, dS(x) - \frac{\kappa}{2} \|c_\rho\|^2 + \frac{\kappa}{2}. \tag{2.6}$$

A density $\rho \in \mathcal{P}_{ac}(\mathbb{S}^d)$ is a critical point of the energy E if the expression

$$\frac{\delta E}{\delta \rho} = \left(\frac{m}{m-1} \right) \rho(x)^{m-1} - \kappa \langle c_\rho, x \rangle \tag{2.7}$$

is equal to a constant, possibly different in each closed connected component of the support of ρ [6, 35]. At critical points, the first variation of the energy (with respect to perturbations that preserve mass) vanishes. As alternative terminology, we will refer to critical points of the energy and equilibrium densities, interchangeably in the paper. Our main goal in this paper is to identify the global energy minimizers, for which reason we will further reduce our attention to a subset of this large class of critical points.

Critical points that are candidates for global minimizers must necessarily satisfy

$$\left(\frac{m}{m-1}\right)\rho(x)^{m-1} - \kappa\langle c_\rho, x \rangle = \lambda, \quad x \in \text{supp}(\rho), \tag{2.8}$$

a condition that demands that the expression in (2.7) takes the *same* constant value λ in each connected component of $\text{supp}(\rho)$. Condition (2.8) can be formally derived by taking perturbations in suitable Wasserstein spaces [6, 19]. More informally, we can explain it by using the physical interpretation of the constant λ , which mathematically plays the role of a Lagrange multiplier in the derivation of the Euler–Lagrange equations. Specifically, λ represents the energy per unit mass felt by a point mass at position x due to interaction with all points in $\text{supp}(\rho)$ [2]. Then, a critical point with different values of λ in different connected components of the support, cannot be a global minimizer (not even a local minimizer) with respect to perturbations that transport mass from the component with larger λ to that with smaller λ .

We first note that the uniform distribution on the sphere,

$$\rho_{\text{uni}}(x) = \frac{1}{|\mathbb{S}^d|}, \quad \forall x \in \mathbb{S}^d, \tag{2.9}$$

where $|\mathbb{S}^d|$ denotes the area of \mathbb{S}^d , is a solution of (2.8) for all $\kappa > 0$. Indeed, in this case, $c_{\rho_{\text{uni}}} = 0$, and the constant λ is given by

$$\lambda_{\text{uni}} = \left(\frac{m}{m-1}\right) \frac{1}{|\mathbb{S}^d|^{m-1}}. \tag{2.10}$$

The following lemma limits the candidates to global energy minimizers to densities that are fully supported on a geodesic disk of the sphere.

Lemma 2.1. *Suppose ρ is a global energy minimizer; in particular it satisfies (2.8) for a constant λ . Then,*

$$\text{supp}(\rho) = \{x : \kappa\langle c_\rho, x \rangle + \lambda \geq 0\}.$$

Proof. Denote by $A := \{x : \kappa\langle c_\rho, x \rangle + \lambda \geq 0\}$. We want to show that $\text{supp}(\rho) = A$. The inclusion $\text{supp}(\rho) \subset A$ is immediate. Indeed, by (2.8), for any $x \in \text{supp}(\rho)$ we have $\rho(x) \geq 0$, which implies that $\kappa\langle c_\rho, x \rangle + \lambda \geq 0$, and hence $x \in A$.

Assume by contradiction that $\text{supp}(\rho)$ is a proper subset of A , so that $|A \setminus \text{supp}(\rho)| > 0$. Take a probability density ψ such that $\text{supp}(\psi) \subset A \setminus \text{supp}(\rho)$, and consider the following perturbation:

$$\rho^\epsilon = (1 - \epsilon)\rho + \epsilon\psi, \quad 0 \leq \epsilon < 1.$$

Since

$$\partial_\epsilon \rho^\epsilon = -\rho + \psi,$$

we calculate using (2.6):

$$\begin{aligned} \left. \frac{d}{d\epsilon} E[\rho^\epsilon] \right|_{\epsilon=0+} &= \frac{m}{m-1} \int_{\mathbb{S}^d} \rho^\epsilon(x)^{m-1} (-\rho(x) + \psi(x)) \, dS(x) - \kappa\langle c_{\rho^\epsilon}, -c_\rho + c_\psi \rangle \Big|_{\epsilon=0} \\ &= -\frac{m}{m-1} \int_{\mathbb{S}^d} \rho(x)^m \, dS(x) + \kappa\|c_\rho\|^2 - \kappa\langle c_\rho, c_\psi \rangle \\ &\geq 0, \end{aligned} \tag{2.11}$$

where for the second equal sign we used that the supports of ρ and ψ are disjoint, and the inequality comes from the fact that ρ is a minimizer.

Multiplying (2.8) by ρ and integrating with respect to x on $\text{supp}(\rho)$, we get

$$\frac{m}{m-1} \int_{\mathbb{S}^d} \rho(x)^m \, dS(x) - \kappa\|c_\rho\|^2 = \lambda,$$

which used in (2.11) leads to

$$-\lambda - \kappa\langle c_\rho, c_\psi \rangle \geq 0.$$

Since ψ is supported on $A \setminus \text{supp}(\rho)$, we infer from the above that

$$\int_{A \setminus \text{supp}(\rho)} (\lambda + \kappa \langle c_\rho, x \rangle) \psi(x) \, dS(x) \leq 0. \tag{2.12}$$

On the other hand, $\lambda + \kappa \langle c_\rho, x \rangle \geq 0$ for all $x \in A$, which together with (2.12) implies that $\lambda + \kappa \langle c_\rho, x \rangle = 0$ a.e. with respect to $\psi(x) \, dS(x)$. This means however that $\psi(x) \, dS(x)$ is supported on ∂A , which is not possible. We conclude from here that $\text{supp}(\rho)$ must be the entire set A . \square

Write the equilibria (2.8) as

$$\rho(x) = \begin{cases} \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} (\lambda + \kappa \langle c_\rho, x \rangle)^{\frac{1}{m-1}}, & \text{for } x \in \text{supp}(\rho), \\ 0, & \text{otherwise.} \end{cases}$$

By lemma 2.1, ρ is fully supported in the geodesic disk given by $\lambda + \kappa \langle c_\rho, x \rangle \geq 0$ —see more discussion below. Also, by its explicit expression we note that ρ is axially symmetric with respect to the direction of its centre of mass.

Without loss of generality, we can assume $c_\rho = \|c_\rho\|x_0$ for some unit vector $x_0 \in \mathbb{S}^d$. Define $\theta_x = \arccos \langle x_0, x \rangle \in [0, \pi]$, to write $\langle c_\rho, x \rangle = \|c_\rho\| \cos \theta_x$. The condition $\lambda + \kappa \langle c_\rho, x \rangle \geq 0$ on $\text{supp}(\rho)$ can be simplified into

$$-\frac{\lambda}{\kappa \|c_\rho\|} \leq \cos \theta_x, \quad \forall x \in \text{supp}(\rho). \tag{2.13}$$

Given that $\cos \theta_x$ ranges in $[-1, 1]$, we now distinguish two types of equilibria that are candidates for global minimizers, in terms of the relative size of λ : equilibria supported on the entire sphere ($\text{supp}(\rho) = \mathbb{S}^d$) when $\lambda \geq \kappa \|c_\rho\|$, and equilibria that are supported on a strict subset of the sphere ($\text{supp}(\rho) \subsetneq \mathbb{S}^d$) when $-\kappa \|c_\rho\| < \lambda < \kappa \|c_\rho\|$. Note that (2.13) has no solutions if $\lambda < -\kappa \|c_\rho\|$. Also, for $\lambda = -\kappa \|c_\rho\|$, (2.13) has only the solution $\theta_x = 0$ and this does not give an admissible density since the support of ρ is one point and it corresponds to a Dirac mass at x_0 .

a. Equilibria fully supported on \mathbb{S}^d ($\lambda \geq \kappa \|c_\rho\|$). In this case $-\frac{\lambda}{\kappa \|c_\rho\|} \leq -1$, so θ_x has full range $[0, \pi]$. The equilibrium is given by

$$\rho(x) = \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} (\lambda + \kappa \|c_\rho\| \cos \theta_x)^{\frac{1}{m-1}}, \quad \forall x \in \mathbb{S}^d. \tag{2.14}$$

To determine ρ one needs to find λ and $\|c_\rho\|$. These can be found by requiring that ρ has unit mass and centre of mass at c_ρ , as following.

From the definition of c_ρ we compute

$$\begin{aligned} \|c_\rho\|^2 &= \left\langle c_\rho, \int_{\mathbb{S}^d} x \rho(x) \, dS(x) \right\rangle \\ &= \int_{\mathbb{S}^d} \langle c_\rho, x \rangle \rho(x) \, dS(x) \\ &= d w_d \|c_\rho\| \int_0^\pi \cos \theta \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} (\lambda + \kappa \|c_\rho\| \cos \theta)^{\frac{1}{m-1}} \sin^{d-1} \theta \, d\theta, \end{aligned}$$

where for the last equal sign we used hyperspherical coordinates and that $|\mathbb{S}^{d-1}| = d w_d$, where w_d denotes the volume of the d -dimensional unit ball ($w_d = \pi^{d/2} / \Gamma(d/2 + 1)$, where Γ denotes the Gamma function). Together with the unit mass condition, this leads to the following two equations to be solved for λ and $\|c_\rho\|$:

$$\begin{aligned} 1 &= d w_d \int_0^\pi \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} (\lambda + \kappa \|c_\rho\| \cos \theta)^{\frac{1}{m-1}} \sin^{d-1} \theta \, d\theta, \\ \|c_\rho\| &= d w_d \int_0^\pi \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} (\lambda + \kappa \|c_\rho\| \cos \theta)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta \, d\theta. \end{aligned} \tag{2.15}$$

b. Equilibria supported on a strict subset of $\mathbb{S}^d(-\kappa\|c_\rho\| < \lambda < \kappa\|c_\rho\|)$. In this case we have $-1 < -\frac{\lambda}{\kappa\|c_\rho\|} < 1$, and (2.13) restricts θ_x to $\left[0, \arccos\left(-\frac{\lambda}{\kappa\|c_\rho\|}\right)\right]$. The equilibrium ρ can be expressed as

$$\rho(x) = \begin{cases} \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} (\lambda + \kappa\|c_\rho\| \cos\theta_x)^{\frac{1}{m-1}}, & \text{if } 0 \leq \theta_x \leq \arccos\left(-\frac{\lambda}{\kappa\|c_\rho\|}\right), \\ 0, & \text{otherwise.} \end{cases} \tag{2.16}$$

Again, λ and $\|c_\rho\|$ need to be found. Denote by $\phi = \arccos\left(-\frac{\lambda}{\kappa\|c_\rho\|}\right)$. Then, similar to how system (2.15) was derived, we find in this case:

$$\begin{aligned} 1 &= dw_d \int_0^\phi \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} (\lambda + \kappa\|c_\rho\| \cos\theta)^{\frac{1}{m-1}} \sin^{d-1}\theta \, d\theta, \\ \|c_\rho\| &= dw_d \int_0^\phi \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} (\lambda + \kappa\|c_\rho\| \cos\theta)^{\frac{1}{m-1}} \sin^{d-1}\theta \cos\theta \, d\theta, \end{aligned} \tag{2.17}$$

which has to be solved for λ and $\|c_\rho\|$.

Remark 2.1. We note that for the free Onsager functional with linear diffusion, all critical points are fully supported on the sphere [17, 26]. In considering nonlinear diffusion, the set of admissible equilibria is richer, which leads to more sophisticated phase transitions, in particular in the case $m > 2$.

3. Stability of the uniform distribution

In this section we will study the full nonlinear stability of the uniform distribution ρ_{uni} given by (2.9). Consider a general family of perturbations $\{\rho^\epsilon\} \subset \mathcal{P}_{ac}(\mathbb{S}^d)$ of the uniform distribution. In particular, we have

$$\rho^0 = \rho_{\text{uni}}, \quad \int_{\mathbb{S}^d} \rho^\epsilon(x) \, dS(x) = 1, \quad \text{for all } \epsilon. \tag{3.18}$$

The energy of ρ^ϵ is given by (use (2.5)):

$$E[\rho^\epsilon] = \frac{1}{m-1} \int_{\mathbb{S}^d} \rho^\epsilon(x)^m \, dS(x) - \frac{\kappa}{2} \iint_{\mathbb{S}^d \times \mathbb{S}^d} \langle x, y \rangle \rho^\epsilon(x) \rho^\epsilon(y) \, dS(x) \, dS(y) + \frac{\kappa}{2}.$$

The uniform distribution is a local minimizer provided

$$\left. \frac{d^2}{d\epsilon^2} E[\rho^\epsilon] \right|_{\epsilon=0} \geq 0,$$

for all perturbations ρ^ϵ that satisfy (3.18).

Denote

$$\delta\rho^\epsilon = \frac{d}{d\epsilon} \rho^\epsilon, \quad \text{and} \quad \delta^2\rho^\epsilon = \frac{d^2}{d\epsilon^2} \rho^\epsilon.$$

Note that by the unit mass condition in (3.18), we have

$$\int_{\mathbb{S}^d} \delta\rho^\epsilon(x) \, dS(x) = 0, \quad \text{and} \quad \int_{\mathbb{S}^d} \delta^2\rho^\epsilon(x) \, dS(x) = 0, \quad \text{for all } \epsilon.$$

Now compute:

$$\begin{aligned} \frac{d}{d\epsilon} E[\rho^\epsilon] &= \frac{1}{m-1} \int_{\mathbb{S}^d} m\rho^\epsilon(x)^{m-1} \delta\rho^\epsilon(x) \, dS(x) - \frac{\kappa}{2} \iint_{\mathbb{S}^d \times \mathbb{S}^d} \langle x, y \rangle (\delta\rho^\epsilon(x) \rho^\epsilon(y) + \rho^\epsilon(x) \delta\rho^\epsilon(y)) \, dS(x) \, dS(y) \\ &= \frac{1}{m-1} \int_{\mathbb{S}^d} m\rho^\epsilon(x)^{m-1} \delta\rho^\epsilon(x) \, dS(x) - \kappa \iint_{\mathbb{S}^d \times \mathbb{S}^d} \langle x, y \rangle \rho^\epsilon(x) \delta\rho^\epsilon(y) \, dS(x) \, dS(y), \end{aligned}$$

and

$$\begin{aligned} \frac{d^2}{d\epsilon^2} E[\rho^\epsilon] &= m \int_{\mathbb{S}^d} \rho^\epsilon(x)^{m-2} (\delta\rho^\epsilon(x))^2 dS(x) + \frac{m}{m-1} \int_{\mathbb{S}^d} \rho^\epsilon(x)^{m-1} \delta^2\rho^\epsilon(x) dS(x) \\ &\quad - \kappa \iint_{\mathbb{S}^d \times \mathbb{S}^d} \langle x, y \rangle (\delta\rho^\epsilon(x) \delta\rho^\epsilon(y) + \rho^\epsilon(x) \delta^2\rho^\epsilon(y)) dS(x) dS(y). \end{aligned} \tag{3.19}$$

Evaluate (3.19) at $\epsilon = 0$. Using that $\rho^0(x)$ is constant and that $\delta^2\rho^\epsilon(x)$ integrates to 0, we find that the second term in the r.h.s. vanishes:

$$\int_{\mathbb{S}^d} \rho^0(x)^{m-1} \delta^2\rho^0(x) dS(x) = \frac{1}{|\mathbb{S}^d|^{m-1}} \int_{\mathbb{S}^d} \delta^2\rho^0(x) dS(x) = 0.$$

Also,

$$\int_{\mathbb{S}^d} x\rho^0(x) dS(x) = 0,$$

and hence,

$$\iint_{\mathbb{S}^d \times \mathbb{S}^d} \langle x, y \rangle \rho^0(x) \delta^2\rho^0(y) dS(x) dS(y) = \left\langle \int_{\mathbb{S}^d} x\rho^0(x) dS(x), \int_{\mathbb{S}^d} y\delta^2\rho^0(y) dS(y) \right\rangle = 0.$$

Now drop the superscript 0 and denote $\delta\rho = \delta\rho^\epsilon|_{\epsilon=0}$. We find from (3.19):

$$\frac{d^2}{d\epsilon^2} E[\rho^\epsilon] \Big|_{\epsilon=0} = \frac{m}{|\mathbb{S}^d|^{m-2}} \int_{\mathbb{S}^d} (\delta\rho(x))^2 dS(x) - \kappa \left\| \int_{\mathbb{S}^d} x\delta\rho(x) dS(x) \right\|^2. \tag{3.20}$$

For the uniform density to be a local minimum, the expression above has to be positive, i.e.

$$\kappa < \frac{m}{|\mathbb{S}^d|^{m-2}} \cdot \frac{\int_{\mathbb{S}^d} (\delta\rho(x))^2 dS(x)}{\left| \int_{\mathbb{S}^d} x\delta\rho(x) dS(x) \right|^2},$$

for all perturbations $\delta\rho$ that have zero mass.

Define the following functional on $\mathcal{A} = \{\psi \in L^2(\mathbb{S}^d) : \int_{\mathbb{S}^d} \psi(x) dS(x) = 0, \psi \not\equiv 0\}$:

$$\mathcal{F}[\psi] = \frac{\int_{\mathbb{S}^d} (\psi(x))^2 dS(x)}{\left| \int_{\mathbb{S}^d} x\psi(x) dS(x) \right|^2}. \tag{3.21}$$

The uniform distribution is a local minimum (and hence, stable) provided

$$\kappa < \frac{m}{|\mathbb{S}^d|^{m-2}} \cdot \inf_{\psi \in \mathcal{A}} \mathcal{F}[\psi], \tag{3.22}$$

and unstable otherwise.

Lemma 3.1. *Let $\mathcal{F} : \mathcal{A} \rightarrow \mathbb{R}$ be given by (3.21). Then,*

$$\inf_{\psi \in \mathcal{A}} \mathcal{F}[\psi] = \frac{d+1}{\mathbb{S}^d}.$$

Proof. We will present a direct proof, though this result can also be inferred from the known facts on the free energy with linear diffusion—see remark 3.1.

By Cauchy–Schwarz inequality, we see immediately that \mathcal{F} is bounded below by 1. Also, \mathcal{F} is lower-semicontinuous and \mathbb{S}^d is compact, from which we infer that \mathcal{F} admits a global minimizer in \mathcal{A} .

Denote by s_ψ the centre of mass of ψ :

$$s_\psi = \int_{\mathbb{S}^d} x\psi(x) dS(x).$$

By calculating the first variation and setting it to zero, i.e.

$$d\mathcal{F}[\psi](\delta\psi) = 0,$$

we find the following Euler–Lagrange equation for the critical points:

$$\int_{\mathbb{S}^d} \left(\psi(x) |s_\psi|^2 - \left(\int_{\mathbb{S}^d} \psi^2(y) dS(y) \right) s_\psi \cdot x \right) \delta\psi(x) dS(x) = 0,$$

for all perturbations $\delta\psi \in \mathcal{A}$. From here we find

$$\psi(x) |s_\psi|^2 - \left(\int_{\mathbb{S}^d} \psi^2(y) dS(y) \right) s_\psi \cdot x = \lambda, \tag{3.23}$$

for a constant λ . By integrating this equation over \mathbb{S}^d on both sides, and using that ψ integrates to 0, we find $\lambda = 0$.

Consider a critical point $\bar{\psi}$ and denote $C := \int_{\mathbb{S}^d} \bar{\psi}^2(y) dS(y)$. From (3.23) (where $\lambda = 0$), we can write

$$\bar{\psi}(x) = \frac{C}{|s_{\bar{\psi}}|^2} s_{\bar{\psi}} \cdot x.$$

By taking the square of this expression and integrating, we find

$$C = \frac{|s_{\bar{\psi}}|^4}{\int_{\mathbb{S}^d} (s_{\bar{\psi}} \cdot y)^2 dS(y)},$$

and hence,

$$\bar{\psi}(x) = \frac{|s_{\bar{\psi}}|^4}{\int_{\mathbb{S}^d} (s_{\bar{\psi}} \cdot y)^2 dS(y)} s_{\bar{\psi}} \cdot x. \tag{3.24}$$

All the critical points of \mathcal{F} are in the form above. Evaluate \mathcal{F} at such $\bar{\psi}$ to find

$$\begin{aligned} \mathcal{F}[\bar{\psi}] &= \frac{C}{|s_{\bar{\psi}}|^2} \\ &= \frac{|s_{\bar{\psi}}|^2}{\int_{\mathbb{S}^d} (s_{\bar{\psi}} \cdot y)^2 dS(y)}. \end{aligned} \tag{3.25}$$

By symmetry, if $y = (y_1, y_2, \dots, y_{d+1}) \in \mathbb{S}^d$, we have

$$\int_{\mathbb{S}^d} y_1^2 dS(y) = \int_{\mathbb{S}^d} y_2^2 dS(y) = \dots = \int_{\mathbb{S}^d} y_{d+1}^2 dS(y).$$

Then, since $y_1^2 + y_2^2 + \dots + y_{d+1}^2 = 1$, it holds that

$$\int_{\mathbb{S}^d} y_i^2 dS(y) = \frac{1}{d+1} \int_{\mathbb{S}^d} (y_1^2 + \dots + y_{d+1}^2) dS(y) = \frac{1}{d+1} \int_{\mathbb{S}^d} dS(y) = \frac{|\mathbb{S}^d|}{d+1},$$

for all $i = 1, 2, \dots, d+1$.

