HW #2 - Solutions

P1 Use the Frenet - Serret equations for $\vec{t}(s) = \vec{r}'(s)$, $\vec{n}(s) = \dfrac{\vec{r}''(s)}{\kappa(s)}$

$\vec{b} = \vec{t} \times \vec{n}$

$$\begin{cases} \vec{t}' = \kappa \vec{n} \\ \vec{n}' = -\kappa \vec{t} - \tau \vec{b} \\ \vec{b}' = \tau \vec{n} \end{cases}$$

Now, $\vec{N}(s,v) = \dfrac{\partial \vec{r}}{\partial s} \times \dfrac{\partial \vec{r}}{\partial v} \Big/ \left\| \dfrac{\partial \vec{r}}{\partial s} \times \dfrac{\partial \vec{r}}{\partial v} \right\|$

$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial v} = \left[\vec{t} + r\left(\cos v\, \vec{n}' + \sin v\, \vec{b}'\right)\right] \times \left[r\left(-\sin v\, \vec{n} + \cos v\, \vec{b}\right)\right]$$

$$= -r\sin v \underbrace{\vec{t} \times \vec{n}}_{\vec{b}} + r\cos v \underbrace{\vec{t} \times \vec{b}}_{-\vec{n}} - r^2 \sin v \cos v\, \vec{n}' \times \vec{n} + r^2 \cos^2 v\, \vec{n}' \times \vec{b}$$

$$- r^2 \sin^2 v \underbrace{\vec{b}' \times \vec{n}}_{0} + r^2 \sin v \cos v\, \vec{b}' \times \vec{b}$$

$$= -r\sin v\, \vec{b} - r\cos v\, \vec{n} - r^2 \sin v \cos v\, (-\kappa \vec{t} - \tau \vec{b}) \times \vec{n} +$$

$$+ r^2 \cos^2 v\, (-\kappa \vec{t} - \tau \vec{b}) \times \vec{b} + r^2 \sin v \cos v\, \tau\, \vec{n} \times \vec{b}$$

$$= -r\sin v\, \vec{b} - r\cos v\, \vec{n} + r^2 \kappa \sin v \cos v\, \vec{b} - r^2 \tau \cancel{\sin v} \cos v\, \vec{t}$$

$$+ r^2 \cos^2 v\, \kappa\, \vec{n} + r^2 \tau \cancel{\sin v} \cos v\, \vec{t}$$

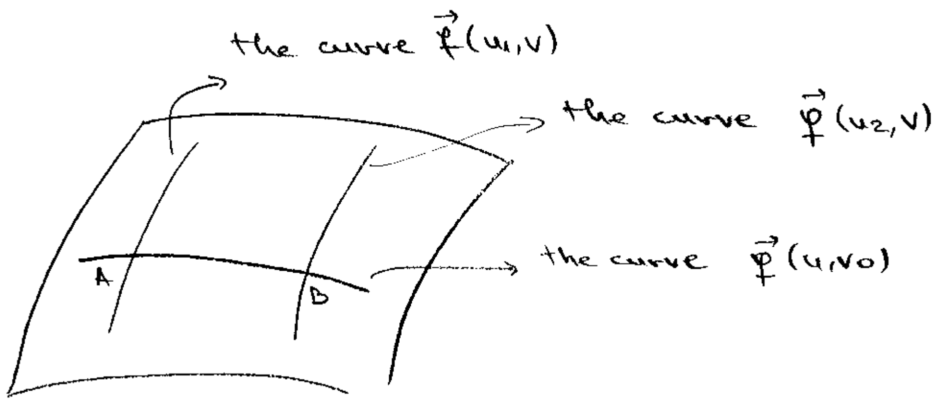
$$= -r\sin v\,(1 - r\kappa \cos v)\,\vec{b} - r\cos v\,(1 - r\kappa\cos v)\,\vec{n}$$

$$= r(1 - r\kappa\cos v)(-\sin v\, \vec{b} - \cos v\, \vec{n})$$

After normalization, since $\left\| \dfrac{\partial \vec{r}}{\partial s} \times \dfrac{\partial \vec{r}}{\partial v} \right\| = |r(1 - r\kappa\cos v)|$

we get the result. (up to a $\pm$ sign)

P2



We have to show that the length of AB does not depend on v_0.

$\vec{f}(u,v_0)$ is a curve parametrized by u: $\begin{cases} x = u\cos v_0 \\ y = u \sin v_0 . \\ z = \log(\cos v_0) + u \end{cases}$

The length of AB is:

$$\mathcal{L}(AB) = \int_{u_1}^{u_2} \| \vec{f}_u(u,v_0) \| \, du = \int_{u_1}^{u_2} \underbrace{\| (\cos v_0, \sin v_0, 1) \|}_{\substack{\| \\ (\cos^2 v_0 + \sin^2 v_0 + 1)^{1/2} = \sqrt{2}}} \, du = \sqrt{2}\,(u_2 - u_1)$$

Indeed, $\mathcal{L}(AB)$ does not depend on v_0! (qed)