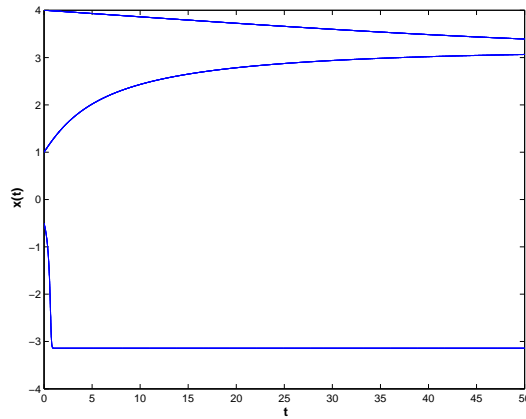


Solutions hw 1

2.2.4

The fixed points for $\dot{x} = \exp(-x)\sin(x)$ are $x^* = n\pi$, where n is an integer. When $n = \dots -4, -2, 0, 2, 4, \dots$, x^* is unstable, while when $n = \dots -3, -1, 1, 3, \dots$, x^* is stable. The graph below shows the solution $x(t)$ for $x(0) = -0.5, 1$ and 4 .



2.2.13

Part a) Write $\dot{x} = \frac{dx}{dt}$, then we need to solve

$$\frac{mdv}{mg - kv^2} = dt.$$

Notice that

$$\frac{m}{mg - kv^2} = \frac{1}{2g} \left(\frac{1}{1 - \sqrt{\frac{k}{mg}}v} + \frac{1}{1 + \sqrt{\frac{k}{mg}}v} \right), \quad (\text{a little tricky})$$

we get solution

$$\sqrt{\frac{m}{4gk}} \left(\log \frac{1 + \sqrt{\frac{k}{mg}}v}{1 - \sqrt{\frac{k}{mg}}v} \right) = t + C.$$

Putting in the condition $v(0) = 0$ we get $C = 0$, therefore the analytical solution is

$$v = \frac{rm}{k} \left(\frac{e^{rt} - e^{-rt}}{e^{rt} + e^{-rt}} \right), \quad \text{where} \quad r = \sqrt{gk/m}.$$

As this course does not emphasize on solving ODE, you can just solve it using some math software such as Matlab, Mathematica, etc.

Part b) When $t \rightarrow \infty$, both e^{-rt} terms in the above vanish and the big fraction becomes 1. The limit is whatever remained which turns out to be $(rm)/k = \sqrt{mg/k}$.

Part c) Now we solve it geometrically, the equation can be written as

$$\dot{v} = g - (k/m)v^2,$$

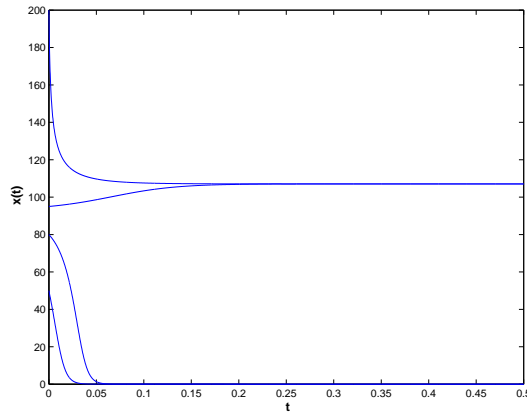
and we set it equal to 0. The graph of \dot{v} versus v is a parabola crossing the x -axis from the above. The terminal velocity is the stable fixed point $v = \sqrt{mg/k}$.

2.3.4

Part a) The effect growth rate is at its highest ($= r$) when $N = b$. If N is either too high or too low, then the effect growth rate will be negative.

Part b) Fixed points are $N_+^* = b + \sqrt{\frac{r}{a}}, N_-^* = b - \sqrt{\frac{r}{a}}$ (provided that $b > \sqrt{\frac{r}{a}}$) and $N_0 = 0$. Here N_+^* and N_0 are stable while N_-^* is unstable.