Therefore, for any $\xi \in \mathbb{R}^{d+1}$, one gets

$$\int_{\mathbb{S}^d} (\xi \cdot y)^2 dS(y) = \sum_{i=1}^{d+1} \int_{\mathbb{S}^d} \xi_i^2 y_i^2 dS(y) = \sum_{i=1}^{d+1} \frac{|\mathbb{S}^d|}{d+1} \xi_i^2 = \frac{|\mathbb{S}^d|}{d+1} |\xi|^2,$$

which used in (3.25) for $\xi = s_{\bar{\psi}}$, yields

$$\mathcal{F}[\bar{\psi}] = \frac{d+1}{|\mathbb{S}^d|}.$$

Note that \mathcal{F} is invariant under scalar multiplication $\psi \rightarrow \lambda\psi$, which is reflected in the fact that the value of $\mathcal{F}[\bar{\psi}]$ does not depend on the magnitude of $s_{\bar{\psi}}$. Put it differently, up to symmetry and a scalar multiplication, the critical points in the form (3.24) are unique. Since \mathcal{F} admits a global minimizer, $\bar{\psi}$ must necessarily achieve this minimum value. □

Remark 3.1. Lemma 3.1 also follows directly from the known results for linear diffusion. Indeed, consider the free energy with linear diffusion, which can be simplified and written as

$$E_1[\rho] = \int_{\mathbb{S}^d} \rho(x) \log \rho(x) \, dS(x) - \frac{\kappa}{2} \left\| \int_{\mathbb{S}^d} x \rho(x) \, dS(x) \right\|^2 + \frac{\kappa}{2}.$$

By taking perturbations ρ^ϵ of the uniform distribution and performing a similar calculation as for $m \neq 1$, we calculate

$$\frac{d^2}{d\epsilon^2} E_1[\rho^\epsilon] \Big|_{\epsilon=0} = \frac{1}{|\mathbb{S}^d|^{-1}} \int_{\mathbb{S}^d} (\delta\rho(x))^2 \, dS(x) - \kappa \left\| \int_{\mathbb{S}^d} x \delta\rho(x) \, dS(x) \right\|^2,$$

which is in fact (3.20) for $m = 1$. Since the uniform distribution is a global energy minimizer for $\kappa \leq d + 1$ [17, 26], the second variation calculated above needs to be non-negative, for all $\delta\rho \in \mathcal{A}$. Hence, $|\mathbb{S}^d| \mathcal{F}[\delta\rho] \geq \kappa$, for all $\delta\rho \in \mathcal{A}$ and $\kappa \geq d + 1$, from which we conclude

$$\inf_{\psi \in \mathcal{A}} \mathcal{F}[\psi] \geq \frac{d+1}{|\mathbb{S}^d|}.$$

The considerations above lead to the following result.

Proposition 3.1 (Stability of the uniform distribution). *The uniform distribution ρ_{uni} is a stable critical point of the energy functional E if and only if $\kappa < \kappa_1$, where*

$$\kappa_1 := \frac{m(d+1)}{|\mathbb{S}^d|^{m-1}}. \tag{3.26}$$

Proof. The proof follows from the previous calculations, by using lemma 3.1 in (3.22). □

Remark 3.2. We note that in the limit $m \searrow 1$, proposition 3.1 recovers the stability result from the classical Onsager free energy with linear diffusion [17, 26]. Indeed, in the case of linear diffusion, the uniform distribution is stable for $\kappa < d + 1$, and unstable otherwise. In the context of polymer orientation [17, 26], this critical interaction strength indicates the phase transition from isotropic to nematic states.

In the next sections, we investigate the existence and bifurcations of critical points of the energy. We will consider the following cases separately i) $1 < m < 2$, ii) $m = 2$, and iii) $m > 2$. The case $m = 2$ is considered separately, as it constitutes a degenerate case. The range $m > 2$ is very different qualitatively, and it is only considered for $d = 2$.

4. Case $1 < m < 2$: critical values of κ and energy minimizers

In this section we investigate the range $1 < m < 2$. We identify two critical values of κ , which we present in order. We then investigate the energy minimizers and also provide some numerical results.

4.1. Bifurcation from the uniform distribution

We first study a bifurcation that occurs at $\kappa = \kappa_1$ (see proposition 3.1). Consider the fully supported equilibrium (2.14), with λ and $\|c_\rho\|$ that solve (2.15). Recall that in this case, $\lambda \geq \kappa \|c_\rho\|$ holds.

For simpler notation, substitute

$$s = \|c_\rho\|, \quad \lambda = -\kappa s \eta. \tag{4.27}$$

To guarantee the condition on λ , we assume $\eta \leq -1$. Then, (2.15) becomes

$$\begin{aligned} 1 &= dw_d \left(\frac{\kappa s (m-1)}{m} \right)^{\frac{1}{m-1}} \int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta \, d\theta, \\ s &= dw_d \left(\frac{\kappa s (m-1)}{m} \right)^{\frac{1}{m-1}} \int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta \, d\theta. \end{aligned}$$

From the above, we can express s and κ in terms of η as follows:

$$s = \frac{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta}{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta}, \tag{4.28a}$$

$$\kappa^{-1} = H(\eta), \tag{4.28b}$$

where $H(\eta)$ is defined for $\eta \leq -1$ by

$$H(\eta) = \frac{m-1}{m} (dw_d)^{m-1} \left(\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta \right) \left(\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^{m-2}. \tag{4.29}$$

Lemma 4.1. *The function $H(\eta)$ given by (4.29) is decreasing for $1 < m < 2$, constant for $m = 2$, and increasing for $m > 2$.*

Proof. The proof is based on a technical and fairly involved result that uses associated Legendre functions of the first kind and the integral form of the Chebyshev inequality. This technical result is stated and proved in appendix A—see lemma A.1. The reader should first check appendix A before continuing to read.

We first rewrite the first integral in the expression of $H(\eta)$ from (4.29) using integration by parts, to get

$$\begin{aligned} \int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta &= \left[(\cos \theta - \eta)^{\frac{1}{m-1}} \times \frac{1}{d} \sin^d \theta \right]_0^\pi \\ &\quad - \int_0^\pi \frac{1}{m-1} (\cos \theta - \eta)^{\frac{1}{m-1}-1} (-\sin \theta) \times \frac{1}{d} \sin^d \theta d\theta \\ &= \frac{1}{d(m-1)} \int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d+1} \theta d\theta. \end{aligned} \tag{4.30}$$

Therefore, we can express $H(\eta)$ as

$$H(\eta) = \frac{1}{md} (dw_d)^{m-1} \left(\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d+1} \theta d\theta \right) \left(\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^{m-2}.$$

Then, for any $\eta < -1$, we have

$$\begin{aligned} \frac{H'(\eta)}{H(\eta)} &= -\frac{2-m}{m-1} \left(\frac{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-2} \sin^{d+1} \theta d\theta}{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d+1} \theta d\theta} \right) \\ &\quad - \frac{m-2}{m-1} \left(\frac{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta}{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta} \right) \\ &= \left(\frac{m-2}{m-1} \right) \left(\frac{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-2} \sin^{d+1} \theta d\theta}{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d+1} \theta d\theta} - \frac{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta}{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta} \right). \end{aligned}$$

Using the notation (A.74) in appendix A, we can further simplify it as

$$\frac{H'(\eta)}{H(\eta)} = -\left(\frac{m-2}{m-1} \right) \frac{\mathcal{H}_{m,d}(-\eta)}{\left(\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d+1} \theta d\theta \right) \left(\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)}.$$

Finally, we apply lemma A.1 in appendix A to get

$$\begin{aligned} \operatorname{sgn} \left(\frac{H'(\eta)}{H(\eta)} \right) &= \operatorname{sgn} \left(-\left(\frac{m-2}{m-1} \right) \frac{\mathcal{H}_{m,d}(-\eta)}{\left(\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d+1} \theta d\theta \right) \left(\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)} \right) \\ &= \operatorname{sgn} \left(\left(\frac{m-2}{m-1} \right) \left(\frac{1}{m-1} + \frac{d-1}{2} \right) \right). \end{aligned}$$

Here, we used that both $\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d+1} \theta d\theta$ and $\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta$ are positive. From $m > 1$, we also get $m-1 > 0$ and $\frac{1}{m-1} + \frac{d-1}{2} > 0$. Therefore, also using that $H(\eta)$ is positive, we find

$$\operatorname{sgn}(H'(\eta)) = \operatorname{sgn}(m-2),$$

which proves the result. □

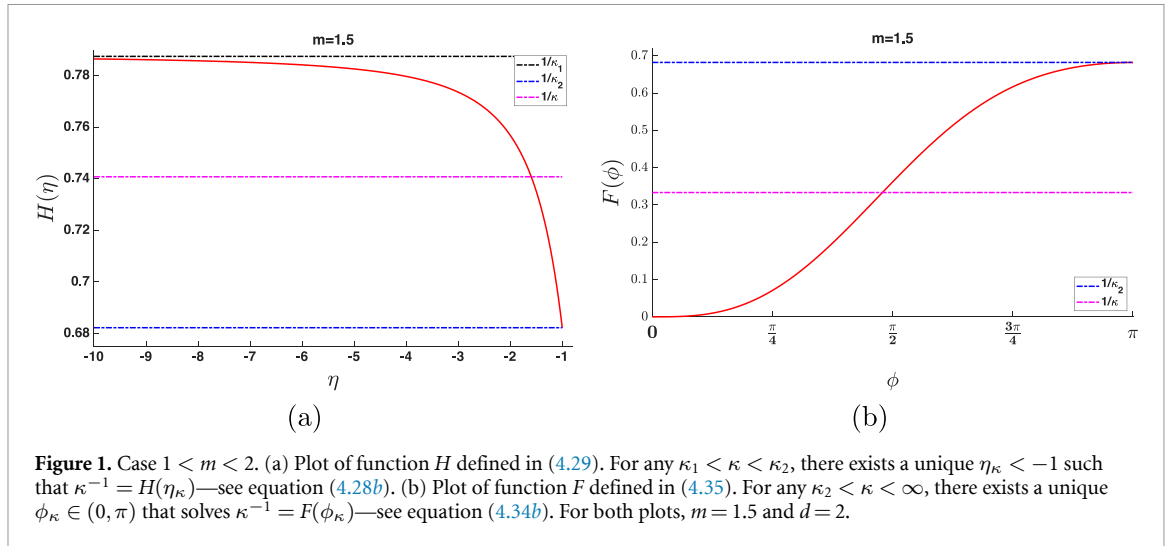


Figure 1. Case $1 < m < 2$. (a) Plot of function H defined in (4.29). For any $\kappa_1 < \kappa < \kappa_2$, there exists a unique $\eta_\kappa < -1$ such that $\kappa^{-1} = H(\eta_\kappa)$ —see equation (4.28b). (b) Plot of function F defined in (4.35). For any $\kappa_2 < \kappa < \infty$, there exists a unique $\phi_\kappa \in (0, \pi)$ that solves $\kappa^{-1} = F(\phi_\kappa)$ —see equation (4.34b). For both plots, $m = 1.5$ and $d = 2$.

Lemma 4.2. *The following holds:*

$$\left(\lim_{\eta \rightarrow -\infty} H(\eta) \right)^{-1} = \kappa_1,$$

where κ_1 is the critical κ identified in proposition 3.1.

Proof. The proof is based on a direct calculation, using Newton’s generalized binomial theorem; see appendix B. □

Remark 4.1. When $\eta \rightarrow -\infty$ and $s \rightarrow 0$ (along with $\lambda \rightarrow \lambda_{\text{uni}}$ defined in (2.10)) one finds the uniform distribution. Therefore, at $\kappa = \kappa_1$, a fully supported equilibrium ρ_κ gets born from the uniform distribution ρ_{uni} .

Since the domain of $H(\eta)$ is $(-\infty, -1]$, and the first critical value κ_1 was derived from the behaviour as $\eta \rightarrow -\infty$, it is natural to investigate the following second critical value of κ defined as

$$\kappa_2 := (H(-1))^{-1}. \tag{4.31}$$

Note that since $H(\eta)$ is decreasing when $1 < m < 2$, we have $\kappa_1 < \kappa_2$. Also, by lemmas 4.1 and 4.2, for any $\kappa \in (\kappa_1, \kappa_2)$, there exists a unique $\eta_\kappa < -1$ such that $\kappa^{-1} = H(\eta_\kappa)$ —see equation (4.28b) and figure 1(a). With η being known, one then finds the corresponding s_κ from (4.28a), and $\lambda_\kappa = -\kappa s_\kappa \eta_\kappa$ by (4.27). All together, they determine uniquely a fully supported equilibrium in the form (2.14).

4.2. Second critical value $\kappa = \kappa_2$

We now study what happens at the critical value $\kappa = \kappa_2$, with κ_2 defined in (4.31).

Consider the equilibria with support strictly contained in \mathbb{S}^d —see (2.16). We will continue using the notation $s = \|c_\rho\|$. Also recall the notation $\phi = \arccos(-\lambda/(\kappa\|c_\rho\|))$, so for strictly supported equilibria we have

$$\lambda = -\kappa s \cos \phi. \tag{4.32}$$

Then, we write system (2.17) as

$$\begin{aligned} 1 &= dw_d \left(\frac{\kappa s (m-1)}{m} \right)^{\frac{1}{m-1}} \int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta, \\ s &= dw_d \left(\frac{\kappa s (m-1)}{m} \right)^{\frac{1}{m-1}} \int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta, \end{aligned} \tag{4.33}$$

from which we can express s and κ in terms of ϕ as follows:

$$s = \frac{\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta}{\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta}, \tag{4.34a}$$

$$\kappa^{-1} = \frac{m-1}{m} (dw_d)^{m-1} \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta \right) \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^{m-2}. \tag{4.34b}$$

Lemma 4.3. *Let $1 < m < 2$ be fixed, and define the following function on $0 < \phi < \pi$:*

$$F(\phi) = \frac{m-1}{m} (dw_d)^{m-1} \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta \right) \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^{m-2}. \tag{4.35}$$

Then, F is an increasing function on $(0, \pi)$.

Proof. The proof is based on calculating F' and assessing its sign; see appendix C. □

Equation (4.34b) can be written as $F(\phi) = \kappa^{-1}$. First note that $F(\pi) = H(-1) = \kappa_2^{-1}$ (see (4.29) and (4.31)). Also, $F(0) = 0$ as we can write

$$F(\phi) = \text{const} \times \frac{\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta}{\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta} \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^{m-1},$$

where the quotient factor in the r.h.s is bounded in absolute value by 1, while the other factor is 0 at $\phi = 0$. Therefore, since F is strictly increasing, we infer that for any $\kappa \in (\kappa_2, \infty)$, there exists a unique $0 < \phi_\kappa < \pi$ that solves $F(\phi_\kappa) = \kappa^{-1}$ —see figure 1(b), and hence it solves (4.34b). Corresponding to ϕ_κ , we calculate s_κ from (4.34a), and then have $\lambda_\kappa = -\kappa s_\kappa \cos \phi_\kappa$ by (4.32). Together they determine a unique equilibrium in the form (2.16).

By combining these considerations with the finding from section 4.1, we arrive at the following result.

Proposition 4.1 (Critical κ and equilibria for $1 < m < 2$). *For any $1 < m < 2$ and $d \geq 1$, there exist two critical values of κ (κ_1 and κ_2 defined by (3.26) and (4.31), respectively) such that:*

- i) *At $\kappa = \kappa_1$, an equilibrium fully supported on \mathbb{S}^d bifurcates from the uniform distribution ρ_{uni} . For any $\kappa \in (\kappa_1, \kappa_2)$, there exists a unique equilibrium in the form (2.14), given by*

$$\rho_\kappa(x) = \left(\frac{m-1}{m} \right)^{\frac{1}{m-1}} (\lambda_\kappa + \kappa s_\kappa \cos \theta_x)^{\frac{1}{m-1}}, \quad \forall x \in \mathbb{S}^d, \tag{4.36}$$

where s_κ and λ_κ are uniquely determined by κ —see (4.27) and (4.28).

- ii) *At $\kappa = \kappa_2$, the fully supported equilibria from part i) turn into equilibria with support strictly contained in \mathbb{S}^d . For any $\kappa \in (\kappa_2, \infty)$, there exists a unique equilibrium in the form (2.16), given by*

$$\rho_\kappa(x) = \begin{cases} \left(\frac{m-1}{m} \right)^{\frac{1}{m-1}} (\lambda_\kappa + \kappa s_\kappa \cos \theta_x)^{\frac{1}{m-1}}, & \text{if } 0 \leq \theta_x \leq \arccos \left(-\frac{\lambda_\kappa}{\kappa s_\kappa} \right), \\ 0, & \text{otherwise,} \end{cases} \tag{4.37}$$

with s_κ and λ_κ uniquely determined by κ —see (4.32) and (4.34).

Remark 4.2. The second critical value κ_2 can be computed explicitly in terms of the Gamma function. Indeed, by direct calculations (see appendix D), we can write κ_2 as

$$\kappa_2 = \frac{m(1+d(m-1))}{(m-1)2^{1+(d-1)(m-1)}|\mathbb{S}^d|^{m-1}} \left(\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{m-1}+d)}{\Gamma(\frac{1}{m-1}+\frac{d}{2})\Gamma(\frac{d+1}{2})} \right)^{m-1}. \tag{4.38}$$

4.3. Energy minimizers

Next, we will show that the equilibria ρ_κ found in proposition 4.1 are more energetically favourable than the uniform distribution ρ_{uni} , for every $\kappa > \kappa_1$ fixed. We caution the reader that for a fixed κ , the energy of the uniform distribution $E[\rho_{\text{uni}}]$ depends on κ (see (2.6)), though this dependence is not explicit in our notations.

The formal result is given by the following theorem.

Theorem 4.1 (Global energy minimizers for $1 < m < 2$)

Let $1 < m < 2$ and $d \geq 1$. The global minimizer of (1.1) on $\mathcal{P}_{ac}(\mathbb{S}^d)$ is

- (1) the uniform distribution ρ_{uni} when $\kappa < \kappa_1$,
- (2) the equilibrium ρ_κ given by (4.36) (for $\kappa_1 < \kappa < \kappa_2$) or by (4.37) (for $\kappa > \kappa_2$).

Moreover, the energy difference $E[\rho_{uni}] - E[\rho_\kappa]$ increases with κ for $\kappa > \kappa_1$.

Proof. When $\kappa < \kappa_1$, the only equilibrium is the uniform distribution and hence, it is the global minimizer. For $\kappa > \kappa_1$, we have two equilibria (ρ_{uni} and ρ_κ) and we will compare directly their energies. We will consider separately the two ranges $\kappa_1 < \kappa < \kappa_2$ and $\kappa > \kappa_2$, with equilibria given by (4.36) and (4.37), respectively.

