HW 1 - Solutions:

P1 (folium of Descartes) $\quad \alpha : (-1, \infty) \to \mathbb{R}^2$

$$\alpha(t) = \left(\frac{3at}{1+t^3}, \ \frac{3at^2}{1+t^3} \right)$$

a) $$\alpha'(t) = 3a \left(\frac{1}{1+t^3} - \frac{t}{(1+t^3)^2} \cdot 3t^2, \ \frac{2t}{1+t^3} - \frac{t^2}{(1+t^3)^2} \cdot 3t^2 \right)$$

$$= 3a \left(\frac{1+t^3 - 3t^3}{(1+t^3)^2}, \ \frac{2t + 2t^4 - 3t^4}{(1+t^3)^2} \right) =$$

$$= 3a \left(\frac{1-2t^3}{(1+t^3)^2}, \ \frac{2t - t^4}{(1+t^3)^2} \right)$$

$\alpha'(0) = 3a(1, 0)$ $\quad$ - tangent to the x axis.

b) as $t \to \infty$ it is clear that $\alpha(t) \to (0,0)$, $\alpha'(t) \to (0,0)$

c) at $t \to -1$ calculate $x(t) + y(t) + a$:

$$\lim_{t \to -1} \left[x(t) + y(t) + a \right] = \lim_{t \to -1} \left[\frac{3at}{1+t^3} + \frac{3at^2}{1+t^3} + a \right] =$$

$$= \lim_{t \to -1} \frac{3at\cancel{(t+1)}}{\cancel{(t+1)}(t^2 - t + 1)} + a = \frac{-3a}{3} + a = 0. \quad \Rightarrow \text{ as } t \to -1 \text{ the}$$

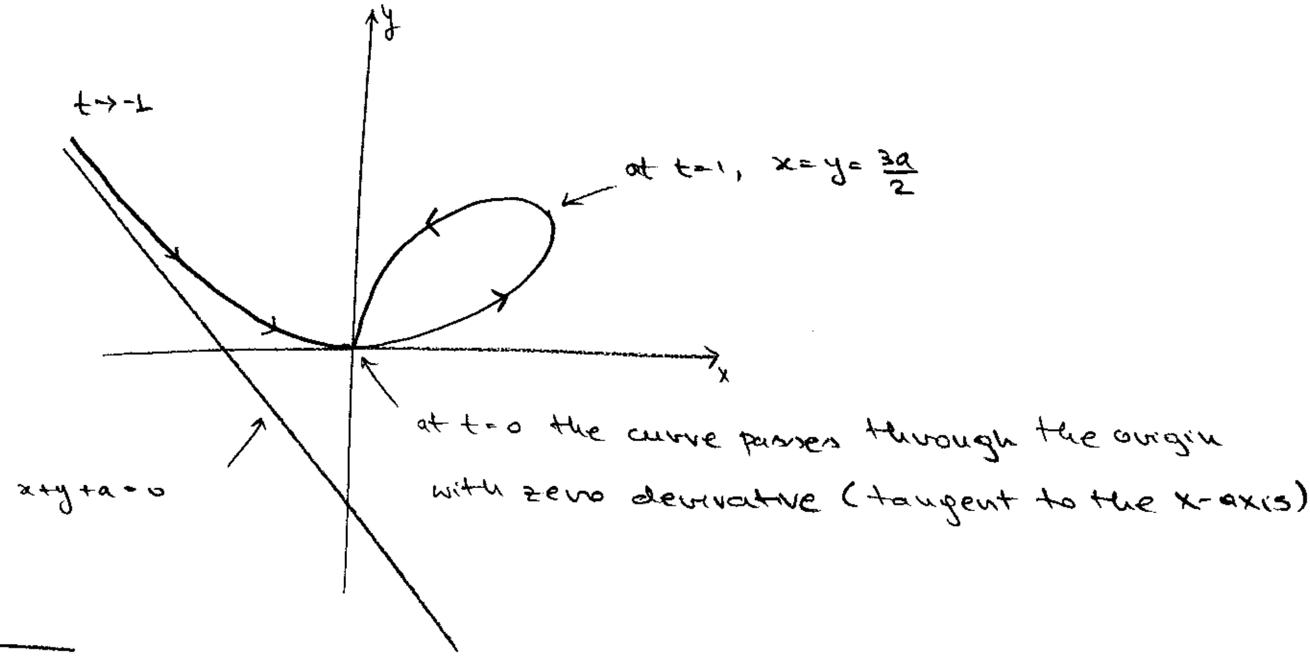
curve approaches the line $x + y + a = 0$

For $\alpha'(t)$: note that $|\alpha'(t)| \longrightarrow \infty$ as $t \to -1$, but

the direction: $\dfrac{y'(t)}{x'(t)} = \dfrac{2t - t^4}{1 - 2t^3}$ approaches (-1) as $t \to -1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\|$ the slope of $x + y + a = 0$

$\Rightarrow$ asymptotic convergence as $t \to -1$.



$\boxed{P2}$ $\quad \vec{\beta}(s) = \vec{\alpha}(s) + \frac{1}{\kappa(s)} \vec{n}(s)$

For a plane curve $\quad \tau = 0, \ \vec{b} = \vec{e}_3$

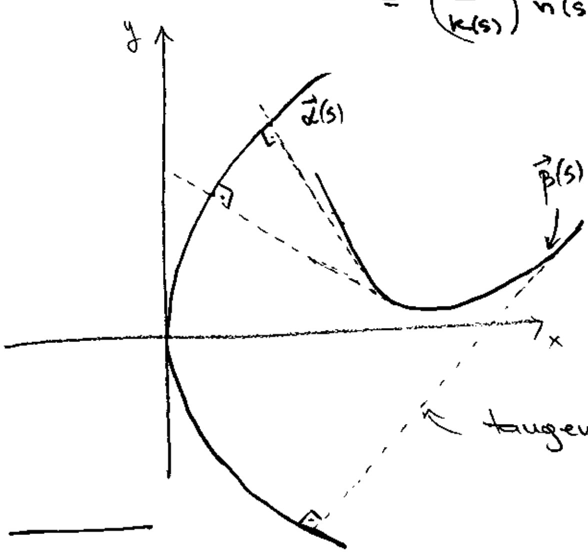
Frenet - Serre system becomes: $\begin{cases} \vec{t}\,' = \kappa \vec{n} \\ \vec{n}\,' = -\kappa \vec{t} \end{cases}$

(a) Compute $\vec{\beta}'(s)$:

$$\vec{\beta}'(s) = \vec{\alpha}'(s) + \frac{1}{\kappa(s)} \vec{n}'(s) + \left(\frac{1}{\kappa(s)}\right)' \vec{n}(s)$$

$$= \vec{t}(s) + \frac{1}{\kappa(s)} \left(-\kappa(s) \vec{t}(s)\right) + \left(\frac{1}{\kappa(s)}\right)' \vec{n}(s)$$

$$= \left(\frac{1}{\kappa(s)}\right)' \vec{n}(s)$$ is in the direction of the principal normal, hence normal to $\vec{\alpha}$



tangents to $\vec{\beta}$ are normal to $\vec{\alpha}$!
