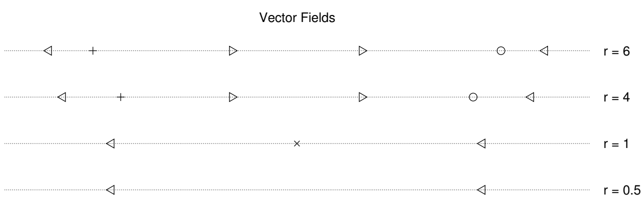
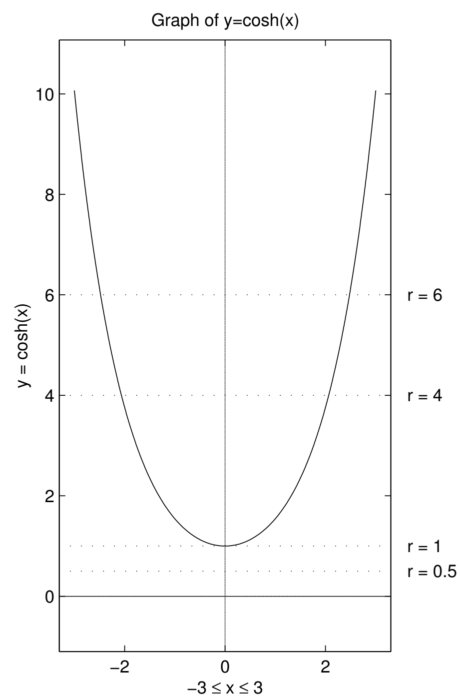
Solutions 2

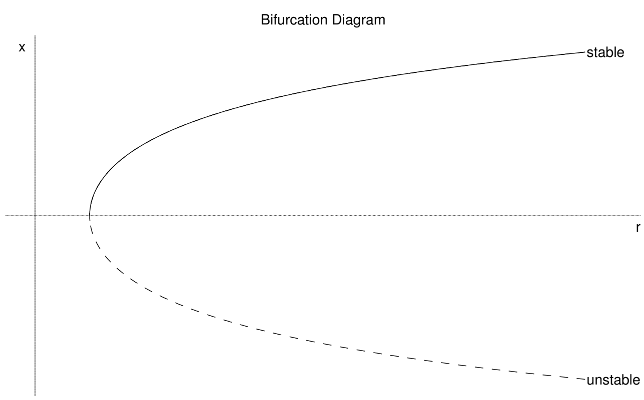
3.1.2

The graph of the function $y = \cosh x$ is shown below on the right, with dotted lines indicating the values of r. It's then clear that x moves to the right when y is below r and vice versa. The vector fields are sketched as follows



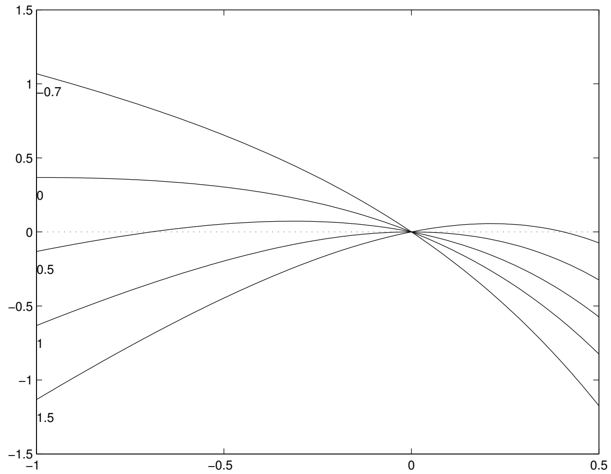


A saddle-node bifurcation occurs at $r = 1$.