Case $\kappa_1 < \kappa < \kappa_2$. In this case the equilibria ρ_κ from (4.36) are fully supported in \mathbb{S}^d . We will express $E[\rho_{uni}] - E[\rho_\kappa]$ in terms of the variable η . By (2.6) we have

$$E[\rho_{uni}] - E[\rho_\kappa] = \frac{1}{m-1} \int_{\mathbb{S}^d} \rho_{uni}^m(x) dS(x) - \frac{1}{m-1} \int_{\mathbb{S}^d} \rho_\kappa^m(x) dS(x) + \frac{\kappa}{2} \|c_{\rho_\kappa}\|^2. \tag{4.39}$$

Since $\frac{1}{m-1} \int_{\mathbb{S}^d} \rho_{uni}^m(x) dS(x)$ is a constant, consider only

$$\frac{\kappa}{2} \|c_{\rho_\kappa}\|^2 - \frac{1}{m-1} \int_{\mathbb{S}^d} \rho_\kappa^m(x) dS(x).$$

Recall the notations $s_\kappa = \|c_{\rho_\kappa}\|$ and $\lambda_\kappa = -\kappa s_\kappa \eta_\kappa$. Using (4.28a), (4.28b) and (4.36), compute

$$\begin{aligned} \frac{\kappa}{2} \|c_{\rho_\kappa}\|^2 - \frac{1}{m-1} \int_{\mathbb{S}^d} \rho_\kappa^m(x) dS(x) &= \frac{m}{2(m-1)(d\omega_d)^{m-1}} \frac{\int_0^\pi (\cos\theta - \eta_\kappa)^{\frac{1}{m-1}} \sin^{d-1}\theta \cos\theta d\theta}{\left(\int_0^\pi (\cos\theta - \eta_\kappa)^{\frac{1}{m-1}} \sin^{d-1}\theta d\theta\right)^m} \\ &\quad - \frac{1}{(m-1)(d\omega_d)^{m-1}} \frac{\int_0^\pi (\cos\theta - \eta_\kappa)^{\frac{m}{m-1}} \sin^{d-1}\theta d\theta}{\left(\int_0^\pi (\cos\theta - \eta_\kappa)^{\frac{1}{m-1}} \sin^{d-1}\theta d\theta\right)^m} \\ &= g_1(\eta_\kappa) g_2(\eta_\kappa), \end{aligned} \tag{4.40}$$

where we denoted

$$\begin{aligned} g_1(\eta) &= m \int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}} \sin^{d-1}\theta \cos\theta d\theta - 2 \int_0^\pi (\cos\theta - \eta)^{\frac{m}{m-1}} \sin^{d-1}\theta d\theta, \\ g_2(\eta) &= \frac{1}{2(m-1)(d\omega_d)^{m-1} \left(\int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}} \sin^{d-1}\theta d\theta\right)^m}. \end{aligned}$$

Calculate

$$\begin{aligned} g_1'(\eta) &= -\frac{m}{m-1} \int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1}\theta \cos\theta d\theta + 2\frac{m}{m-1} \int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}} \sin^{d-1}\theta d\theta \\ &= -\frac{m}{m-1} \int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1}\theta ((\cos\theta - \eta) + \eta) d\theta + 2\frac{m}{m-1} \int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}} \sin^{d-1}\theta d\theta \\ &= -\frac{\eta m}{m-1} \int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1}\theta d\theta + \frac{m}{m-1} \int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1}\theta (\cos\theta - \eta) d\theta \\ &= -\frac{2\eta m}{m-1} \int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1}\theta d\theta + \frac{m}{m-1} \int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1}\theta \cos\theta d\theta \end{aligned} \tag{4.41}$$

and

$$g_2'(\eta) = \frac{m}{2(m-1)^2(d\omega_d)^{m-1}} \frac{\int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1}\theta d\theta}{\left(\int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}} \sin^{d-1}\theta d\theta\right)^{m+1}}. \tag{4.42}$$

Combining (4.41) and (4.42) we then find

$$\begin{aligned}
 (g_1(\eta)g_2(\eta))' &= g_1'(\eta)g_2(\eta) + g_1(\eta)g_2'(\eta) \\
 &= \frac{\frac{m}{m-1} \int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta}{2(m-1)(d\omega_d)^{m-1} \left(\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^m} \\
 &\quad \times \left(-2\eta + \frac{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta \cos \theta d\theta}{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta} \right. \\
 &\quad \left. + \frac{m \int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta - 2 \int_0^\pi (\cos \theta - \eta)^{\frac{m}{m-1}} \sin^{d-1} \theta d\theta}{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta} \right).
 \end{aligned}$$

Now use

$$\begin{aligned}
 \int_0^\pi (\cos \theta - \eta)^{\frac{m}{m-1}} \sin^{d-1} \theta d\theta &= \int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} (\cos \theta - \eta) \sin^{d-1} \theta d\theta \\
 &= \int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \cos \theta d\theta - \eta \int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta,
 \end{aligned}$$

to write

$$\begin{aligned}
 (g_1(\eta)g_2(\eta))' &= \frac{\frac{m}{m-1} \int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta}{2(m-1)(d\omega_d)^{m-1} \left(\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^m} \\
 &\quad \times \left(-2\eta + \frac{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta \cos \theta d\theta}{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta} + \frac{(m-2) \int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta}{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta} + 2\eta \right) \\
 &= \frac{\frac{m}{m-1} \int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta}{2(m-1)(d\omega_d)^{m-1} \left(\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^m} \\
 &\quad \times \left(\frac{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta \cos \theta d\theta}{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta} + \frac{(m-2) \int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta}{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta} \right). \tag{4.43}
 \end{aligned}$$

By a direct calculation, using (4.29), one can compute

$$\frac{H'(\eta)}{H(\eta)} = -\frac{1}{m-1} \left(\frac{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta \cos \theta d\theta}{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta} + (m-2) \frac{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta}{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta} \right). \tag{4.44}$$

Then, by combining (4.43) and (4.44), we write

$$(g_1(\eta)g_2(\eta))' = h(\eta) \left(-(m-1) \frac{H'(\eta)}{H(\eta)} \right), \tag{4.45}$$

where we used the notation

$$\begin{aligned}
 h(\eta) &:= \frac{\frac{m}{m-1} \int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta}{2(m-1)(d\omega_d)^{m-1} \left(\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^m} \times \frac{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta}{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta} \\
 &= \frac{m}{2(m-1)^2 (d\omega_d)^{m-1}} \frac{\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta}{\left(\int_0^\pi (\cos \theta - \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^m}.
 \end{aligned} \tag{4.46}$$

Note that $h(\eta) \geq 0$ —to show the positivity of the numerator one can use the integration by parts in (4.30). Also, $H(\eta) \geq 0$, and hence, $(g_1(\eta)g_2(\eta))'$ and $H'(\eta)$ have opposite signs, for all $m > 1$ and $-\infty < \eta < -1$. By lemma 4.1, $H(\eta)$ is strictly decreasing for $1 < m < 2$, and therefore $g_1(\eta)g_2(\eta)$ is strictly increasing in η . Finally, by (4.39) and (4.40), we conclude that for any $1 < m < 2$, the energy difference $E[\rho_{\text{uni}}] - E[\rho_\kappa]$ increases with η_κ (and hence with κ), as κ ranges in $\kappa_1 \leq \kappa \leq \kappa_2$.

Case $\kappa > \kappa_2$. The calculations are very similar to the fully supported case, and we will only summarize them here.

We estimate again $E[\rho_{\text{uni}}] - E[\rho_\kappa]$ using (4.39). In this case the equilibria ρ_κ from (4.37) are supported on a geodesic disk of radius ϕ_κ . Recall the notations $s_\kappa = \|\mathbf{c}_{\rho_\kappa}\|$ and $\lambda_\kappa = -\kappa s_\kappa \cos \phi_\kappa$. Using (4.34a), (4.34b) and (4.37), we compute (similar to (4.40)):

$$\begin{aligned} \frac{\kappa}{2} \|\mathbf{c}_{\rho_\kappa}\|^2 - \frac{1}{m-1} \int_{\mathbb{S}^d} \rho_\kappa^m(x) \, dS(x) &= \frac{m}{2(m-1)(d\omega_d)^{m-1}} \frac{\int_0^{\phi_\kappa} (\cos \theta - \cos \phi_\kappa)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta \, d\theta}{\left(\int_0^{\phi_\kappa} (\cos \theta - \cos \phi_\kappa)^{\frac{1}{m-1}} \sin^{d-1} \theta \, d\theta\right)^m} \\ &\quad - \frac{1}{(m-1)(d\omega_d)^{m-1}} \frac{\int_0^{\phi_\kappa} (\cos \theta - \cos \phi_\kappa)^{\frac{m}{m-1}} \sin^{d-1} \theta \, d\theta}{\left(\int_0^{\phi_\kappa} (\cos \theta - \cos \phi_\kappa)^{\frac{1}{m-1}} \sin^{d-1} \theta \, d\theta\right)^m} \\ &= \bar{g}_1(\phi_\kappa) \bar{g}_2(\phi_\kappa), \end{aligned} \tag{4.47}$$

where

$$\begin{aligned} \bar{g}_1(\phi) &= m \int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta \, d\theta - 2 \int_0^\phi (\cos \theta - \cos \phi)^{\frac{m}{m-1}} \sin^{d-1} \theta \, d\theta, \\ \bar{g}_2(\phi) &= \frac{1}{2(m-1)(d\omega_d)^{m-1} \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \, d\theta\right)^m}. \end{aligned} \tag{4.48}$$

In computing $(\bar{g}_1(\phi)\bar{g}_2(\phi))'$ we notice that the same calculations as in (4.41) and (4.42) appear, with η being replaced by $\cos \phi$ and an additional $-\sin \phi$ that comes from the derivative of $\cos \phi$. We find

$$\begin{aligned} (\bar{g}_1(\phi)\bar{g}_2(\phi))' &= \frac{-\sin \phi \frac{m}{m-1} \int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta \, d\theta}{2(m-1)(d\omega_d)^{m-1} \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \, d\theta\right)^m} \\ &\quad \times \left(\frac{\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta \cos \theta \, d\theta}{\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta \, d\theta} + \frac{(m-2) \int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta \, d\theta}{\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \, d\theta} \right) \\ &= \bar{h}(\phi) \left(-(m-1) \frac{F'(\phi)}{F(\phi)} \right), \end{aligned} \tag{4.49}$$

where we used (C.77) and the notation

$$\begin{aligned} \bar{h}(\phi) &:= \frac{\frac{m}{m-1} \int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta \, d\theta}{2(m-1)(d\omega_d)^{m-1} \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \, d\theta\right)^m} \times \frac{\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta \, d\theta}{\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta \, d\theta} \\ &= \frac{m}{2(m-1)^2 (d\omega_d)^{m-1}} \frac{\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta \, d\theta}{\left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \, d\theta\right)^m}. \end{aligned} \tag{4.50}$$

We note that $\bar{h}(\phi) \geq 0$ —see the integration by parts (C.78) by which one can show the positivity of the numerator. As $F(\phi) \geq 0$, we then find that $(\bar{g}_1(\phi)\bar{g}_2(\phi))'$ and $F'(\phi)$ have opposite signs, for all $m > 1$ and $0 < \phi < \pi$. By lemma 4.3, $F(\phi)$ is strictly increasing in ϕ for $1 < m < 2$, and therefore $\bar{g}_1(\eta)\bar{g}_2(\eta)$ is strictly decreasing in ϕ . By (4.39) and (4.47), we conclude that for any $1 < m < 2$, the energy difference $E[\rho_{\text{uni}}] - E[\rho_\kappa]$ decreases with ϕ_κ . As κ takes values from κ_2 to ∞ , ϕ_κ ranges from π to 0 (hence ϕ_κ decreases with κ). We conclude that $E[\rho_{\text{uni}}] - E[\rho_\kappa]$ increases with κ , for $\kappa > \kappa_2$.

Conclusion. To conclude, we have shown that the energy difference $E[\rho_{\text{uni}}] - E[\rho_\kappa]$ increases with κ , as κ ranges in $\kappa_1 \leq \kappa \leq \kappa_2$ (where ρ_κ has full support in \mathbb{S}^d), as well as in $\kappa > \kappa_2$ (where ρ_κ has strict support in \mathbb{S}^d). Since at $\kappa = \kappa_1$, ρ_κ coincides with ρ_{uni} , the energy difference is zero there. We conclude that

$$E[\rho_{\text{uni}}] > E[\rho_\kappa], \quad \text{for all } \kappa > \kappa_1.$$

□

Remark 4.3. We note that calculations in the proof of theorem 4.1 which lead to (4.45) (for equilibria of full support) and (4.49) (for equilibria of strict support), apply to any $m > 1$. Hence, for easier reference, we summarize the conclusions inferred from (4.45) and (4.49): a) $(g_1(\eta)g_2(\eta))'$ and $H'(\eta)$ have opposite signs, for all $m > 1$ and $-\infty < \eta < -1$, and b) $(\bar{g}_1(\phi)\bar{g}_2(\phi))'$ and $F'(\phi)$ have opposite signs, for all $m > 1$ and $0 < \phi < \pi$. These results will be used in sections 5 and 6 for the case $m \geq 2$.

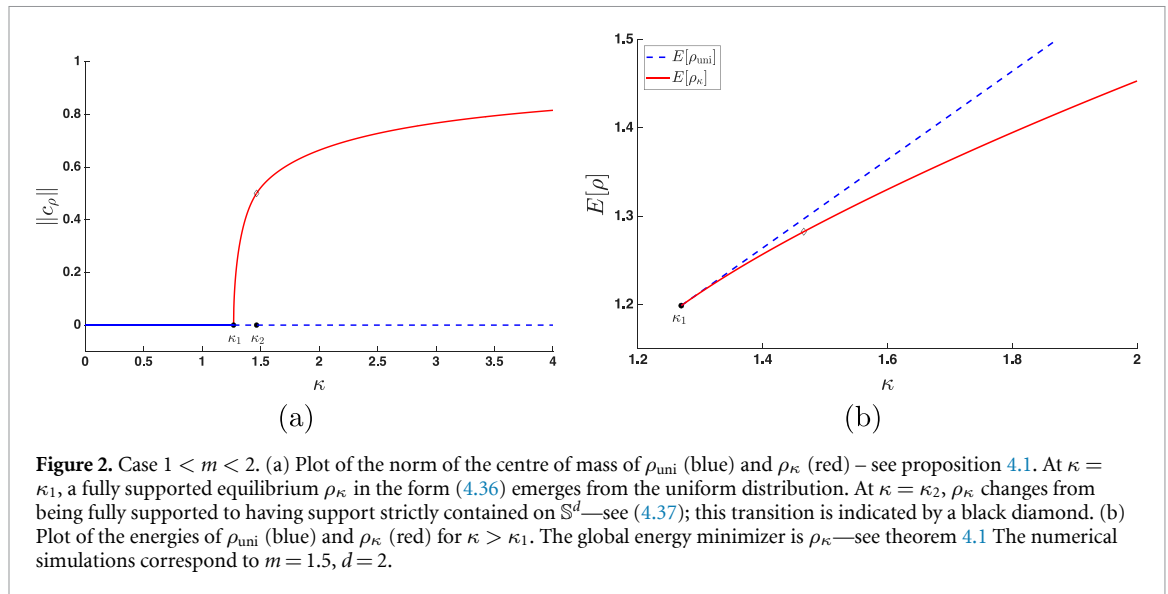


Figure 2. Case $1 < m < 2$. (a) Plot of the norm of the centre of mass of ρ_{uni} (blue) and ρ_κ (red) – see proposition 4.1. At $\kappa = \kappa_1$, a fully supported equilibrium ρ_κ in the form (4.36) emerges from the uniform distribution. At $\kappa = \kappa_2$, ρ_κ changes from being fully supported to having support strictly contained on \mathbb{S}^d —see (4.37); this transition is indicated by a black diamond. (b) Plot of the energies of ρ_{uni} (blue) and ρ_κ (red) for $\kappa > \kappa_1$. The global energy minimizer is ρ_κ —see theorem 4.1 The numerical simulations correspond to $m = 1.5$, $d = 2$.

Remark 4.4 (Linear diffusion limit). In the linear diffusion case ($m = 1$), there exists only one critical value κ_1 , and no second critical value κ_2 . When $\kappa \leq \kappa_1$, the uniform distribution is the unique critical point and also the global minimizer. For $\kappa > \kappa_1$, there are exactly two types of critical points: the uniform distribution and a fully supported non-uniform equilibrium ρ_κ , where ρ_κ is unique up to rotations; moreover, ρ_κ is the global minimizer in this case. These results have been established in [17, 26].

By taking $m \searrow 1$ in our results, we can verify that the limit is indeed consistent with the linear diffusion case. In particular, by letting $m \searrow 1$ in the expression for κ_2 given in (4.38) (where κ_2 is written explicitly in terms of m), we find

$$\lim_{m \searrow 1} \kappa_2 = \infty,$$

showing that no second critical κ exists for linear diffusion. This limit can be justified as follows.

By the log-convexity of the Gamma function, we have

$$\log \Gamma\left(\frac{1}{2}\right) + \log \Gamma\left(\frac{1}{m-1} + d\right) > \log \Gamma\left(\frac{1}{m-1} + \frac{d}{2}\right) + \log \Gamma\left(\frac{d+1}{2}\right),$$

for any $d \geq 1$ and $m > 1$, and it implies $\left(\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{m-1}+d)}{\Gamma(\frac{1}{m-1}+\frac{d}{2})\Gamma(\frac{d+1}{2})}\right)^{m-1} \geq 1$. By (4.38), we then find

$$\lim_{m \searrow 1} \kappa_2 \geq \lim_{m \searrow 1} \frac{m(1+d(m-1))}{(m-1)2^{1+(d-1)(m-1)}|\mathbb{S}^d|^{m-1}} = \infty.$$

4.4. Numerical illustrations

We illustrate proposition 4.1 with some numerical results. In the numerics we have used $m = 1.5$ and $d = 2$. Figure 2(a) illustrates the bifurcation at κ_1 and also indicates the second critical value κ_2 . For this choice of m and d we have $\kappa_1 \approx 1.2694$ and $\kappa_2 \approx 1.4658$. In blue we plot the uniform distribution, which is stable for $\kappa < \kappa_1$, and unstable otherwise. In red we plot $s_\kappa = \|\rho_\kappa\|$ for the equilibria (4.36) ($\kappa_1 < \kappa < \kappa_2$) and (4.37) ($\kappa > \kappa_2$). The transition at κ_2 is indicated by a black diamond. Also, theorem 4.1 is illustrated in figure 2(b); note how the energy difference $E[\rho_{\text{uni}}] - E[\rho_\kappa]$ increases with κ .

In figure 3(a) we show the size of the support ϕ_κ of equilibria (4.37), and in figure 3(b) we plot the equilibria at several values of κ . Note that at $\kappa = \kappa_1$, the equilibria is simply the uniform distribution (i.e. constant on \mathbb{S}^d). As κ increases, ρ_κ becomes more and more concentrated at $\theta = 0$. However, as seen in plot (a), the size of the support ϕ_κ has a rather slow convergence to 0.

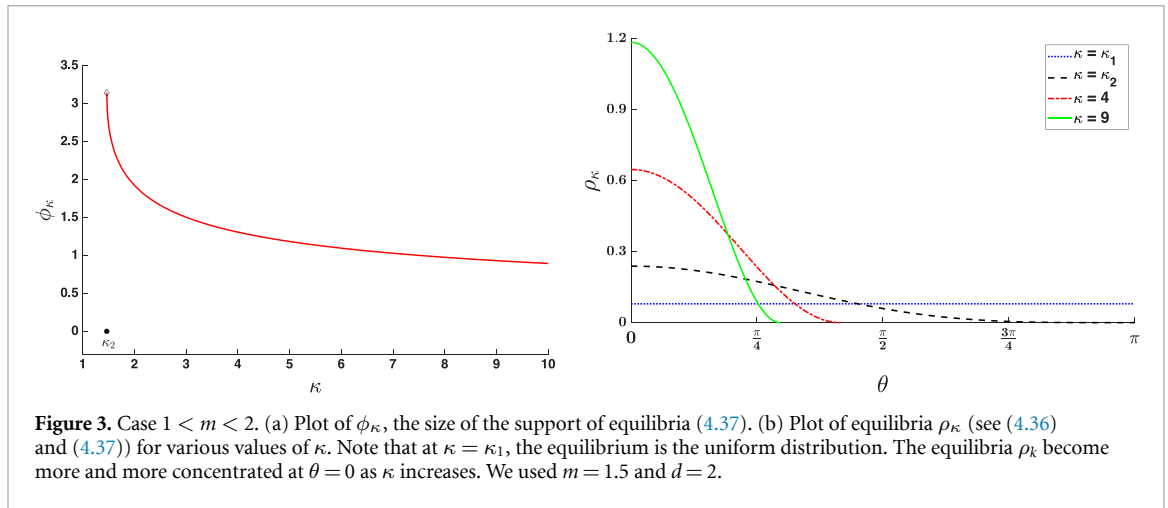


Figure 3. Case $1 < m < 2$. (a) Plot of ϕ_κ , the size of the support of equilibria (4.37). (b) Plot of equilibria ρ_κ (see (4.36) and (4.37)) for various values of κ . Note that at $\kappa = \kappa_1$, the equilibrium is the uniform distribution. The equilibria ρ_k become more and more concentrated at $\theta = 0$ as κ increases. We used $m = 1.5$ and $d = 2$.

5. Case $m = 2$

We consider separately the case $m = 2$, which turns out to be degenerate in a sense that we will explain.

5.1. Equilibria and critical κ

Consider first equilibria that are fully supported on \mathbb{S}^d —see case a) in section 2, in particular equations (2.14) and (2.15). Use again the notation $s = \|c_\rho\|$. Equations (2.15) become

$$1 = \frac{dw_d}{2} \int_0^\pi (\lambda + \kappa s \cos \theta) \sin^{d-1} \theta d\theta,$$

$$s = \frac{dw_d}{2} \int_0^\pi (\lambda + \kappa s \cos \theta) \sin^{d-1} \theta \cos \theta d\theta.$$

However, some of the integrals above are zero, so the system simplifies to

$$1 = \lambda \left(\frac{dw_d}{2} \int_0^\pi \sin^{d-1} \theta d\theta \right), \tag{5.51}$$

$$s = \kappa s \left(\frac{dw_d}{2} \int_0^\pi \sin^{d-1} \theta \cos^2 \theta d\theta \right).$$

From the first equation in (5.51), one can determine λ :

$$\lambda = \left(\frac{dw_d}{2} \int_0^\pi \sin^{d-1} \theta d\theta \right)^{-1} = \frac{2}{|\mathbb{S}^d|}. \tag{5.52}$$

Using

$$\int_0^\pi \cos^2 \theta \sin^{d-1} \theta d\theta = \frac{1}{d+1} \int_0^\pi \sin^{d-1} \theta d\theta, \tag{5.53}$$

the second equation in (5.51) can also be written as

$$s = \kappa s \frac{|\mathbb{S}^d|}{2(d+1)}. \tag{5.54}$$

For $m = 2$, the uniform distribution loses stability (see (3.26)) at

$$\kappa_1 = \frac{2(d+1)}{|\mathbb{S}^d|}. \tag{5.55}$$

Then, the second equation in (5.51) (or equivalently, (5.54)) is satisfied trivially for $\kappa = \kappa_1$, and it has only the solution $s = 0$ (corresponding to the uniform distribution) for $\kappa \neq \kappa_1$.