Part c) The graph below shows the solution $N(t)$ for $N(0) = 50, 80, 95$ and 200 for $r = 1, a = 0.02, b = 100$.



Part d) Note that when $N(0) > N_-^*$ the behaviour of $N(t)$ will be the same as the the solution of the logistic equation (approaches a non-zero fixed point). The different here is that when $N(0) < N_-^*$, then $N(t) \rightarrow 0$.

2.4.2

The fixed points are 0, 1 and 2. Since $f'(x) = x(x-1) + x(x-2) + (x-1)(x-2)$, we have

$$f'(0) = 2, \quad 0 \text{ is unstable,}$$

$$f'(1) = -1, \quad 1 \text{ is stable,}$$

$$f'(2) = 2, \quad 2 \text{ is unstable.}$$

2.4.8

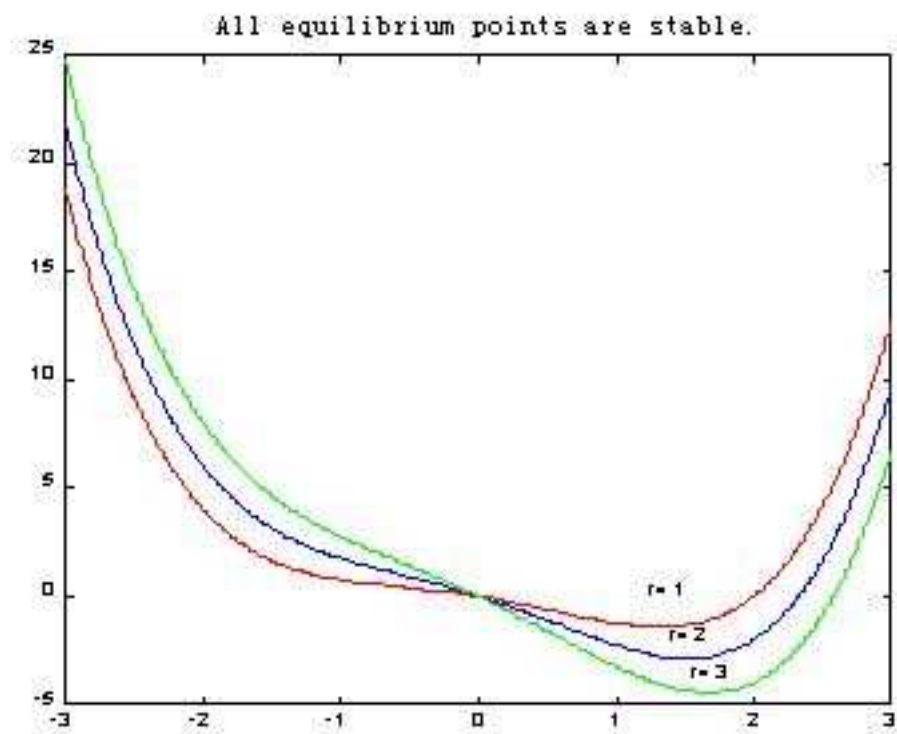
Letting $\dot{N} = 0$ we get $N = 1/b$. Taking derivative:

$$f'(N) = -a \ln(bN) - \frac{a}{b},$$

then $f'(1/b) = -\frac{a}{b} < 0$, $1/b$ is stable.