Recall that for fully supported equilibria, $\lambda \geq \kappa s$. Hence, for $\kappa = \kappa_1$, using (5.52) we get that s can be any value that satisfies:

$$0 \leq s \leq \frac{\lambda}{\kappa_1} = \frac{1}{d+1}.$$

Therefore, for $\kappa = \kappa_1$ there is a continuum family of fully supported equilibria in the form (2.14), where λ is given by (5.52) and s can take any value in the range $0 \leq s \leq \frac{1}{d+1}$.

Look now into equilibria with support strictly contained in \mathbb{S}^d —see case b) in section 2, equations (2.16) and (2.17). In this case, $-\kappa s < \lambda < \kappa s$. Equations (2.17) become

$$\begin{aligned} 1 &= \frac{dw_d}{2} \int_0^\phi (\lambda + \kappa s \cos \theta) \sin^{d-1} \theta d\theta, \\ s &= \frac{dw_d}{2} \int_0^\phi (\lambda + \kappa s \cos \theta) \sin^{d-1} \theta \cos \theta d\theta. \end{aligned} \tag{5.56}$$

By substituting $\lambda = -\kappa s \cos \phi$ in (5.56), we get

$$\begin{aligned} 1 &= \frac{dw_d}{2} \kappa s \left(-\cos \phi \int_0^\phi \sin^{d-1} \theta d\theta + \int_0^\phi \cos \theta \sin^{d-1} \theta d\theta \right), \\ s &= \frac{dw_d}{2} \kappa s \left(-\cos \phi \int_0^\phi \sin^{d-1} \theta \cos \theta d\theta + \int_0^\phi \cos^2 \theta \sin^{d-1} \theta d\theta \right). \end{aligned} \tag{5.57}$$

By noting that

$$\int_0^\phi \cos^2 \theta \sin^{d-1} \theta d\theta - \cos \phi \int_0^\phi \sin^{d-1} \theta \cos \theta d\theta = \frac{1}{d} \int_0^\phi \sin^{d+1} \theta d\theta,$$

and cancelling s , the second equation in (5.57) reduces to

$$1 = \frac{w_d \kappa}{2} \int_0^\phi \sin^{d+1} \theta d\theta,$$

or equivalently,

$$\kappa^{-1} = \frac{w_d}{2} \int_0^\phi \sin^{d+1} \theta d\theta. \tag{5.58}$$

This is in fact equation (4.34b) for $m = 2$. Note that

$$\frac{w_d}{2} \int_0^\pi \sin^{d+1} \theta d\theta = \frac{w_d}{2} \frac{d}{d+1} \int_0^\pi \sin^{d-1} \theta d\theta = \kappa_1^{-1}.$$

Then, since ϕ ranges from 0 to π , if $\kappa^{-1} > \kappa_1^{-1}$ (or $\kappa < \kappa_1$), there is no ϕ that satisfies (5.58). On the other hand, for any $\kappa > \kappa_1$, there is a unique $\phi \in [0, \pi]$; this follows from the intermediate value property and the property of increasing functions.

With ϕ determined uniquely in terms of κ , we can find s from the first equation in (5.57):

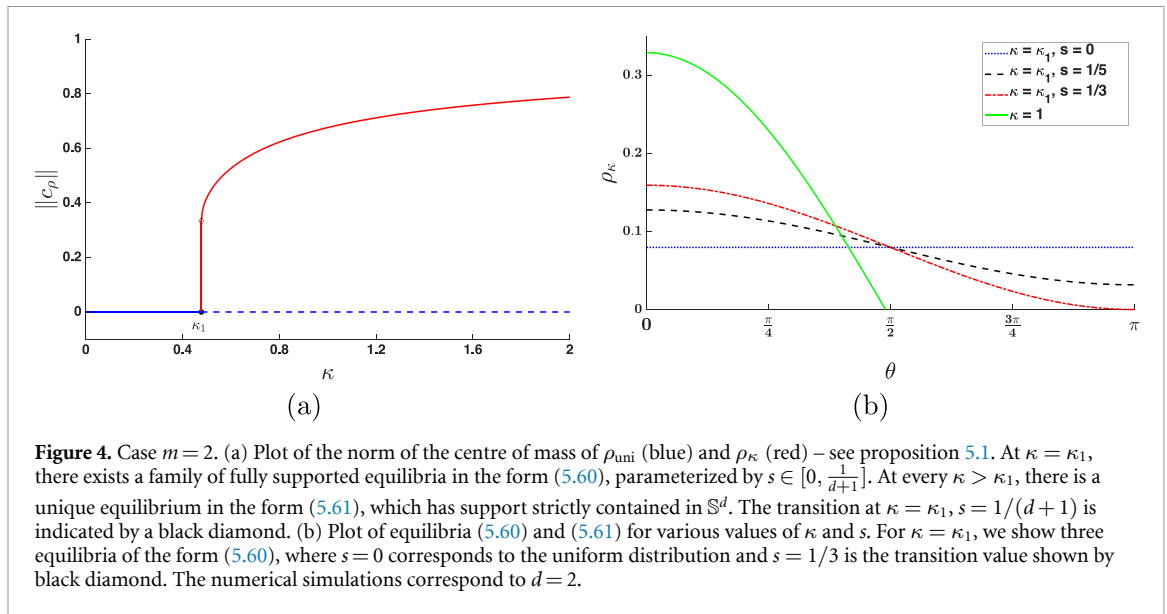
$$s = \frac{2}{dw_d \kappa} \left(-\cos \phi \int_0^\phi \sin^{d-1} \theta d\theta + \int_0^\phi \cos \theta \sin^{d-1} \theta d\theta \right)^{-1}, \tag{5.59}$$

and then set $\lambda = -\kappa s \cos \phi$.

Remark 5.1. For $m = 2$, the function $H(\eta)$ from (4.29) is constant in η , as the second term in the r.h.s. is 1, and the first term does not depend on η , since

$$\int_0^\pi (\cos \theta - \eta) \sin^{d-1} \theta \cos \theta d\theta = \int_0^\pi \cos^2 \theta \sin^{d-1} \theta d\theta = \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{d}{2})}{\Gamma(\frac{d+3}{2})}.$$

Hence, $\kappa_1 = \kappa_2$ in this case; note that one can also check that $\kappa_1 = \kappa_2$ by substituting $m = 2$ into (3.26) and (4.38).



The considerations above can be collected in the following proposition.

Proposition 5.1 (Critical κ and equilibria for $m = 2$). For $m = 2$ and $d \geq 1$, there exists only one critical value κ_1 (see (3.26) for $m = 2$) such that:

- i) The uniform distribution ρ_{uni} is the only equilibrium for $\kappa < \kappa_1$. At $\kappa = \kappa_1$, there exists a family of fully supported equilibria in the form (2.14) given by

$$\rho_{\kappa_1}(x; s) = \frac{1}{2} (\lambda + \kappa_1 s \cos \theta_x), \quad \forall x \in \mathbb{S}^d, \tag{5.60}$$

where λ is given by (5.52) and s can take any value in the range $0 \leq s \leq \frac{1}{d+1}$.

- ii) For any $\kappa > \kappa_1$ there exists a unique equilibrium with support strictly contained in \mathbb{S}^d . This equilibrium is given by (see (2.16)):

$$\rho_{\kappa}(x) = \begin{cases} \frac{1}{2} (\lambda_{\kappa} + \kappa s_{\kappa} \cos \theta_x), & \text{if } 0 \leq \theta_x \leq \arccos\left(-\frac{\lambda_{\kappa}}{\kappa s_{\kappa}}\right), \\ 0, & \text{otherwise,} \end{cases} \tag{5.61}$$

with s_{κ} and λ_{κ} uniquely determined by κ —see (5.58) and (5.59).

Remark 5.2. The bifurcation for $m = 2$ is degenerate in the sense that the two critical values κ_1 and κ_2 identified for $1 < m < 2$ (section 4), are in fact the same here. We also point out that in the limit $\kappa \searrow \kappa_1$, $\rho_{\kappa}(\cdot)$ from (5.61) does not approach the uniform distribution, but it approaches instead $\rho_{\kappa_1}(\cdot; \frac{1}{d+1})$ from (5.60) – see also figure 4.

We present some numerical simulations in figure 4. Figure 4(a) illustrates the degenerate bifurcation that occurs at $\kappa_1 \approx 0.4775$ (in the numerics we used $d = 2$). The vertical segment at $\kappa = \kappa_1$, with $1 \leq s \leq 1/3$, corresponds to the family of equilibria (5.60). At $\kappa = \kappa_1$ and $s = 1/(d+1)$, there is a transition (indicated by a black diamond) to equilibria of the form (5.61). Several equilibria in the form (5.60) and (5.61) are shown in figure 4(b).

5.2. Energy minimizers

The global energy minimizers for the case $m = 2$ are presented in the following theorem.

Theorem 5.1 (Global energy minimizers for $m = 2$). Let $m = 2$ and $d \geq 1$. The global minimizer of (1.1) on $\mathcal{P}_{ac}(\mathbb{S}^d)$ is

- (1) the uniform distribution ρ_{uni} when $\kappa < \kappa_1$,
- (2) any equilibrium of the form (5.60) with $s \in [0, \frac{1}{d+1}]$, when $\kappa = \kappa_1$,

(3) the equilibrium ρ_κ given by (5.61) when $\kappa > \kappa_1$.

Moreover, the energy difference $E[\rho_{\text{uni}}] - E[\rho_\kappa]$ increases with κ for $\kappa > \kappa_1$.

Proof. For $\kappa < \kappa_1$, ρ_{uni} is the only equilibrium, and hence the global minimizer. We will next discuss the cases $\kappa = \kappa_1$ and $\kappa > \kappa_1$.

Case $\kappa = \kappa_1$. Consider a generic equilibrium in the form (5.60), corresponding to $\kappa = \kappa_1$ and some $0 \leq s \leq \frac{1}{d+1}$ (see the vertical segment in figure 4(a)), and compute its energy.

The entropy can be computed as follows:

$$\begin{aligned} \int_{\mathbb{S}^d} \rho_{\kappa_1}(x; s)^2 dS(x) &= \frac{dw_d}{4} \int_0^\pi (\lambda + \kappa_1 s \cos \theta)^2 \sin^{d-1} \theta d\theta \\ &= \frac{dw_d}{4} \int_0^\pi \left(\lambda^2 \sin^{d-1} \theta + \kappa_1^2 s^2 \cos^2 \theta \sin^{d-1} \theta \right) d\theta \\ &= \frac{dw_d}{4} \left(\frac{\kappa_1^2}{(d+1)^2} + \kappa_1^2 s^2 \frac{1}{d+1} \right) \int_0^\pi \sin^{d-1} \theta d\theta, \end{aligned}$$

where for the last equality we used (5.53) and that $\lambda = \frac{\kappa_1}{d+1}$ (see (5.52) and (5.55)). Also, using the expression of κ_1 from (5.55), and the fact that $|\mathbb{S}^d| = dw_d \int_0^\pi \sin^{d-1} \theta d\theta$, we can write the above as

$$\int_{\mathbb{S}^d} \rho_{\kappa_1}(x; s)^2 dS(x) = \frac{\kappa_1}{2(d+1)} + \frac{\kappa_1 s^2}{2}.$$

The interaction energy (see (2.6)) is given by

$$-\frac{\kappa_1 s^2}{2} + \frac{\kappa_1}{2}.$$

By combining the two components, we find the total energy

$$E[\rho_{\kappa_1}(\cdot, s)] = \frac{\kappa_1}{2(d+1)} + \frac{\kappa_1}{2} = \frac{\kappa_1(d+2)}{2(d+1)}.$$

Note that the energy does not depend on s . It means that when $\kappa = \kappa_1$, each equilibrium in the form (5.60) (with $0 \leq s \leq \frac{1}{d+1}$) is a global energy minimizer.

Case $\kappa > \kappa_1$. One can show that $E[\rho_{\text{uni}}] > E[\rho_\kappa]$ in the same way as this was shown for theorem (4.1) (see remark 4.3) in the case $\kappa > \kappa_2$; recall that $m = 2$ is a degenerate case where $\kappa_1 = \kappa_2$. Indeed, following the same calculations as in the proof of theorem (4.1) (case $\kappa > \kappa_2$) using $m = 2$ —see (4.47) and the calculations that follow, the desired conclusion results from the monotonicity of the function $F(\phi)$.

Though not covered by lemma 4.3, the monotonicity of the function $F(\phi)$ for $m = 2$ is immediate. Indeed, for $m = 2$ (4.35) reduces to

$$F(\phi) = \frac{1}{2} dw_d \int_0^\phi (\cos \theta - \cos \phi) \sin^{d-1} \theta \cos \theta d\theta,$$

which further simplifies to (see calculation leading to (5.58)):

$$F(\phi) = \frac{w_d}{2} \int_0^\phi \sin^{d+1} \theta d\theta.$$

To conclude, $F(\phi)$ is strictly increasing with ϕ , and as in the proof of theorem (4.1) we infer that the energy difference $E[\rho_{\text{uni}}] - E[\rho_\kappa]$ decreases with ϕ_κ . As κ increases from κ_1 to ∞ , ϕ_κ decreases from π to 0. Hence, ϕ_κ decreases with κ , and we conclude that $E[\rho_{\text{uni}}] - E[\rho_\kappa]$ increases with κ , for $\kappa > \kappa_1$. □

6. Case $m > 2$ and $d = 2$

In this section we consider general exponent $m > 2$, but restrict to the case $d = 2$ only. From the point of view of applications to rod-like polymer orientation [17], we argue that this is the most relevant case to consider. The limitation to $d = 2$ is due to the difficulty in assessing the monotonicity of the function $F(\phi)$ (see (4.35)) for general dimension d . As shown below, $F(\phi)$ is no longer monotonic for $d = 2$, and numerical checks indicate that it is no longer monotonic for general d as well.

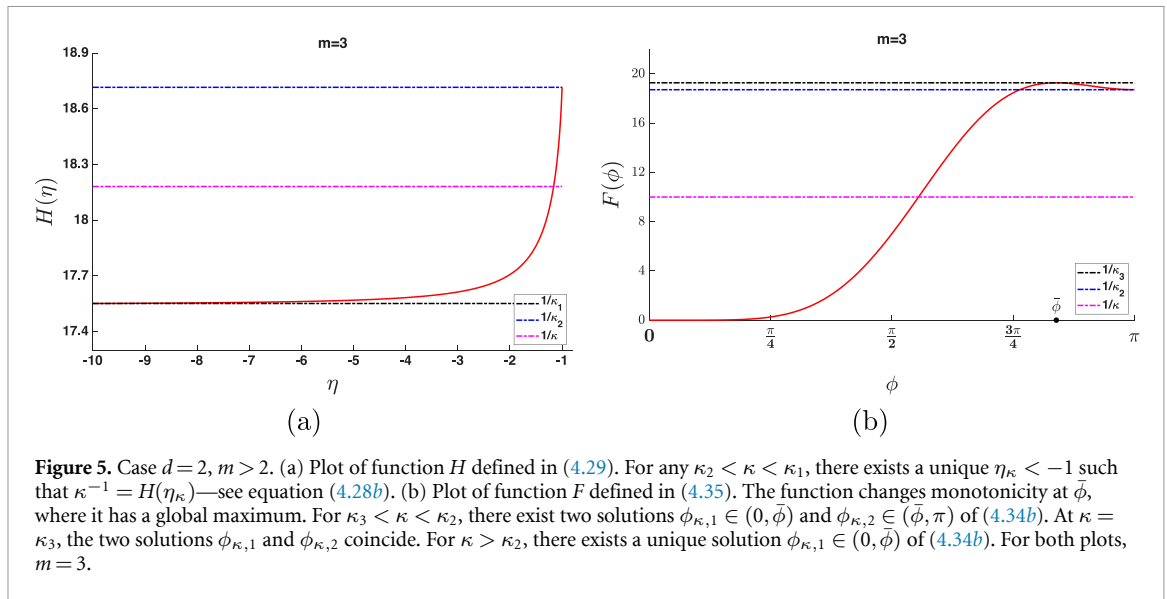


Figure 5. Case $d = 2, m > 2$. (a) Plot of function H defined in (4.29). For any $\kappa_2 < \kappa < \kappa_1$, there exists a unique $\eta_\kappa < -1$ such that $\kappa^{-1} = H(\eta_\kappa)$ —see equation (4.28b). (b) Plot of function F defined in (4.35). The function changes monotonicity at $\bar{\phi}$, where it has a global maximum. For $\kappa_3 < \kappa < \kappa_2$, there exist two solutions $\phi_{\kappa,1} \in (0, \bar{\phi})$ and $\phi_{\kappa,2} \in (\bar{\phi}, \pi)$ of (4.34b). At $\kappa = \kappa_3$, the two solutions $\phi_{\kappa,1}$ and $\phi_{\kappa,2}$ coincide. For $\kappa > \kappa_2$, there exists a unique solution $\phi_{\kappa,1} \in (0, \bar{\phi})$ of (4.34b). For both plots, $m = 3$.

6.1. Phase transitions and existence of equilibria

Start with equilibria of full support. Following section 4.1 (see (4.28) and (4.29)), the problem reduces to solving $H(\eta) = \kappa^{-1}$. Note that when $m > 2$, the function $H(\eta)$ is increasing (see lemma 4.1 and figure 5(a)). Hence, by using the same notations for κ_1 and κ_2 (see lemma 4.2 and equation (4.31)), in this case we have $\kappa_2 < \kappa_1$. For any $\kappa_2 < \kappa < \kappa_1$, there exists a unique solution of $H(\eta) = \kappa^{-1}$ in $(-\infty, -1)$ (i.e. of (4.28b)) – see figure 5(a) for an illustration. In particular, there exists a unique equilibrium with full support in S^d in the form (2.14); in proposition 6.1 below (see (6.64)) this equilibrium is referred to as $\rho_{\kappa,2}$. Therefore, at $\kappa = \kappa_1$ we find again a bifurcation from the uniform density ρ_{uni} , but in this case a fully supported equilibrium in the form (2.14) emerges as κ decreases through κ_1 . This is one of the major differences from the case $1 < m < 2$.

At $\kappa = \kappa_2$, fully supported equilibria transition to equilibria of strict support. Finding the support of the latter equilibria reduces to solving $F(\phi) = \kappa^{-1}$ (see (4.34) and (4.35)). And here arises the second fundamental difference from the case $1 < m < 2$: $F(\phi)$ is no longer monotone when $m > 2$. In $d = 2$ the monotonicity of F can be established easily, by a direct calculation, as given in the following lemma.

Lemma 6.1. *Let $m > 2, d = 2$, and the function $F(\phi)$ for $0 < \phi < \pi$ given by (4.35). Then, there is $\bar{\phi} \in (0, \pi)$ such that F is increasing on $(0, \bar{\phi})$ and decreasing on $(\bar{\phi}, \pi)$.*

Proof. When $d = 2$, the integrals in (4.35) can be computed explicitly by substitution. We find

$$F(\phi) = \frac{(m-1)^2}{m(2m-1)} \left(\frac{m-1}{m}\right)^{m-1} (2\pi)^{m-1} (1 - \cos\phi)^m \left(\frac{1}{m-1} + 1 + \cos\phi\right).$$

Denote the constant

$$C := \frac{(m-1)^2}{m(2m-1)} \left(\frac{m-1}{m}\right)^{m-1} (2\pi)^{m-1}.$$

Then, the derivative of F is computed as

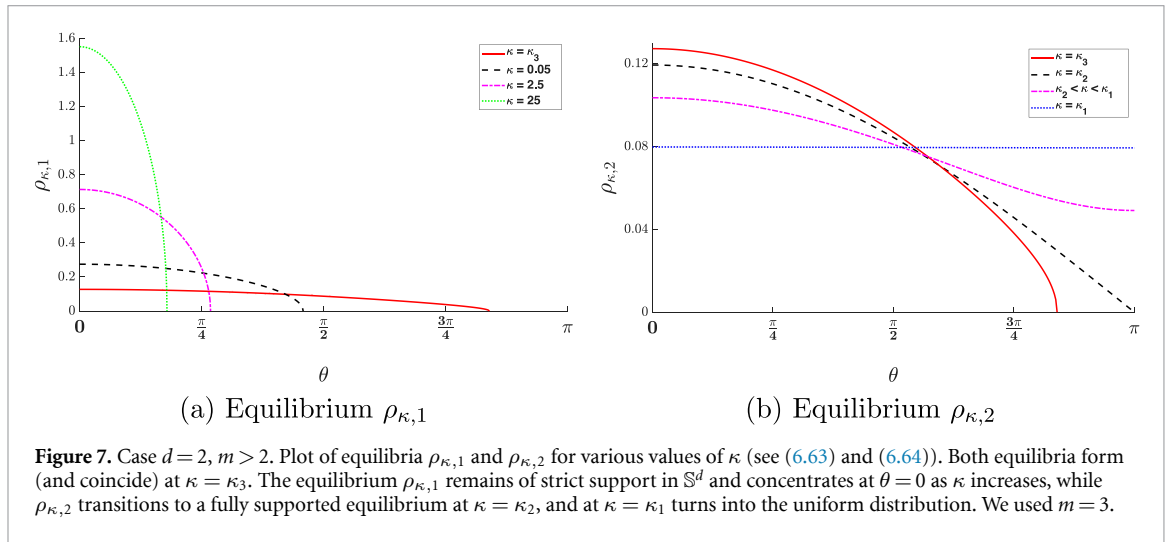
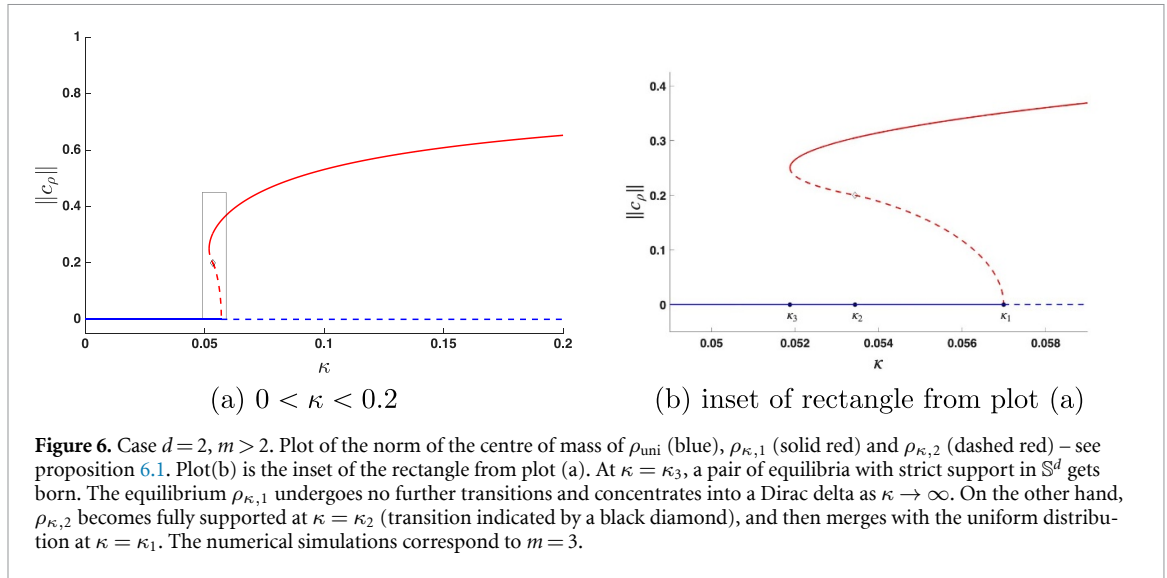
$$F'(\phi) = C(m+1) \sin\phi (1 - \cos\phi)^{m-1} \left(\frac{m^2 - m + 1}{m^2 - 1} + \cos\phi\right).$$

Hence, the sign of F' depends on the behaviour of the function

$$g(m) = \frac{m^2 - m + 1}{m^2 - 1}.$$

For $1 < m < 2$, we have $g(m) > 1$, so $F'(\phi) > 0$ (cf lemma 4.3). For $m > 2$, we have $g(m) < 1$, which makes F' change sign from positive to negative at

$$\bar{\phi} := \pi - \arccos(g(m)).$$



We conclude that F is increasing on $(0, \bar{\phi})$, decreasing on $(\bar{\phi}, \pi)$, and at $\phi = \bar{\phi}$ it has a global maximum – see figure 5(b). □

Remark 6.1. Define a third critical value of κ by

$$\kappa_3 := (F(\bar{\phi}))^{-1}. \tag{6.62}$$

Then, at $\kappa = \kappa_3$, the equation $F(\phi) = \kappa^{-1}$ has one solution given by $\bar{\phi}$ —see the black dash-dotted line in figure 5(b). Also, as $\kappa_2 = 1/F(\pi) > \kappa_3$, for any $\kappa_3 < \kappa < \kappa_2$, the equation $F(\phi) = \kappa^{-1}$ has two solutions: $\phi_{\kappa,1} \in (0, \bar{\phi})$ and $\phi_{\kappa,2} \in (\bar{\phi}, \pi)$. Finally, for $\kappa > \kappa_2$, there exists only one solution $\phi_{\kappa,1} \in (0, \bar{\phi})$ of $F(\phi) = \kappa^{-1}$ —see the magenta dash-dotted line in figure 5(b). As κ increases to ∞ , $\phi_{\kappa,1}$ decreases to 0.

The considerations above can be put together in the following proposition. We also refer to figures 6 and 7 for some numerical illustrations which support the theoretical findings. For numerical results we used $m = 3, d = 2$, for which $\kappa_1 \approx 0.0569, \kappa_2 \approx 0.0534, \kappa_3 \approx 0.0518$.

Proposition 6.1 (Critical κ and equilibria for $m > 2, d = 2$). For $d = 2$ and any $m > 2$, there exist three critical values of κ ($\kappa_1 > \kappa_2 > \kappa_3$ defined by (3.26), (4.31) and (6.62), respectively) such that:

- i) At $\kappa = \kappa_3$, a pair of equilibria with support strictly contained in \mathbb{S}^d gets born. For any $\kappa \in (\kappa_3, \kappa_2)$, there exist two solutions $0 < \phi_{\kappa,1} < \phi_{\kappa,2} < \pi$ of (4.34b); at $\kappa = \kappa_3$, the two solutions $\phi_{\kappa,1}$ and $\phi_{\kappa,2}$ coincide, with

common value $\bar{\phi}$. Correspondingly, there exist two equilibria $\rho_{\kappa,1}$ and $\rho_{\kappa,2}$ of the form (see (2.16)):

$$\rho_{\kappa,i}(x) = \begin{cases} \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} (\lambda_{\kappa,i} + \kappa s_{\kappa,i} \cos \theta_x)^{\frac{1}{m-1}}, & \text{if } 0 \leq \theta_x \leq \phi_{\kappa,i}, \\ 0, & \text{otherwise,} \end{cases} \tag{6.63}$$

with $s_{\kappa,i}$ and $\lambda_{\kappa,i}$ uniquely determined by $\phi_{\kappa,i}$ ($i = 1, 2$) – see (4.32) and (4.34a).

ii) At $\kappa = \kappa_2$, the equilibrium $\rho_{\kappa,2}$ from part i) becomes fully supported on \mathbb{S}^d (i.e. $\phi_{\kappa_2,2} = \pi$). For $\kappa_2 < \kappa < \kappa_1$, $\rho_{\kappa,2}$ has the form (see (2.14)):

$$\rho_{\kappa,2}(x) = \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} (\lambda_{\kappa,2} + \kappa s_{\kappa,2} \cos \theta_x)^{\frac{1}{m-1}}, \quad \forall x \in \mathbb{S}^d, \tag{6.64}$$

where $s_{\kappa,2}$ and $\lambda_{\kappa,2}$ are uniquely determined by κ —see (4.27) and (4.28a). On the other hand, $\rho_{\kappa,1}$ undergoes no transition at $\kappa = \kappa_2$, and retains the form (6.63).

iii) At $\kappa = \kappa_1$, the fully supported equilibrium $\rho_{\kappa,2}$ from part ii) merges with the uniform distribution ρ_{uni} , and vanishes. And again, $\rho_{\kappa,1}$ undergoes no transition at $\kappa = \kappa_1$; as κ increases to infinity, $\phi_{\kappa,1}$ approaches 0 and $\rho_{\kappa,1}$ concentrates into a Dirac delta.

6.2. Energy minimizers

It remains to discuss the energy minimizers. As noted in remark 4.3, the calculations in the proof of theorem 4.1 work for $m > 2$ as well, with ρ_{κ} replaced by either $\rho_{\kappa,1}$ or $\rho_{\kappa,2}$, depending on the context. In the subsequent discussion, we will be referring back to notations and calculations used in the proof of theorem 4.1.

For equilibria of full support $\rho_{\kappa,2}$ from (6.64) ($\kappa_2 < \kappa < \kappa_1$), the energy difference $E[\rho_{\text{uni}}] - E[\rho_{\kappa,2}]$ depends on the behaviour of the function $g_1(\eta)g_2(\eta)$. By remark 4.3, since $H(\eta)$ is increasing, $g_1(\eta)g_2(\eta)$ is decreasing with η . Hence, as κ increases from κ_2 to κ_1 (or equivalently, η decreases from -1 to $-\infty$), $E[\rho_{\text{uni}}] - E[\rho_{\kappa,2}]$ increases from a negative value to zero.

For equilibria of strict support (6.63), the energy difference $E[\rho_{\text{uni}}] - E[\rho_{\kappa,i}]$ ($i = 1, 2$) depends on the behaviour of the function $\bar{g}_1(\phi)\bar{g}_2(\phi)$. In turn, the latter depends on the monotonicity of $F(\phi)$, as given by lemma 6.1. We have, for each equilibria:

- a) $\rho_{\kappa,2}$ (ϕ increases from $\bar{\phi}$ to π). Since $F(\phi)$ is decreasing, $\bar{g}_1(\phi)\bar{g}_2(\phi)$ increases on $(\bar{\phi}, \pi)$ (cf remark 4.3). Hence, $E[\rho_{\text{uni}}] - E[\rho_{\kappa,2}]$ increases as ϕ increases from $\bar{\phi}$ to π (or equivalently, κ increases from κ_3 to κ_2). Combined with the previous observation (for the range $\kappa_2 < \kappa < \kappa_1$), we find that as κ increases from κ_3 to κ_1 , $E[\rho_{\text{uni}}] - E[\rho_{\kappa,2}]$ increases from a negative value to zero.
- b) $\rho_{\kappa,1}$ (ϕ decreases from $\bar{\phi}$ to 0). Since $F(\phi)$ is increasing from 0 to $\bar{\phi}$, $\bar{g}_1(\phi)\bar{g}_2(\phi)$ decreases on $(0, \bar{\phi})$ (cf remark 4.3). Hence, $E[\rho_{\text{uni}}] - E[\rho_{\kappa,1}]$ increases as ϕ decreases from $\bar{\phi}$ to 0 (or equivalently, as κ increases from κ_3 to ∞). Given that at $\kappa = \kappa_3$, $\rho_{\kappa,1}$ and $\rho_{\kappa,2}$ coincide, using part (a) we infer that $E[\rho_{\text{uni}}] - E[\rho_{\kappa,1}] < 0$ at $\kappa = \kappa_3$. Therefore, $E[\rho_{\text{uni}}] - E[\rho_{\kappa,1}]$ increases from a negative value at $\kappa = \kappa_3$, but we have no more information yet on how the energies compare for a general $\kappa > \kappa_3$.

Theorem 6.1 (Global energy minimizers for $m > 2, d = 2$). *Let $d = 2$ and $m > 2$. Then, there exists a fourth critical value of κ (which we denote by κ_c), with $\kappa_3 < \kappa_c < \kappa_1$, such that the global minimizer of (1.1) on $\mathcal{P}_{ac}(\mathbb{S}^d)$ is*

- (1) the uniform distribution ρ_{uni} when $\kappa < \kappa_c$,
- (2) the equilibrium $\rho_{\kappa,1}$ given by (6.63) when $\kappa > \kappa_c$.

Moreover, the energy difference $E[\rho_{\text{uni}}] - E[\rho_{\kappa,1}]$ increases with κ for $\kappa > \kappa_3$.

Proof. The considerations we made before listing the theorem imply that ρ_{uni} is more energetically favourable than $\rho_{\kappa,2}$ for all $\kappa_3 < \kappa < \kappa_1$. Also, ρ_{uni} is more energetically favourable than $\rho_{\kappa,1}$ at $\kappa = \kappa_3$, and the difference $E[\rho_{\text{uni}}] - E[\rho_{\kappa,1}]$ increases for $\kappa > \kappa_3$ (but we have not yet quantified how). These behaviours can be observed in figure 8; note that for a better visualization we separated the energy plots in three parts, for different ranges of κ . To establish which equilibrium is the global minimizer we still have to compare $E[\rho_{\text{uni}}]$ and $E[\rho_{\kappa,1}]$, and $E[\rho_{\kappa,1}]$ and $E[\rho_{\kappa,2}]$. We will do this separately.

Compare $E[\rho_{\text{uni}}]$ and $E[\rho_{\kappa,1}]$. We use (4.39) and (4.47) (see also (4.48)) to write

$$E[\rho_{\text{uni}}] - E[\rho_{\kappa,1}] = \frac{1}{m-1} \int_{\mathbb{S}^d} \rho_{\text{uni}}^m(x) \, dS(x) + \bar{g}_1(\phi_{\kappa,1}) \bar{g}_2(\phi_{\kappa,1}).$$

Also recall that $\phi_{\kappa,1}$ decreases to 0 as κ increases to ∞ . For this reason, we will check the behaviour of $\bar{g}_1(\phi)\bar{g}_2(\phi)$ as ϕ tends to zero.

Write $\bar{g}_1(\phi)$ from (4.48) (with $d = 2$) as

$$\begin{aligned} \bar{g}_1(\phi) &= (m-2) \int_0^\phi (\cos\theta - \cos\phi)^{\frac{1}{m-1}} \sin\theta \cos\theta \, d\theta \\ &\quad + 2\cos\phi \int_0^\phi (\cos\theta - \cos\phi)^{\frac{1}{m-1}} \sin\theta \, d\theta. \end{aligned}$$

Then, using $m > 2$ we get

$$\begin{aligned} \bar{g}_1(\phi)\bar{g}_2(\phi) &= \frac{(m-2) \int_0^\phi (\cos\theta - \cos\phi)^{\frac{1}{m-1}} \sin\theta \cos\theta \, d\theta + 2\cos\phi \int_0^\phi (\cos\theta - \cos\phi)^{\frac{1}{m-1}} \sin\theta \, d\theta}{2(m-1)(d\omega_d)^{m-1} \left(\int_0^\phi (\cos\theta - \cos\phi)^{\frac{1}{m-1}} \sin\theta \, d\theta \right)^m} \\ &\geq \frac{2\cos\phi \int_0^\phi (\cos\theta - \cos\phi)^{\frac{1}{m-1}} \sin\theta \, d\theta}{2(m-1)(d\omega_d)^{m-1} \left(\int_0^\phi (\cos\theta - \cos\phi)^{\frac{1}{m-1}} \sin\theta \, d\theta \right)^m} \\ &= C \frac{\cos\phi}{\left(\int_0^\phi (\cos\theta - \cos\phi)^{\frac{1}{m-1}} \sin\theta \, d\theta \right)^{m-1}}, \end{aligned}$$

where C is a positive constant.

As the r.h.s. above tends to ∞ as $\phi \rightarrow 0$, we infer that

$$\lim_{\phi \rightarrow 0} \bar{g}_1(\phi)\bar{g}_2(\phi) = \infty.$$

The limit $\phi \rightarrow 0$ corresponds to $\kappa \rightarrow \infty$. Hence, $E[\rho_{\text{uni}}] - E[\rho_{\kappa,1}]$ is negative at $\kappa = \kappa_3$, increases on (κ_3, ∞) , and approaches ∞ as $\kappa \rightarrow \infty$. We conclude that there is a fourth critical value, which we denote by κ_c such that $E[\rho_{\text{uni}}] = E[\rho_{\kappa,1}]$ at $\kappa = \kappa_c$. Also, $E[\rho_{\text{uni}}] < E[\rho_{\kappa,1}]$ for $\kappa_3 < \kappa < \kappa_c$, and $E[\rho_{\text{uni}}] > E[\rho_{\kappa,1}]$ for $\kappa > \kappa_c$ —see figure 8, in particular plot (b) and its inset.

Compare $E[\rho_{\kappa,1}]$ and $E[\rho_{\kappa,2}]$. We will consider two cases: a) $\kappa_3 < \kappa < \kappa_2$, and b) $\kappa_2 < \kappa < \kappa_1$.

Case a) $\kappa_3 < \kappa < \kappa_2$. We use (2.6) and (4.47), to write

$$E[\rho_{\kappa,2}] - E[\rho_{\kappa,1}] = \bar{g}_1(\phi_{\kappa,1})\bar{g}_2(\phi_{\kappa,1}) - \bar{g}_1(\phi_{\kappa,2})\bar{g}_2(\phi_{\kappa,2}). \tag{6.65}$$

We will show that $E[\rho_{\kappa,2}] - E[\rho_{\kappa,1}]$ increases as κ increases from κ_3 to κ_2 .

Compute the derivative of $\bar{g}_1(\phi_{\kappa,i})\bar{g}_2(\phi_{\kappa,i})$ with respect to κ , for $i = 1, 2$. By chain rule,

$$\frac{d}{d\kappa} (\bar{g}_1(\phi_{\kappa,i})\bar{g}_2(\phi_{\kappa,i})) = \frac{d(\bar{g}_1(\phi_{\kappa,i})\bar{g}_2(\phi_{\kappa,i}))}{d\phi_{\kappa,i}} \frac{d\phi_{\kappa,i}}{d\kappa}. \tag{6.66}$$

Now differentiate

$$F(\phi_{\kappa,i})^{-1} = \kappa,$$

with respect to κ , to get

$$-\frac{F'(\phi_{\kappa,i})}{F^2(\phi_{\kappa,i})} \frac{d\phi_{\kappa,i}}{d\kappa} = 1.$$

From the above we get

$$\frac{d\phi_{\kappa,i}}{d\kappa} = -\frac{F^2(\phi_{\kappa,i})}{F'(\phi_{\kappa,i})}. \tag{6.67}$$

By (4.49), we also have

$$\frac{d(\bar{g}_1(\phi_{\kappa,i})\bar{g}_2(\phi_{\kappa,i}))}{d\phi_{\kappa,i}} = -(m-1)\bar{h}(\phi_{\kappa,i}) \frac{F'(\phi_{\kappa,i})}{F(\phi_{\kappa,i})}, \tag{6.68}$$

with \bar{h} given by (4.50). Then, by combining (6.66), (6.67) and (6.68), we find

$$\begin{aligned} \frac{d}{d\kappa} (\bar{g}_1(\phi_{\kappa,i}) \bar{g}_2(\phi_{\kappa,i})) &= (m-1) \bar{h}(\phi_{\kappa,i}) F(\phi_{\kappa,i}) \\ &= (m-1) \bar{h}(\phi_{\kappa,i}) \kappa^{-1}. \end{aligned} \tag{6.69}$$

The equation above holds for $i = 1, 2$. Now, from (6.65) and (6.69) we get

$$\frac{d}{d\kappa} (E[\rho_{\kappa,2}] - E[\rho_{\kappa,1}]) = \kappa^{-1} (m-1) (\bar{h}(\phi_{\kappa,1}) - \bar{h}(\phi_{\kappa,2})). \tag{6.70}$$

Note that at each κ fixed, we have $\phi_{\kappa,1} < \phi_{\kappa,2}$. If we show that $\bar{h}(\phi)$ is decreasing in ϕ , then together with (6.70), we infer that $E[\rho_{\kappa,2}] - E[\rho_{\kappa,1}]$ increases with κ .

We will find that $\bar{h}'(\phi) < 0$ by a direct calculation. By (4.50) with $d = 2$, write

$$\bar{h}(\phi) = \bar{C} \frac{\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin \theta \cos \theta d\theta}{\left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin \theta d\theta\right)^m},$$

where $\bar{C} > 0$ is a constant. Both the numerator and the denominator above can be computed exactly. Indeed, by direct integration we find

$$\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin \theta \cos \theta d\theta = \frac{m-1}{2m-1} (1 - \cos \phi)^{\frac{2m-1}{m-1}} + \frac{m-1}{m} \cos \phi (1 - \cos \phi)^{\frac{m}{m-1}},$$

and

$$\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin \theta d\theta = \frac{m-1}{m} (1 - \cos \phi)^{\frac{m}{m-1}},$$

and hence, by redenoting the constant \bar{C} several times to include new constant terms, we have

$$\begin{aligned} \bar{h}(\phi) &= \bar{C} \frac{\frac{1}{2m-1} (1 - \cos \phi)^{\frac{2m-1}{m-1}} + \frac{1}{m} \cos \phi (1 - \cos \phi)^{\frac{m}{m-1}}}{(1 - \cos \phi)^{\frac{m^2}{m-1}}} \\ &= \bar{C} \left(\frac{1}{2m-1} (1 - \cos \phi)^{1-m} + \frac{1}{m} \cos \phi (1 - \cos \phi)^{-m} \right) \\ &= \bar{C} (1 - \cos \phi)^{-m} (m + (m-1) \cos \phi). \end{aligned}$$

Then, compute

$$\begin{aligned} \bar{h}'(\phi) &= -\bar{C} \sin \phi (1 - \cos \phi)^{-m} \left(\frac{m(m + (m-1) \cos \phi)}{1 - \cos \phi} + m - 1 \right) \\ &= -\bar{C} \sin \phi (1 - \cos \phi)^{-m-1} \left(m^2 + m - 1 + (m-1)^2 \cos \phi \right). \end{aligned}$$

Finally, we have

$$\begin{aligned} m^2 + m - 1 + (m-1)^2 \cos \phi &\geq m^2 + m - 1 - (m-1)^2 \\ &= 3m - 2, \end{aligned}$$

and hence in the regime $m > 2$, $\bar{h}'(\phi) < 0$.