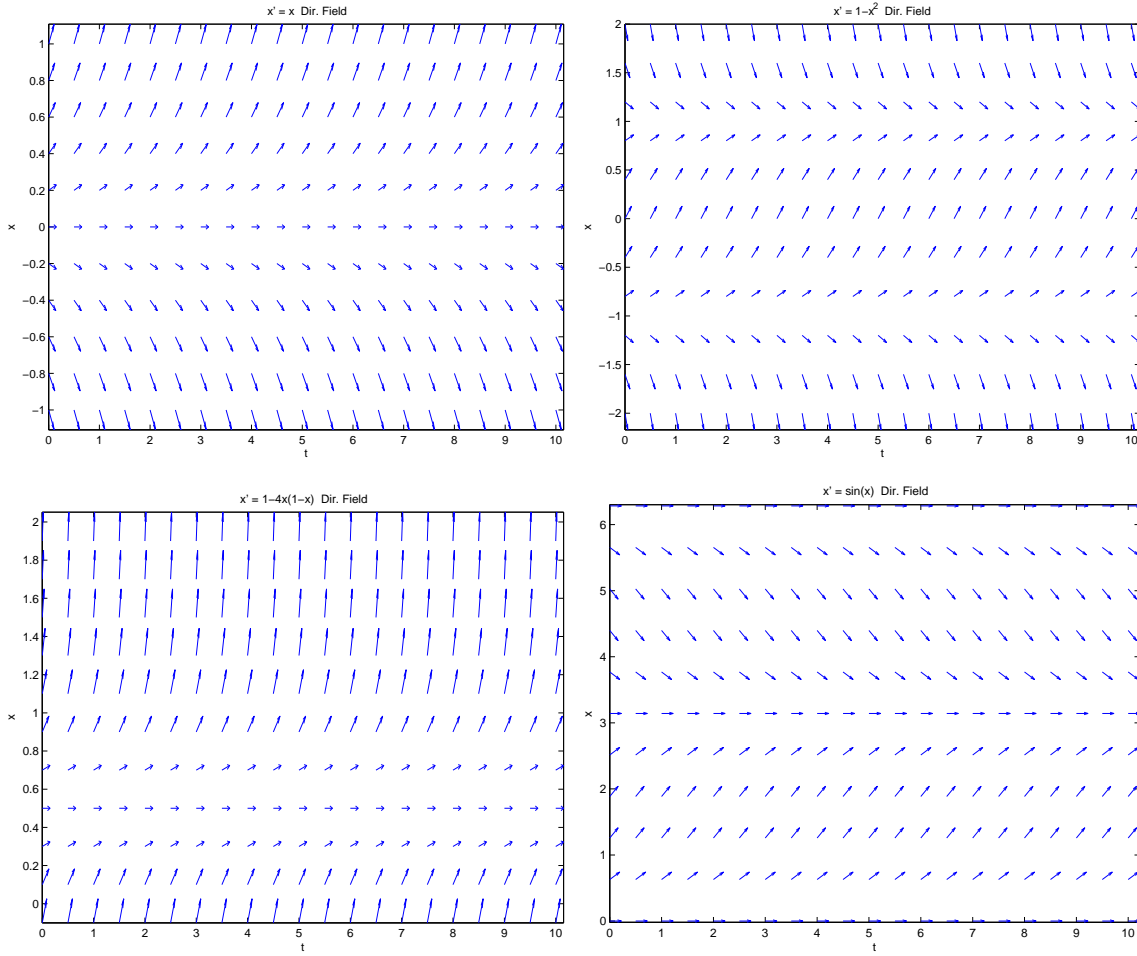
2.7.6

Similar to the above $V(x) = -rx - \frac{1}{2}x^2 + \frac{1}{4}x^4$. Graphs of $V(x)$ for some r values are shown in the figure. The equilibrium points are the local minima.



2.8.2

Plots of the slope fields for a) $\dot{x} = x$ (top left), b) $\dot{x} = 1 - x^2$ (top right), c) $\dot{x} = 1 - 4x(1 - x)$ (bottom right) and d) $\dot{x} = \sin(x)$ (bottom left).



2.8.3

Part a) The solution for $\dot{x} = -x$, $x(0) = 1$ is $x(t) = \exp(-t)$ and the exact value for $x(1)$ is e^{-1} .

Part b & c) Left: The solution found using Euler method with step size $\Delta t = 0.01$. Right: Log-log plot of the error E is a function of Δt (solid lines). Note that the dotted line represents the plot of Δt^{-1} due to the fact that the rate of convergence for Euler method is first order.