Case b) $\kappa_2 < \kappa < \kappa_1$. In this range of κ , $\rho_{\kappa,2}$ is fully supported on \mathbb{S}^d . We use (2.6), (4.40) and (4.47) to write

$$E[\rho_{\kappa,2}] - E[\rho_{\kappa,1}] = \bar{g}_1(\phi_{\kappa,1}) \bar{g}_2(\phi_{\kappa,1}) - g_1(\eta_\kappa) g_2(\eta_\kappa). \tag{6.71}$$

We will show again that $E[\rho_{\kappa,2}] - E[\rho_{\kappa,1}]$ increases as κ increases from κ_2 to κ_1 .

From computations similar to those leading to (6.67) and (6.69), we get

$$\frac{d\eta_\kappa}{d\kappa} = -\frac{H^2(\eta_\kappa)}{H'(\eta_\kappa)},$$

and by chain rule,

$$\frac{d}{d\kappa} (g_1(\eta_\kappa)g_2(\eta_\kappa)) = (m - 1)h(\eta_\kappa)\kappa^{-1},$$

with h given by (4.46). Then, together with (6.69), we write (6.71) as

$$\frac{d}{d\kappa} (E[\rho_{\kappa,2}] - E[\rho_{\kappa,1}]) = \kappa^{-1} (m - 1) (\bar{h}(\phi_{\kappa,1}) - h(\eta_\kappa)). \tag{6.72}$$

By (4.46) with $d = 2$, we have

$$h(\eta) = C \frac{\int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}} \sin\theta \cos\theta d\theta}{\left(\int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}} \sin\theta d\theta\right)^m},$$

for $C > 0$ constant. Inspect first the numerator

$$f(\eta) := \int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}} \sin\theta \cos\theta d\theta.$$

Compute the derivative:

$$f'(\eta) = -\frac{1}{m-1} \int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}-1} \sin\theta \cos\theta d\theta.$$

We will show that $\int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}-1} \sin\theta \cos\theta d\theta \leq 0$, and hence $f'(\eta) \geq 0$. Note first that as $m > 2$, $\frac{1}{m-1} - 1 < 0$. Then, for any $0 \leq \tilde{\theta} \leq \frac{\pi}{2}$, we have: $\left(\cos\left(\frac{\pi}{2} - \tilde{\theta}\right) - \eta\right)^{\frac{1}{m-1}-1} \leq \left(\cos\left(\frac{\pi}{2} + \tilde{\theta}\right) - \eta\right)^{\frac{1}{m-1}-1}$ (by the monotonicity of cosine and $\frac{1}{m-1} - 1 < 0$), $\sin\left(\frac{\pi}{2} - \tilde{\theta}\right) = \sin\left(\frac{\pi}{2} + \tilde{\theta}\right)$ and $0 \leq \cos\left(\frac{\pi}{2} - \tilde{\theta}\right) = -\cos\left(\frac{\pi}{2} + \tilde{\theta}\right)$. From these relations, we infer

$$\int_0^{\frac{\pi}{2}} (\cos\theta - \eta)^{\frac{1}{m-1}-1} \sin\theta \cos\theta d\theta \leq -\int_{\frac{\pi}{2}}^\pi (\cos\theta - \eta)^{\frac{1}{m-1}-1} \sin\theta \cos\theta d\theta,$$

where in the integrals we change variables $\theta = \frac{\pi}{2} - \tilde{\theta}$ and $\theta = \frac{\pi}{2} + \tilde{\theta}$ respectively. Hence, we find

$$\begin{aligned} \int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}-1} \sin\theta \cos\theta d\theta &= \int_0^{\frac{\pi}{2}} (\cos\theta - \eta)^{\frac{1}{m-1}-1} \sin\theta \cos\theta d\theta \\ &\quad + \int_{\frac{\pi}{2}}^\pi (\cos\theta - \eta)^{\frac{1}{m-1}-1} \sin\theta \cos\theta d\theta \\ &\leq 0. \end{aligned}$$

It is also immediate that $\bar{f}(\eta) := \frac{1}{\left(\int_0^\pi (\cos\theta - \eta)^{\frac{1}{m-1}} \sin\theta d\theta\right)^m}$ is an increasing function of η , for $-\infty < \eta < -1$. Therefore, as $h(\eta)$ is a product of two increasing functions, it is also increasing. Recall that η_κ decreases from -1 to $-\infty$ as κ increases from κ_2 to κ_1 . So overall, $h(\eta_\kappa)$ decreases as κ increases from κ_2 to κ_1 , and in particular we have

$$h(-1) \geq h(\eta_\kappa), \quad \text{for all } \kappa_2 < \kappa < \kappa_1.$$

On the other hand, by the monotonicity of \bar{h} (established as in case a)), we have

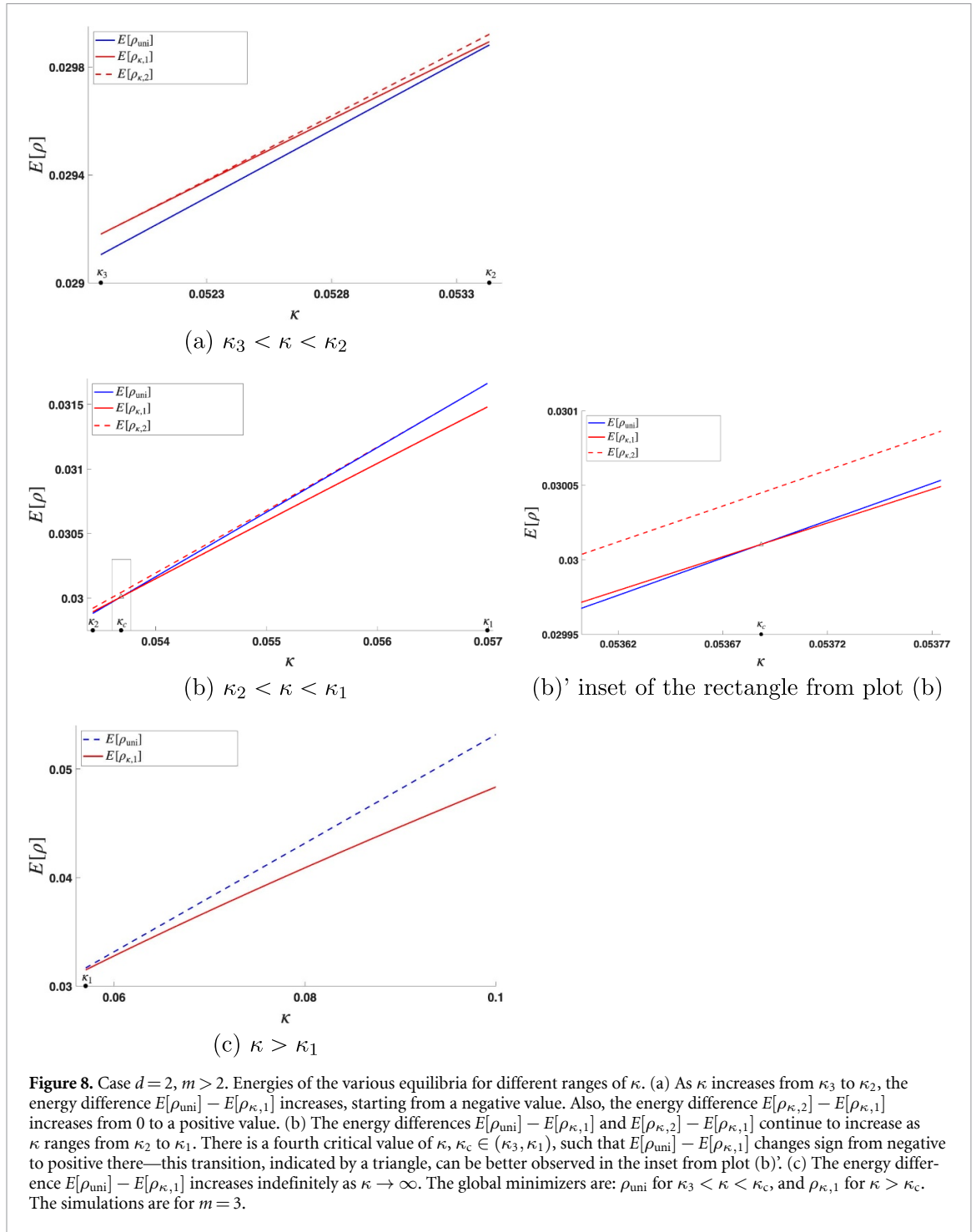
$$\bar{h}(\phi_{\kappa,1}) > \bar{h}(\phi_{\kappa_2,1}), \quad \text{for all } \kappa_2 < \kappa < \kappa_1,$$

and case a), at $\kappa = \kappa_2$ it holds that

$$\bar{h}(\phi_{\kappa_2,1}) > \bar{h}(\phi_{\kappa_2,2}).$$

Finally, observe that $\phi_{\kappa_2,2} = \pi$ and $\bar{h}(\phi_{\kappa_2,2}) = h(-1)$, and by combining the inequalities above we find

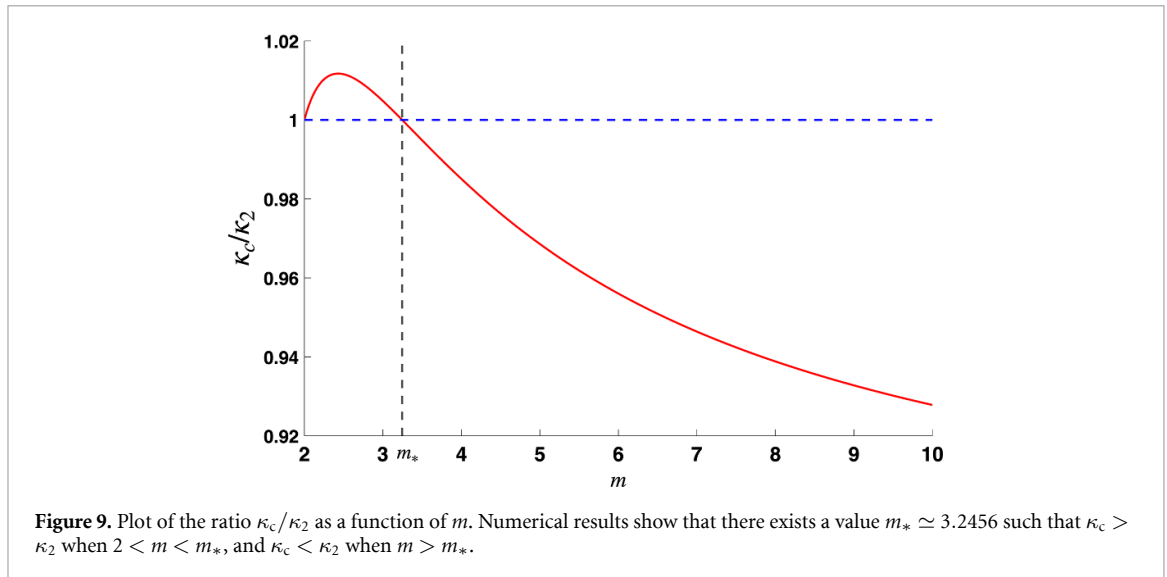
$$\bar{h}(\phi_{\kappa,1}) > h(\eta_\kappa), \quad \text{for all } \kappa_2 < \kappa < \kappa_1.$$



From this result and (6.72), we infer that $E[\rho_{\kappa,2}] - E[\rho_{\kappa,1}]$ increases with κ on (κ_2, κ_1) .

The comparison of energies above provides the global minimum as in the statement of the theorem. Finally, using the fact that ρ_{uni} and $\rho_{\kappa,2}$ coincide at $\kappa = \kappa_1$, we infer that $E[\rho_{\text{uni}}] > E[\rho_{\kappa,1}]$ at $\kappa = \kappa_1$. Hence, the fourth critical value κ_c (where $E[\rho_{\text{uni}}] - E[\rho_{\kappa,1}]$ changes sign) is in the range $\kappa_3 < \kappa_c < \kappa_1$. \square

The numerical illustration in figure 8 shows separately three ranges of κ . Various facts that appear in the proof of theorem 6.1 can be observed in the figure; e.g. the monotonicity with respect to κ of $E[\rho_{\text{uni}}] - E[\rho_{\kappa,1}]$, $E[\rho_{\text{uni}}] - E[\rho_{\kappa,2}]$ and $E[\rho_{\kappa,2}] - E[\rho_{\kappa,1}]$. In figure 6.1(b) we indicate by a triangle the point where $E[\rho_{\text{uni}}] = E[\rho_{\kappa,1}]$, i.e. at the fourth critical value κ_c .



Remark 6.2 (Comparison between κ_c and κ_2). The critical value κ_c introduced in theorem 6.1 can be computed as

$$\kappa_c = \frac{2m - 1}{(2\pi)^{m-1} (m + (m - 1) \cos \phi_c) \left(\frac{m-1}{m} (1 - \cos \phi_c)\right)^m},$$

where $\phi_c \in [0, \pi]$ is the unique solution of the equation

$$\frac{m - 1}{4(2m - 1)} ((2 - m) - (m + 1) \cos \phi_c) = \left(\frac{m - 1}{m}\right)^m \left(\frac{1 - \cos \phi_c}{2}\right)^m. \tag{6.73}$$

The details of this calculation can be found in appendix E. Then, we can numerically compare κ_c with κ_2 for $m > 2$, using the ratio (see also appendix E for details):

$$\frac{\kappa_c}{\kappa_2} = \left(\frac{2}{1 - \cos \phi_c}\right)^m \frac{1}{m + (m - 1) \cos \phi_c}.$$

Since ϕ_c cannot be solved explicitly and equation (6.73) is not easily tractable, an analytic study of the ratio κ_c/κ_2 appears difficult. We therefore resort to numerical approximation—see plot of κ_c/κ_2 in figure 9. Our numerical results indicate that there exists a unique value $m = m_* \simeq 3.2456$ for which $\kappa_c = \kappa_2$, and also,

$$\begin{aligned} \kappa_c &> \kappa_2, & \text{for } 2 < m < m_*, \\ \kappa_c &< \kappa_2, & \text{for } m > m_*. \end{aligned}$$

Finally, as $m \searrow 2$, all critical values κ_1 , κ_2 , and κ_3 converge to the same limit, and κ_c converges to the same limit as well since $\kappa_3 < \kappa_c < \kappa_1$. In particular, $\kappa_c/\kappa_2 \rightarrow 1$ as $m \searrow 2$.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Supporting result for lemma 4.1

Lemma A.1. For any given real number $m > 1$ and natural number d , define the following function on $\xi > 1$:

$$\begin{aligned} \mathcal{H}_{m,d}(\xi) = & \left(\int_0^\pi (\xi + \cos\theta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta \right) \left(\int_0^\pi (\xi + \cos\theta)^{\frac{1}{m-1}-1} \sin^{d+1} \theta d\theta \right) \\ & - \left(\int_0^\pi (\xi + \cos\theta)^{\frac{1}{m-1}-2} \sin^{d+1} \theta d\theta \right) \left(\int_0^\pi (\xi + \cos\theta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right). \end{aligned} \tag{A.74}$$

Then, we distinguish the following cases:

- (1) if $\frac{1}{m-1} + \frac{d-1}{2} < 0$ then $\mathcal{H}_{m,d}(\xi) > 0$ for all $\xi > 1$,
- (2) if $\frac{1}{m-1} + \frac{d-1}{2} = 0$ then $\mathcal{H}_{m,d}(\xi) = 0$ for all $\xi > 1$, and
- (3) if $\frac{1}{m-1} + \frac{d-1}{2} > 0$ then $\mathcal{H}_{m,d}(\xi) < 0$ for all $\xi > 1$.

In other words, we have

$$\text{sgn}(\mathcal{H}_{m,d}(\xi)) = -\text{sgn}\left(\frac{1}{m-1} + \frac{d-1}{2}\right), \quad \text{for all } \xi > 1.$$

Proof. From section 3.666, equation (2) of [28], we have:

$$\int_0^\pi (\cosh\beta + \sinh\beta \cos\theta)^{\mu+\nu} \sin^{-2\nu} \theta d\theta = \frac{\sqrt{\pi}}{2^\nu} \sinh^\nu \beta \Gamma\left(\frac{1}{2} - \nu\right) P_\mu^\nu(\cosh\beta) \quad \forall \text{Re}(\nu) < \frac{1}{2},$$

where P_μ^ν denotes the associated Legendre function of the first kind. By factoring out $\sinh\beta$ from the l.h.s., the equation can be written as

$$\int_0^\pi (\coth\beta + \cos\theta)^{\mu+\nu} \sin^{-2\nu} \theta d\theta = \frac{\sqrt{\pi}}{2^\nu} \sinh^{-\mu} \beta \Gamma\left(\frac{1}{2} - \nu\right) P_\mu^\nu(\cosh\beta). \tag{A.75}$$

Denote $\mu_0 = \frac{1}{m-1} - 2 + \frac{d+1}{2}$, $\nu_0 = -\frac{d+1}{2}$ and substitute $\xi = \coth\beta$ in (A.74), and use (A.75) to simplify each integral in $\mathcal{H}_{m,d}(\xi)$ as follows:

$$\begin{cases} \int_0^\pi (\xi + \cos\theta)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta &= \frac{\sqrt{\pi}}{2^{\nu_0+1}} \sinh^{-\mu_0} \beta \Gamma\left(\frac{1}{2} - (\nu_0 + 1)\right) P_{\mu_0}^{\nu_0+1}(\cosh\beta), \\ \int_0^\pi (\xi + \cos\theta)^{\frac{1}{m-1}-1} \sin^{d+1} \theta d\theta &= \frac{\sqrt{\pi}}{2^{\nu_0}} \sinh^{-(\mu_0+1)} \beta \Gamma\left(\frac{1}{2} - \nu_0\right) P_{\mu_0+1}^{\nu_0}(\cosh\beta), \\ \int_0^\pi (\xi + \cos\theta)^{\frac{1}{m-1}-2} \sin^{d+1} \theta d\theta &= \frac{\sqrt{\pi}}{2^{\nu_0}} \sinh^{-\mu_0} \beta \Gamma\left(\frac{1}{2} - \nu_0\right) P_{\mu_0}^{\nu_0}(\cosh\beta), \\ \int_0^\pi (\xi + \cos\theta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta &= \frac{\sqrt{\pi}}{2^{\nu_0+1}} \sinh^{-(\mu_0+1)} \beta \Gamma\left(\frac{1}{2} - (\nu_0 + 1)\right) P_{\mu_0+1}^{\nu_0+1}(\cosh\beta). \end{cases}$$

Using the above, $\mathcal{H}_{m,d}(\xi)$ from (A.74) can be written in terms of β as

$$\begin{aligned} \mathcal{H}_{m,d}(\xi) = & \frac{\pi}{2^{2\nu_0+1}} \sinh^{-(2\mu_0+1)} \beta \Gamma\left(\frac{1}{2} - \nu_0\right) \Gamma\left(-\frac{1}{2} - \nu_0\right) \\ & \times \left(P_{\mu_0+1}^{\nu_0+1}(\cosh\beta) P_{\mu_0+1}^{\nu_0}(\cosh\beta) - P_{\mu_0}^{\nu_0}(\cosh\beta) P_{\mu_0+1}^{\nu_0+1}(\cosh\beta) \right). \end{aligned}$$

Here, the range of β is $(0, \infty)$ since the range of ξ is $(1, \infty)$. Equation (1) in section 8.715 of [28] provides the following integral representation of the associated Legendre function of the first kind:

$$P_\mu^\nu(\cosh\beta) = \frac{\sqrt{2} \sinh^\nu \beta}{\sqrt{\pi} \Gamma\left(\frac{1}{2} - \nu\right)} \int_0^\beta \frac{\cosh\left(\mu + \frac{1}{2}\right) t}{(\cosh\beta - \cosh t)^{\nu+\frac{1}{2}}} dt.$$

By using the integral expression above, we can rewrite $\mathcal{H}_{m,d}(\xi)$ as

$$\begin{aligned} \mathcal{H}_{m,d}(\xi) &= \frac{\pi}{2^{2\nu_0+1}} \sinh^{-(2\nu_0+1)} \beta \Gamma\left(\frac{1}{2} - \nu_0\right) \Gamma\left(-\frac{1}{2} - \nu_0\right) \times \frac{2 \sinh^{2\nu_0+1} \beta}{\pi \Gamma\left(\frac{1}{2} - \nu_0\right) \Gamma\left(-\frac{1}{2} - \nu_0\right)} \\ &\times \left(\int_0^\beta \frac{\cosh\left(\mu_0 + \frac{1}{2}\right) t}{(\cosh \beta - \cosh t)^{\nu_0 + \frac{3}{2}}} dt \int_0^\beta \frac{\cosh\left(\mu_0 + \frac{3}{2}\right) t}{(\cosh \beta - \cosh t)^{\nu_0 + \frac{1}{2}}} dt \right. \\ &\left. - \int_0^\beta \frac{\cosh\left(\mu_0 + \frac{1}{2}\right) t}{(\cosh \beta - \cosh t)^{\nu_0 + \frac{1}{2}}} dt \int_0^\beta \frac{\cosh\left(\mu_0 + \frac{3}{2}\right) t}{(\cosh \beta - \cosh t)^{\nu_0 + \frac{3}{2}}} dt \right). \end{aligned}$$

Finally, to assess the sign of $\mathcal{H}_{m,d}(\xi)$, we will investigate the sign of

$$\begin{aligned} \tilde{\mathcal{H}}_{m,d}(\beta) &:= \int_0^\beta \frac{\cosh\left(\mu_0 + \frac{1}{2}\right) t}{(\cosh \beta - \cosh t)^{\nu_0 + \frac{3}{2}}} dt \int_0^\beta \frac{\cosh\left(\mu_0 + \frac{3}{2}\right) t}{(\cosh \beta - \cosh t)^{\nu_0 + \frac{1}{2}}} dt \\ &- \int_0^\beta \frac{\cosh\left(\mu_0 + \frac{1}{2}\right) t}{(\cosh \beta - \cosh t)^{\nu_0 + \frac{1}{2}}} dt \int_0^\beta \frac{\cosh\left(\mu_0 + \frac{3}{2}\right) t}{(\cosh \beta - \cosh t)^{\nu_0 + \frac{3}{2}}} dt, \end{aligned}$$

for $\beta \in (0, \infty)$.

The sign of

$$A := \mu_0 + 1 = \frac{1}{m-1} + \frac{d-1}{2}$$

will be crucial in determining the sign of $\tilde{\mathcal{H}}_{m,d}(\beta)$ (and hence, of $\mathcal{H}_{m,d}(\xi)$). We express $\tilde{\mathcal{H}}_{m,d}(\beta)$ using A instead of μ_0 as follows:

$$\begin{aligned} \tilde{\mathcal{H}}_{m,d}(\beta) &= \int_0^\beta \frac{\cosh\left(A - \frac{1}{2}\right) t}{(\cosh \beta - \cosh t)^{\nu_0 + \frac{3}{2}}} dt \int_0^\beta \frac{\cosh\left(A + \frac{1}{2}\right) t}{(\cosh \beta - \cosh t)^{\nu_0 + \frac{1}{2}}} dt \\ &- \int_0^\beta \frac{\cosh\left(A - \frac{1}{2}\right) t}{(\cosh \beta - \cosh t)^{\nu_0 + \frac{1}{2}}} dt \int_0^\beta \frac{\cosh\left(A + \frac{1}{2}\right) t}{(\cosh \beta - \cosh t)^{\nu_0 + \frac{3}{2}}} dt. \end{aligned}$$

Now define, for $t > 0$:

$$\zeta(t) = \int_0^t \frac{\cosh\left(A + \frac{1}{2}\right) \tau}{(\cosh \beta - \cosh \tau)^{\nu_0 + \frac{3}{2}}} d\tau, \quad f(t) = \cosh \beta - \cosh t, \quad g(t) = \frac{\cosh\left(A - \frac{1}{2}\right) t}{\cosh\left(A + \frac{1}{2}\right) t}.$$

Hence, we write

$$\tilde{\mathcal{H}}_{m,d}(\beta) = \int_0^\beta f(t) d\zeta(t) \int_0^\beta g(t) d\zeta(t) - \int_0^\beta f(t) g(t) d\zeta(t) \int_0^\beta d\zeta(t).$$

The function f is decreasing on $t \geq 0$. Also, as shown below, the function g is increasing for $A < 0$, is constant when $A = 0$, and is decreasing for $A > 0$. The conclusion of the lemma then follows from the integral form of the Chebyshev inequality [28, section 11.31] (we account here for the monotonicities of the functions f and g). We find:

Case 1: $A < 0$. Since f is decreasing and g is increasing, $\mathcal{H}_{m,d}(\xi) > 0$ for all $\xi > 1$.

Case 2: $A = 0$. In this case, $g(t) = 1$ and $\mathcal{H}_{m,d}(\xi) \equiv 0$ for all $\xi > 1$.

Case 3: $A > 0$. Since f and g are both decreasing, $\mathcal{H}_{m,d}(\xi) < 0$ for all $\xi > 1$.

It remains to show the monotonicity of the function g . We calculate:

$$\frac{g'(t)}{g(t)} = \left(A - \frac{1}{2}\right) \tanh\left(A - \frac{1}{2}\right) t - \left(A + \frac{1}{2}\right) \tanh\left(A + \frac{1}{2}\right) t. \tag{A.76}$$

Since $g(t)$ is positive, the sign of $g'(t)$ is given by the sign of the r.h.s of (A.76).

From basic identities of hyperbolic functions, we have for any $u, v \in \mathbb{R}$:

$$\begin{aligned} &(u - v) \tanh(u - v) - (u + v) \tanh(u + v) \\ &= \frac{(u - v) \sinh(u - v) \cosh(u + v) - (u + v) \sinh(u + v) \cosh(u - v)}{\cosh(u + v) \cosh(u - v)} \end{aligned}$$

$$\begin{aligned}
 &= \frac{\frac{u-v}{2} (\sinh(2u) - \sinh(2v)) - \frac{u+v}{2} (\sinh(2u) + \sinh(2v))}{\frac{1}{2} (\cosh(2u) + \cosh(2v))} \\
 &= -\frac{2v \sinh(2u) + 2u \sinh(2v)}{\cosh(2u) + \cosh(2v)}.
 \end{aligned}$$

Now substitute $u = At$ and $v = \frac{t}{2}$ into the equation above to get

$$\begin{aligned}
 &\left(A - \frac{1}{2}\right) \tanh\left(A - \frac{1}{2}\right) t - \left(A + \frac{1}{2}\right) \tanh\left(A + \frac{1}{2}\right) t \\
 &= -\frac{1}{t} \times \frac{t \sinh(2At) + 2At \sinh(t)}{\cosh(2At) + \cosh(t)} \\
 &= -\frac{\sinh(2At) + 2A \sinh(t)}{\cosh(2At) + \cosh(t)}.
 \end{aligned}$$

The denominator of the fraction in the r.h.s. is always positive, while the sign of the numerator depends on A . We conclude that for any $t > 0$, the expression $\left(A - \frac{1}{2}\right) \tanh\left(A - \frac{1}{2}\right) t - \left(A + \frac{1}{2}\right) \tanh\left(A + \frac{1}{2}\right) t$ is positive for $A < 0$, is zero for $A = 0$, and is negative for $A > 0$. The monotonicity of g follows then from (A.76). □

Appendix B. Proof of lemma 4.2

We show that

$$\lim_{\eta \rightarrow \infty} H(-\eta) = \kappa_1^{-1},$$

where κ_1 is given by (3.26) and the function H by (4.29). Compute

$$\begin{aligned}
 H(-\eta) &= \frac{m-1}{m} (dw_d)^{m-1} \left(\int_0^\pi (\cos\theta + \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos\theta d\theta \right) \left(\int_0^\pi (\cos\theta + \eta)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^{m-2} \\
 &= \frac{m-1}{m} (dw_d)^{m-1} \eta \left(\int_0^\pi \left(\frac{\cos\theta}{\eta} + 1\right)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos\theta d\theta \right) \left(\int_0^\pi \left(\frac{\cos\theta}{\eta} + 1\right)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^{m-2}.
 \end{aligned}$$

Substitute $h = \frac{1}{\eta}$ and take $h \rightarrow 0$ by l'Hôpital's rule, to get

$$\begin{aligned}
 \lim_{\eta \rightarrow \infty} H(-\eta) &= \frac{m-1}{m} (dw_d)^{m-1} \lim_{h \rightarrow 0} \frac{\left(\int_0^\pi (h \cos\theta + 1)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos\theta d\theta \right) \left(\int_0^\pi (h \cos\theta + 1)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^{m-2}}{h} \\
 &= \frac{m-1}{m} (dw_d)^{m-1} \times \lim_{h \rightarrow 0} \left(\left(\int_0^\pi \frac{\cos\theta}{m-1} (h \cos\theta + 1)^{\frac{1}{m-1}-1} \sin^{d-1} \theta \cos\theta d\theta \right) \left(\int_0^\pi (h \cos\theta + 1)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^{m-2} \right. \\
 &\quad \left. + \left(\int_0^\pi (h \cos\theta + 1)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos\theta d\theta \right) \times (m-2) \left(\int_0^\pi (h \cos\theta + 1)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^{m-1} \right. \\
 &\quad \left. \times \left(\int_0^\pi \frac{\cos\theta}{m-1} (h \cos\theta + 1)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta \right) \right).
 \end{aligned}$$

Then, by taking the $h \rightarrow 0$ limit, we find

$$\begin{aligned}
 \lim_{\eta \rightarrow \infty} H(-\eta) &= \frac{m-1}{m} (dw_d)^{m-1} \left(\left(\int_0^\pi \frac{\cos\theta}{m-1} \sin^{d-1} \theta \cos\theta d\theta \right) \left(\int_0^\pi \sin^{d-1} \theta d\theta \right)^{m-2} \right. \\
 &\quad \left. + \left(\int_0^\pi \sin^{d-1} \theta \cos\theta d\theta \right) \times (m-2) \left(\int_0^\pi \sin^{d-1} \theta d\theta \right)^{m-1} \times \left(\int_0^\pi \frac{\cos\theta}{m-1} \sin^{d-1} \theta d\theta \right) \right) \\
 &= \frac{1}{m} (dw_d)^{m-1} \left(\int_0^\pi \sin^{d-1} \theta \cos^2 \theta d\theta \right) \left(\int_0^\pi \sin^{d-1} \theta d\theta \right)^{m-2},
 \end{aligned}$$

as $\int_0^\pi \sin^{d-1} \theta \cos\theta d\theta = 0$.

Finally, by (5.53) we get

$$\lim_{\eta \rightarrow \infty} H(-\eta) = \frac{\left(dw_d \int_0^\pi \sin^{d-1} \theta d\theta \right)^{m-1}}{m(d+1)} = \frac{|\mathbb{S}^d|^{m-1}}{m(d+1)} = \kappa_1^{-1}.$$

This proves the claim.

Appendix C. Proof of lemma 4.3

Compute from (4.35):

$$\begin{aligned} \frac{F'(\phi)}{F(\phi)} &= \frac{\sin \phi}{m-1} \times \frac{\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta \cos \theta d\theta}{\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta} \\ &+ \frac{(m-2) \sin \phi}{m-1} \times \frac{\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta}{\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta}. \end{aligned} \tag{C.77}$$

Hence, if we show that the following is positive:

$$\begin{aligned} G(\phi) &:= \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta \cos \theta d\theta \right) \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right) \\ &+ (m-2) \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta \right) \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta \right), \end{aligned}$$

then we get that $F'(\phi)$ is positive, and the conclusion follows.

To simplify $G(\phi)$, we use the following two identities obtained from integration by parts:

$$\begin{aligned} &\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta \cos \theta d\theta \\ &= \left[\frac{1}{d} (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^d \theta \right]_{\theta=0}^{\theta=\phi} + \frac{1}{d} \left(\frac{1}{m-1} - 1 \right) \int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-2} \sin^{d+1} \theta d\theta \\ &= \frac{2-m}{d(m-1)} \int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-2} \sin^{d+1} \theta d\theta, \end{aligned}$$

and

$$\begin{aligned} &\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta \\ &= \left[\frac{1}{d} (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^d \theta \right]_{\theta=0}^{\theta=\phi} + \frac{1}{d} \left(\frac{1}{m-1} \right) \int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d+1} \theta d\theta \\ &= \frac{1}{d(m-1)} \int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d+1} \theta d\theta. \end{aligned} \tag{C.78}$$

Then, we have

$$\begin{aligned} G(\phi) &= \frac{2-m}{d(m-1)} \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-2} \sin^{d+1} \theta d\theta \right) \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right) \\ &- \frac{2-m}{d(m-1)} \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta \right) \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d+1} \theta d\theta \right). \end{aligned}$$

Since $\frac{2-m}{d(m-1)} > 0$, we eventually need to show

$$\begin{aligned} &\left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-2} \sin^{d+1} \theta d\theta \right) \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right) \\ &- \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta \right) \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d+1} \theta d\theta \right) \geq 0. \end{aligned} \tag{C.79}$$

With $\phi \in (0, \pi]$ fixed, write

$$\begin{aligned} &2 \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-2} \sin^{d+1} \theta d\theta \right) \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right) \\ &- 2 \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta \right) \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d+1} \theta d\theta \right) \end{aligned}$$

$$\begin{aligned}
 &= \int_0^\phi \int_0^\phi (\cos \theta_1 - \cos \phi)^{\frac{1}{m-1}-2} \sin^{d+1} \theta_1 (\cos \theta_2 - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta_2 d\theta_1 d\theta_2 \\
 &+ \int_0^\phi \int_0^\phi (\cos \theta_2 - \cos \phi)^{\frac{1}{m-1}-2} \sin^{d+1} \theta_2 (\cos \theta_1 - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta_1 d\theta_1 d\theta_2 \\
 &- \int_0^\phi \int_0^\phi (\cos \theta_1 - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta_1 (\cos \theta_2 - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d+1} \theta_2 d\theta_1 d\theta_2 \\
 &- \int_0^\phi \int_0^\phi (\cos \theta_2 - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta_2 (\cos \theta_1 - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d+1} \theta_1 d\theta_1 d\theta_2 \\
 &= \int_0^\phi \int_0^\phi (\cos \theta_1 - \cos \phi)^{\frac{1}{m-1}-2} (\cos \theta_2 - \cos \phi)^{\frac{1}{m-1}-2} \sin^{d-1} \theta_1 \sin^{d-1} \theta_2 \times \left((\cos \theta_2 - \cos \phi)^2 \sin^2 \theta_1 \right. \\
 &\quad \left. + (\cos \theta_1 - \cos \phi)^2 \sin^2 \theta_2 - (\cos \theta_1 - \cos \phi) (\cos \theta_2 - \cos \phi) (\sin^2 \theta_1 + \sin^2 \theta_2) \right) d\theta_1 d\theta_2.
 \end{aligned}$$

Using

$$\begin{aligned}
 &(\cos \theta_2 - \cos \phi)^2 \sin^2 \theta_1 + (\cos \theta_1 - \cos \phi)^2 \sin^2 \theta_2 - (\cos \theta_1 - \cos \phi) (\cos \theta_2 - \cos \phi) (\sin^2 \theta_1 + \sin^2 \theta_2) \\
 &= \sin^2 \theta_1 (\cos \theta_2 - \cos \phi) (\cos \theta_2 - \cos \theta_1) + \sin^2 \theta_2 (\cos \theta_1 - \cos \phi) (\cos \theta_1 - \cos \theta_2) \\
 &= (\cos \theta_2 - \cos \theta_1) \sin^2 \theta_1 \sin^2 \theta_2 \left(\frac{\cos \theta_2 - \cos \phi}{\sin^2 \theta_2} - \frac{\cos \theta_1 - \cos \phi}{\sin^2 \theta_1} \right),
 \end{aligned}$$

we write the above as

$$\begin{aligned}
 &2 \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-2} \sin^{d+1} \theta d\theta \right) \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right) \\
 &\quad - 2 \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d-1} \theta d\theta \right) \left(\int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}-1} \sin^{d+1} \theta d\theta \right) \tag{C.80} \\
 &= \int_0^\phi \int_0^\phi (\cos \theta_1 - \cos \phi)^{\frac{1}{m-1}-2} (\cos \theta_2 - \cos \phi)^{\frac{1}{m-1}-2} \sin^{d-1} \theta_1 \sin^{d-1} \theta_2 \\
 &\quad \times (\cos \theta_2 - \cos \theta_1) \sin^2 \theta_1 \sin^2 \theta_2 \left(\frac{\cos \theta_2 - \cos \phi}{\sin^2 \theta_2} - \frac{\cos \theta_1 - \cos \phi}{\sin^2 \theta_1} \right) d\theta_1 d\theta_2.
 \end{aligned}$$

Note that $\frac{\cos \theta - \cos \phi}{\sin^2 \theta}$ is decreasing on $\theta \in (0, \pi)$, as

$$\frac{d}{d\theta} \left(\frac{\cos \theta - \cos \phi}{\sin^2 \theta} \right) = \frac{-\cos^2 \theta + 2 \cos \phi \cos \theta - 1}{\sin^3 \theta} = -\frac{(\cos \theta - \cos \phi)^2 + \sin^2 \phi}{\sin^3 \theta} \leq 0.$$

Since $\cos \theta$ is also decreasing on $[0, \pi]$, we have

$$(\cos \theta_2 - \cos \theta_1) \left(\frac{\cos \theta_2 - \cos \phi}{\sin^2 \theta_2} - \frac{\cos \theta_1 - \cos \phi}{\sin^2 \theta_1} \right) \geq 0, \quad \forall \theta_1, \theta_2 \in [0, \phi].$$

Therefore, from (C.80) we can conclude that (C.79) holds, which completes the proof.

Appendix D. Calculations in support of remark 4.2

By (4.31), we have

$$\kappa_2^{-1} = \frac{m-1}{m} (dw_d)^{m-1} \left(\int_0^\pi (\cos \theta + 1)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta \right) \left(\int_0^\pi (\cos \theta + 1)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \right)^{m-2},$$

Recall the identity

$$\int_0^\pi \cos^a \left(\frac{\theta}{2} \right) \sin^b \left(\frac{\theta}{2} \right) d\theta = \frac{\Gamma \left(\frac{a+1}{2} \right) \Gamma \left(\frac{b+1}{2} \right)}{\Gamma \left(\frac{a+b+2}{2} \right)}.$$

Using several basic trigonometry formulas

$$\cos \theta + 1 = 2 \cos^2 \frac{\theta}{2}, \quad \sin \theta = 2 \sin \left(\frac{\theta}{2} \right) \cos \left(\frac{\theta}{2} \right), \quad \cos \theta = \cos^2 \left(\frac{\theta}{2} \right) - \sin^2 \left(\frac{\theta}{2} \right),$$

and the property $\Gamma(z + 1) = z\Gamma(z)$ for all $z > 0$, we calculate

$$\begin{aligned} & \int_0^\pi (\cos \theta + 1)^{\frac{1}{m-1}} \sin^{d-1} \theta \cos \theta d\theta \\ &= 2^{\frac{1}{m-1}} \int_0^\pi \cos^{\frac{2}{m-1}} \left(\frac{\theta}{2}\right) \sin^{d-1} \theta \cos \theta d\theta \\ &= 2^{\frac{1}{m-1} + d-1} \int_0^\pi \left(\cos^{\frac{2}{m-1} + d+1} \left(\frac{\theta}{2}\right) \sin^{d-1} \left(\frac{\theta}{2}\right) - \cos^{\frac{2}{m-1} + d-1} \left(\frac{\theta}{2}\right) \sin^{d+1} \left(\frac{\theta}{2}\right) \right) d\theta \\ &= 2^{\frac{1}{m-1} + d-1} \left(\frac{\Gamma\left(\frac{1}{m-1} + \frac{d+2}{2}\right) \Gamma\left(\frac{d}{2}\right) - \Gamma\left(\frac{1}{m-1} + \frac{d}{2}\right) \Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{1}{m-1} + d + 1\right)} \right) \\ &= 2^{\frac{1}{m-1} + d-1} \frac{\Gamma\left(\frac{1}{m-1} + \frac{d}{2}\right) \Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{1}{m-1} + d\right)} \left(\frac{\frac{1}{m-1} + \frac{d}{2} - \frac{d}{2}}{\frac{1}{m-1} + d} \right) \\ &= 2^{\frac{1}{m-1} + d-1} \frac{\Gamma\left(\frac{1}{m-1} + \frac{d}{2}\right) \Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{1}{m-1} + d\right)} \times \frac{1}{1 + d(m-1)}. \end{aligned}$$

Similarly,

$$\begin{aligned} & \int_0^\pi (\cos \theta + 1)^{\frac{1}{m-1}} \sin^{d-1} \theta d\theta \\ &= 2^{\frac{1}{m-1}} \int_0^\pi \cos^{\frac{2}{m-1}} \left(\frac{\theta}{2}\right) \sin^{d-1} \theta d\theta \\ &= 2^{\frac{1}{m-1} + d-1} \int_0^\pi \cos^{\frac{2}{m-1} + d-1} \left(\frac{\theta}{2}\right) \sin^{d-1} \left(\frac{\theta}{2}\right) d\theta \\ &= 2^{\frac{1}{m-1} + d-1} \frac{\Gamma\left(\frac{1}{m-1} + \frac{d}{2}\right) \Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{1}{m-1} + d\right)}. \end{aligned}$$

Therefore, we can find an explicit expression of κ_2 in terms of the Gamma function:

$$\kappa_2^{-1} = \frac{m-1}{m} (dw_d)^{m-1} 2^{1+(d-1)(m-1)} \left(\frac{\Gamma\left(\frac{1}{m-1} + \frac{d}{2}\right) \Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{1}{m-1} + d\right)} \right)^{m-1} \times \frac{1}{1 + d(m-1)}.$$

Finally, using

$$|\mathbb{S}^d| = \frac{dw_d \sqrt{\pi} \Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)},$$

one arrives at (4.38).

Appendix E. Calculations that support remark 6.2

Part 1. Calculation of κ_c

To find the value of κ_c , we compare $E[\rho_{\text{uni}}]$ and $E[\rho_{\kappa,1}]$, where ρ_{uni} is the uniform distribution on \mathbb{S}^d and $\rho_{\kappa,1}$ is defined in (6.63). The energy of the uniform distribution can be calculated directly, as given by

$$E[\rho_{\text{uni}}] = \frac{(4\pi)^{1-m}}{m-1} + \frac{\kappa}{2}.$$

To calculate $E[\rho_{\kappa,1}]$, we first simplify the equations in (4.33). Substitute $d = 2$ in (4.33) to get

$$\begin{aligned} 1 &= 2\pi \left(\frac{\kappa_S(m-1)}{m} \right)^{\frac{1}{m-1}} \int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin \theta d\theta, \\ s &= 2\pi \left(\frac{\kappa_S(m-1)}{m} \right)^{\frac{1}{m-1}} \int_0^\phi (\cos \theta - \cos \phi)^{\frac{1}{m-1}} \sin \theta \cos \theta d\theta. \end{aligned} \tag{E.81}$$

Each integral in the r.h.s. of (E.81) can be computed explicitly by substitution. We find

$$\begin{aligned} 1 &= 2\pi \left(\frac{\kappa s(m-1)}{m} \right)^{\frac{1}{m-1}} \times \frac{m-1}{m} (1 - \cos \phi)^{\frac{m}{m-1}} \\ &= 2\pi (\kappa s)^{\frac{1}{m-1}} \left(\frac{m-1}{m} (1 - \cos \phi) \right)^{\frac{m}{m-1}}, \end{aligned}$$

and

$$\begin{aligned} s &= 2\pi \left(\frac{\kappa s(m-1)}{m} \right)^{\frac{1}{m-1}} \times \left(\frac{m-1}{2m-1} (1 - \cos \phi)^{\frac{2m-1}{m-1}} + \cos \phi \frac{m-1}{m} (1 - \cos \phi)^{\frac{m}{m-1}} \right) \\ &= 2\pi (\kappa s)^{\frac{1}{m-1}} \left(\frac{m-1}{m} (1 - \cos \phi) \right)^{\frac{m}{m-1}} \times \frac{m + (m-1) \cos \phi}{2m-1}. \end{aligned}$$

Using these simplified equations for $s = s_{\kappa,1}$ and $\phi = \phi_{\kappa,1}$, we write $s_{\kappa,1}$ and κ in terms of $\phi_{\kappa,1}$ as

$$s_{\kappa,1} = \frac{m + (m-1) \cos \phi_{\kappa,1}}{2m-1}, \tag{E.82a}$$

$$\kappa = \frac{2m-1}{(2\pi)^{m-1} (m + (m-1) \cos \phi_{\kappa,1}) \left(\frac{m-1}{m} (1 - \cos \phi_{\kappa,1}) \right)^m}. \tag{E.82b}$$

We now calculate $E[\rho_{\kappa,1}]$ using (6.63) and (4.32) to get

$$\begin{aligned} E[\rho_{\kappa,1}] &= \frac{2\pi}{m-1} \int_0^{\phi_{\kappa,1}} \left(\left(\frac{m-1}{m} \right)^{\frac{1}{m-1}} (\lambda_{\kappa,1} + \kappa s_{\kappa,1} \cos \theta)^{\frac{1}{m-1}} \right)^m \sin \theta d\theta - \frac{\kappa s_{\kappa,1}^2}{2} + \frac{\kappa}{2} \\ &= \frac{2\pi}{m-1} \left(\frac{\kappa s_{\kappa,1} (m-1)}{m} \right)^{\frac{m}{m-1}} \int_0^{\phi_{\kappa,1}} (\cos \theta - \cos \phi_{\kappa,1})^{\frac{m}{m-1}} \sin \theta d\theta - \frac{\kappa s_{\kappa,1}^2}{2} + \frac{\kappa}{2} \\ &= \frac{\kappa s_{\kappa,1}}{m} \left(2\pi \left(\frac{\kappa s_{\kappa,1} (m-1)}{m} \right)^{\frac{1}{m-1}} \int_0^{\phi_{\kappa,1}} (\cos \theta - \cos \phi_{\kappa,1})^{\frac{m}{m-1}} \sin \theta d\theta \right) - \frac{\kappa s_{\kappa,1}^2}{2} + \frac{\kappa}{2} \\ &= \frac{\kappa s_{\kappa,1}}{m} (s_{\kappa,1} - \cos \phi_{\kappa,1}) - \frac{\kappa s_{\kappa,1}^2}{2} + \frac{\kappa}{2}, \end{aligned} \tag{E.83}$$

where in the last equality we used the identity obtained by subtracting from the second equation in (E.81), the first equation in (E.81) multiplied by $\cos \phi$ (use $s = s_{\kappa,1}$ and $\phi = \phi_{\kappa,1}$). Using (E.83), we calculate the difference between the two energies:

$$\begin{aligned} E[\rho_{\kappa,1}] - E[\rho_{\text{uni}}] &= \frac{\kappa s_{\kappa,1}}{m} (s_{\kappa,1} - \cos \phi_{\kappa,1}) - \frac{\kappa s_{\kappa,1}^2}{2} - \frac{(4\pi)^{1-m}}{m-1} \\ &= \kappa s_{\kappa,1} \left(\frac{s_{\kappa,1} - \cos \phi_{\kappa,1}}{m} - \frac{s_{\kappa,1}}{2} \right) - \frac{(4\pi)^{1-m}}{m-1} \\ &= \frac{1}{(2\pi)^{m-1} \left(\frac{m-1}{m} (1 - \cos \phi_{\kappa,1}) \right)^m} \left(\frac{1 - \cos \phi_{\kappa,1}}{2m-1} - \frac{m + (m-1) \cos \phi_{\kappa,1}}{2(2m-1)} \right) - \frac{(4\pi)^{1-m}}{m-1} \\ &= (4\pi)^{1-m} \left(\frac{2^{m-1}}{\left(\frac{m-1}{m} (1 - \cos \phi_{\kappa,1}) \right)^m} \left(\frac{(2-m) - (m+1) \cos \phi_{\kappa,1}}{2(2m-1)} \right) - \frac{1}{m-1} \right), \end{aligned}$$

where in the third equality we used (E.82a) and (E.82b).

Since at $\kappa = \kappa_c$, $E[\rho_{\kappa_c,1}] - E[\rho_{\text{uni}}] = 0$, we have

$$2^{m-1} (m-1) \left(\frac{(2-m) - (m+1) \cos \phi_c}{2(2m-1)} \right) = \left(\frac{m-1}{m} (1 - \cos \phi_c) \right)^m,$$

where we denoted $\phi_c := \phi_{\kappa_c,1}$. Therefore, we find that ϕ_c satisfies the equation

$$\frac{m-1}{4(2m-1)} ((2-m) - (m+1) \cos \phi_c) = \left(\frac{m-1}{m} \right)^m \left(\frac{1 - \cos \phi_c}{2} \right)^m. \tag{E.84}$$

Lemma E.1. Equation (E.84) has only one solution ϕ_c in $[0, \pi]$.

Proof. We denote $X = \frac{1-\cos\phi_c}{2}$, and write equation (E.84) as

$$I(X) := \frac{m-1}{4(2m-1)} (2(m+1)X - (2m-1)) - \left(\frac{m-1}{m}\right)^m X^m = 0.$$

The question reduces to show that the equation above has only one solution $X \in [0, 1]$. To use the intermediate value property, we calculate

$$I(0) = -\frac{m-1}{4}, \quad \text{and} \quad I(1) = \frac{3(m-1)}{4(2m-1)} - \left(\frac{m-1}{m}\right)^m.$$

Note that $I(0) < 0$. We will show $I(1) > 0$, which is equivalent to

$$J(m) := \frac{2m-1}{m-1} \left(\frac{m-1}{m}\right)^m < \frac{3}{4}.$$

From a simple calculation, we have

$$\frac{d}{dm} \log J(m) := \frac{2}{2m-1} + \log(m-1) - \log(m).$$

Also, the second derivative can be calculated and estimated as

$$\begin{aligned} \frac{d^2}{dm^2} \log J(m) &= -\frac{4}{(2m-1)^2} + \frac{1}{m-1} - \frac{1}{m} = -\frac{4}{4m^2-4m+1} + \frac{1}{m^2-m} \\ &> -\frac{4}{4m^2-4m} + \frac{1}{m^2-m} = 0. \end{aligned}$$

By the strict convexity of $\log J(m)$ together with

$$\lim_{m \rightarrow \infty} \frac{d}{dm} \log J(m) = 0,$$

we infer that $\frac{d}{dm} \log J(m) < 0$ for any $m > 2$. Finally, we can conclude that

$$\frac{3}{4} = J(2) > J(m), \quad \forall m > 2.$$

Therefore, we have $I(1) > 0$ and by the intermediate value property, there exists at least one solution $X \in [0, 1]$. The uniqueness of the solution comes directly from the convexity of $I(X)$ on $[0, 1]$, together with $I(0) < 0$ and $I(1) > 0$. □

Due to the previous lemma, we know that (E.84) has a unique solution ϕ_c , and by substituting this solution into (E.82b) we get

$$\kappa_c = \frac{2m-1}{(2\pi)^{m-1} (m + (m-1) \cos \phi_c) \left(\frac{m-1}{m} (1 - \cos \phi_c)\right)^m}.$$

Part 2. Comparison between κ_c and κ_2

We substitute $d=2$ into (4.31) to get

$$\begin{aligned} \kappa_2^{-1} &= H(-1) \\ &= \frac{m-1}{m} (2\pi)^{m-1} \left(\int_0^\pi (\cos \theta + 1)^{\frac{1}{m-1}} \sin \theta \cos \theta d\theta \right) \left(\int_0^\pi (\cos \theta + 1)^{\frac{1}{m-1}} \sin \theta d\theta \right)^{m-2} \\ &= \frac{m-1}{m} (2\pi)^{m-1} \left(\frac{2^{\frac{m}{m-1}} (m-1)}{m(2m-1)} \right) \left(\frac{m-1}{m} 2^{\frac{m}{m-1}} \right)^{m-2} \\ &= \frac{2^m (2\pi)^{m-1}}{2m-1} \left(\frac{m-1}{m} \right)^m, \end{aligned}$$

where we used two identities from integration by substitution ($a = \cos \theta + 1$):

$$\begin{aligned} \int_0^\pi (\cos\theta + 1)^{\frac{1}{m-1}} \sin\theta \cos\theta d\theta &= \int_0^2 a^{\frac{1}{m-1}} (a-1) da \\ &= \frac{m-1}{2m-1} 2^{\frac{2m-1}{m-1}} - \frac{m-1}{m} 2^{\frac{m}{m-1}} \\ &= \frac{2^{\frac{m}{m-1}} (m-1)}{m(2m-1)}, \end{aligned}$$

and

$$\int_0^\pi (\cos\theta + 1)^{\frac{1}{m-1}} \sin\theta d\theta = \int_0^2 a^{\frac{1}{m-1}} da = \frac{m-1}{m} 2^{\frac{m}{m-1}}.$$

Therefore, we get the explicit form of κ_2 :

$$\kappa_2 = \frac{2m-1}{2^m (2\pi)^{m-1}} \left(\frac{m}{m-1} \right)^m.$$

From the explicit forms of κ_2 and κ_c , the ratio between κ_c and κ_2 can be written as

$$\frac{\kappa_c}{\kappa_2} = \left(\frac{2}{1 - \cos\phi_c} \right)^m \times \frac{1}{m + (m-1)\cos\phi_c}.$$

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References

- [1] Ambrosio L, Gigli N and Savaré G 2005 *Gradient Flows in Metric Spaces and in the Space of Probability Measures (Lectures in Mathematics ETH Zürich)* (Birkhäuser Verlag)
- [2] Bernoff A J and Topaz C M 2011 A primer of swarm equilibria *SIAM J. Appl. Dyn. Syst.* **10** 212–50
- [3] Burger M, Fetecau R C and Huang Y 2014 Stationary states and asymptotic behavior of aggregation models with nonlinear local repulsion *SIAM J. Appl. Dyn. Syst.* **13** 397–424
- [4] Calvez V, Carrillo J and Hoffmann F 2017 Equilibria of homogeneous functionals in the fair-competition regime *Nonlinear Anal.* **159** 85–128
- [5] Carrillo J A, Craig K and Yao Y 2019 Aggregation-diffusion equations: dynamics, asymptotics and singular limits *Active Particles, Volume 2: Advances in Theory, Models and Applications (Modeling and Simulation in Science, Engineering and Technology)* ed N Bellomo P Degond and E Tadmor pp 65–108
- [6] Carrillo J A, Delgadino M and Patacchini F S 2019 Existence of ground states for aggregation-diffusion equations *Anal. Appl.* **17** 393–423
- [7] Carrillo J A, Fernández-Jiménez A and Gómez-Castro D 2024 Partial mass concentration for fast-diffusions with non-local aggregation terms *J. Diff. Equ.* **409** 700–73
- [8] Carrillo J A, Fetecau R C and Park H 2025 Existence of ground states for free energies on the hyperbolic space *Can. J. Math.* 1–43
- [9] Carrillo J A, Fetecau R C and Park H 2025 Global minimizers for fast diffusion versus nonlocal interactions on negatively curved manifolds (arXiv: 2503.19154)
- [10] Carrillo J A, Hittmeir S, Volzone B and Yao Y 2019 Nonlinear aggregation-diffusion equations: radial symmetry and long time asymptotics *Invent. Math.* **218** 889–977
- [11] Carrillo J A, Hoffmann F, Mainini E and Volzone B 2018 Ground states in the diffusion-dominated regime *Calc. Var. Partial Differ. Equ.* **57** 127
- [12] Carrillo J A, McCann R J and Villani C 2006 Contractions in the 2-Wasserstein length space and thermalization of granular media *Arch. Ration. Mech. Anal.* **179** 217–63
- [13] Chi D, Choi S-H and Ha S-Y 2014 Emergent behavior of a holonomic particle system on a sphere *J. Math. Phys.* **55** 052703
- [14] Constantin P, Kevrekidis I G and Titi E S 2004 Asymptotic states of a Smoluchowski equation *Arch. Ration. Mech. Anal.* **174** 365–84
- [15] Degond P, Frouvelle A and Liu J-G 2014 A note on phase transitions for the Smoluchowski equation with dipolar potential *Proc. Fourteenth Int. Conf. on Hyperbolic Problems: Theory, Numerics and Application (AIMS on Applied Mathematics vol 8)* ed F Ancona, A Bressan, P Marcati and A Marson (AIMS) pp 179–92
- [16] Delgadino M G, Yan X and Yao Y 2022 Uniqueness and nonuniqueness of steady states of aggregation-diffusion equations *Commun. Pure Appl. Math.* **75** 3–59
- [17] Fatkullin I and Slastikov V 2005 Critical points of the Onsager functional on a sphere *Nonlinearity* **18** 2565
- [18] Fetecau R C, Ha S-Y and Park H 2021 An intrinsic aggregation model on the special orthogonal group SO(3): well-posedness and collective behaviours *J. Nonlinear Sci.* **31** 74
- [19] Fetecau R C and Park H 2023 Equilibria and energy minimizers for an interaction model on the hyperbolic space *Physica D* **446** 133670

- [20] Fetecau R C and Park H 2023 Long-time behaviour of interaction models on Riemannian manifolds with bounded curvature *J. Geom. Anal.* **33** 218
- [21] Fetecau R C and Park H 2024 Aggregation-diffusion energies on Cartan–Hadamard manifolds of unbounded curvature *J. Geom. Anal.* **34** 356
- [22] Fetecau R C and Park H 2025 Ground states for aggregation–diffusion models on Cartan–Hadamard manifolds *J. London Math. Soc.* **111** e70079
- [23] Fetecau R C, Park H and Patacchini F S 2021 Well-posedness and asymptotic behaviour of an aggregation model with intrinsic interactions on sphere and other manifolds *Anal. Appl.* **19** 965–1017
- [24] Fetecau R C and Patacchini F S 2021 Well-posedness of an interaction model on Riemannian manifolds *Commun. Pure Appl. Anal.* **21** 3559–85
- [25] Fetecau R C and Zhang B 2019 Self-organization on Riemannian manifolds *J. Geom. Mech.* **11** 397–426
- [26] Frouvelle A and Liu J-G 2012 Dynamics in a kinetic model of oriented particles with phase transition *SIAM J. Math. Anal.* **44** 791–826
- [27] Geshkovski B, Letrouit C, Polyanskiy Y and Rigollet P 2025 A mathematical perspective on transformers *Bull. Am. Math. Soc.* **62** 427–79
- [28] Gradshteyn I S and Ryzhik I M 2014 *table of Integrals, Series and Products* 8th edn (Academic)
- [29] Guan S, Wang W and Zhao Q 2025 On stability of critical points for the high dimensional Onsager functional with Maier-Saupe potential *Commun. Math. Sci.* **23** 2319–34
- [30] Ha S-Y, Hwang S, Kim D, Kim S-C and Min C 2020 Emergent behaviors of a first-order particle swarm model on the hyperboloid *J. Math. Phys.* **61** 042701
- [31] Ha S-Y, Kang M and Kim D 2022 Emergent behaviors of high-dimensional Kuramoto models on Stiefel manifolds *Automatica* **136** 110072
- [32] Ha S-Y, Ko D and Ryoo S W 2018 On the relaxation dynamics of Lohe oscillators on some Riemannian manifolds *J. Stat. Phys.* **172** 1427–78
- [33] Holm D D and Putkaradze V 2006 Formation of clumps and patches in self-aggregation of finite-size particles *Physica D* **220** 183–96
- [34] Huang H, Liu J-G and Lu J 2019 Learning interacting particle systems: diffusion parameter estimation for aggregation equations *Math. Models Methods Appl. Sci.* **29** 1–29
- [35] Kaib G 2017 Stationary states of an aggregation equation with degenerate diffusion and bounded attractive potential *SIAM J. Math. Anal.* **49** 272–96
- [36] Keller E F and Segel L A 1970 Initiation of slime mold aggregation viewed as an instability *J. Theor. Biol.* **26** 399–415
- [37] Liu H, Zhang H and Zhang P 2005 Axial symmetry and classification of stationary solutions of Doi-Onsager equation on the sphere with Maier-Saupe potential *Commun. Math. Sci.* **3** 201–18
- [38] Lohe M 2009 Non-Abelian Kuramoto models and synchronization *J. Phys. A: Math. Theor.* **42** 395101
- [39] Bongini M, Fornasier M, Hansen M and Maggioni M 2017 Inferring interaction rules from observations of evolutive systems I: the variational approach *Math. Models Methods Appl. Sci.* **27** 909–51
- [40] Maggioni M, Miller J, Qiu H and Zhong M 2021 Learning interaction kernels for agent systems on Riemannian manifolds *Proc. 38th Int. Conf. on Machine Learning* vol 139 pp 7290–300
- [41] Markdahl J, Thunberg J and Gonçalves J 2018 Almost global consensus on the n -sphere *IEEE Trans. Autom. Control* **63** 1664–75
- [42] Motsch S and Tadmor E 2014 Heterophilious dynamics enhances consensus *SIAM Rev.* **56** 577–621
- [43] Olfati-Saber R 2006 Swarms on sphere: a programmable swarm with synchronous behaviors like oscillator networks *Proc. 45th IEEE Conf. on Decision and Control* pp 5060–6
- [44] Onsager L 1949 The effects of shape on the interaction of colloidal particles *Ann. New York Acad. Sci.* **51** 627–59
- [45] Patlak C S 1953 Random walk with persistence and external bias *Bull. Math. Biophys.* **15** 311–38
- [46] Rotskoff G and Vanden-Eijnden E 2022 Trainability and accuracy of artificial neural networks: an interacting particle system approach *Commun. Pure Appl. Math.* **75** 1889–935
- [47] Topaz C M, Bertozzi A L and Lewis M A 2006 A nonlocal continuum model for biological aggregation *Bull. Math. Biol.* **68** 1601–23
- [48] Wu L and Slepčev D 2015 Nonlocal interaction equations in environments with heterogeneities and boundaries *Commun. Partial Differ. Equ.* **40** 1241–81