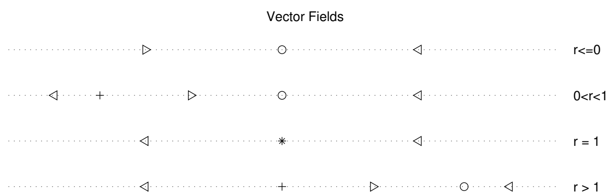


3.2.4

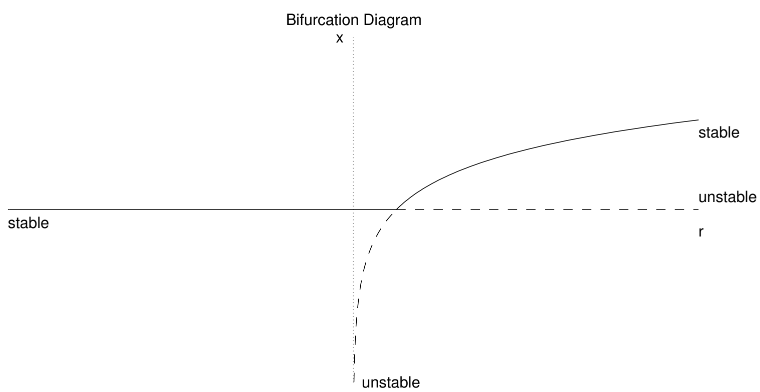
First we plot the graph of $\dot{x}$ versus x for various r and get the following picture:



It's then clear that the vector fields can be described qualitatively as follows:



Now we can draw the bifurcation diagram

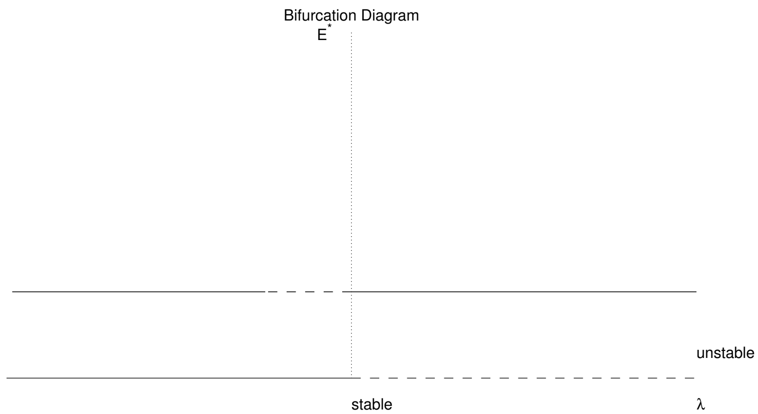


3.3.2

Part a) Assume that $\dot{P} \approx 0, \dot{D} \approx 0$ then, to first order, $ED = P$ and $\lambda + 1 - \lambda EP = D$. Substitute the $D = \frac{E}{P}$ into the second equation we will have $P = \frac{(\lambda+1)E}{1+\lambda E^2}$. Since $\dot{E} = \kappa(P - E)$, the evolution equation of E is thus $E = \frac{\lambda\kappa(1-E^2)E}{1+\lambda E^2}$.

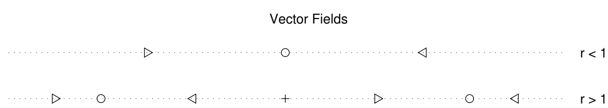
Part b) The fixed points of E are $E^* = 0$ and $E^* = 1$.

Part c) Bifurcation diagram.

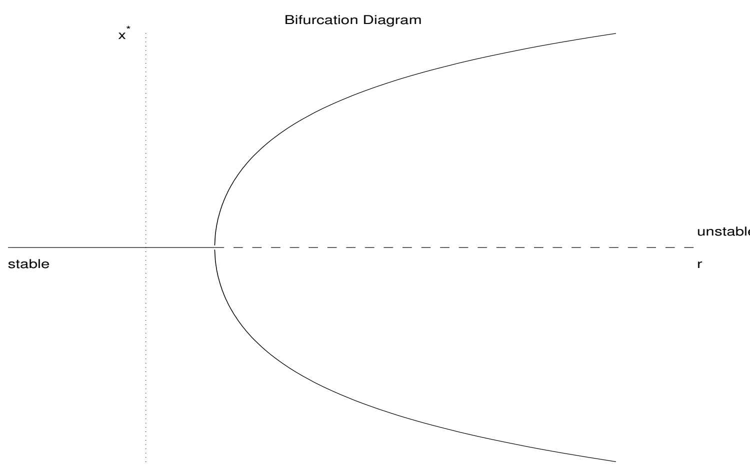


3.4.2

Fixed points of x and r are related by $r = \frac{\sinh x}{x}$, its graph shows that the critical value is $r = 1$. There are two qualitatively different vector fields:

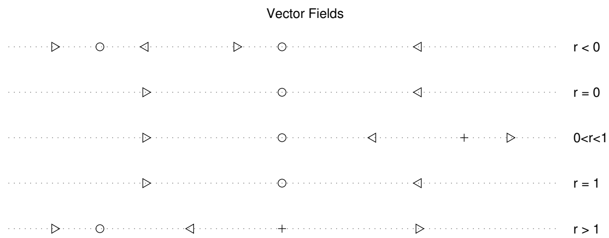


It's readily seen that 0 changes from stable to unstable after r passes critical value 1 and two more stable fixed points are created. This is a pitchfork bifurcation, it's supercritical.

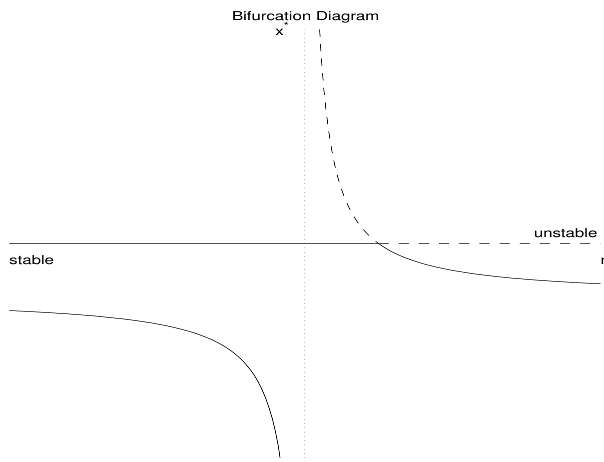


3.4.6

We start by solving equation $rx = x/(1+x)$, 0 is always a solution and another solution is given by $x = 1/r - 1$ as long as r is nonzero. The critical values for r are 0 and 1, representing the cases in which there is only one fixed point. The vector fields can be sketched as follows

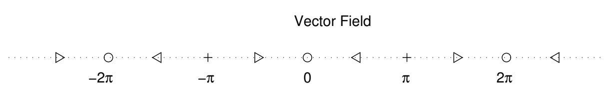


At $r = 1$ two fixed points changed their types, this is a transcritical bifurcation.

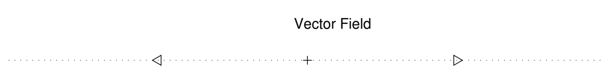


3.4.11

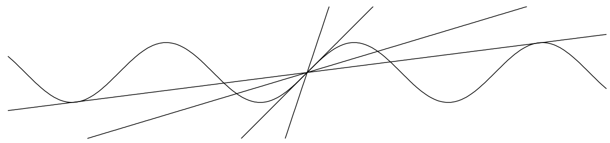
Part a) Fixed points are integral multiples of π.



Part b) When $r > 1$, the absolute value of x is always greater than the absolute value of $\sin x$ unless $x = 0$, which is the only fixed point. The derivative of $(rx - \sin x)$ is $r - 1$ at point $x = 0$, since it's positive $x = 0$ is unstable.



Part c) As r decreases, the graph of $y = rx$ has more intersections with the graph of $y = \sin x$, i.e. more fixed points are created. At an intersection point $x = c$, if $(y = \sin x)$ crosses $(y = rx)$ from the below, then $x = c$ is unstable, and vice versa. When $(y = rx)$ touches $(y = \sin x)$ at a new point, a bifurcation occurs, and after the bifurcation the smaller fixed point will be unstable.



Notice that 0 is always a fixed point and it changes from unstable to stable as r passes 1. We conclude that when r decreases from ∞ to 0, there is a subcritical pitchfork bifurcation at $r = 1$ and saddle-node bifurcations when $0 < r < 1$.

Part d) When $r \ll 1$, $y = rx$ touches $y = \sin x$ at approximately the peaks of its graph, i.e. $x = \frac{\pi}{2} + 2k\pi$, where k is a positive integer. Therefore bifurcations occur near $r = \frac{2\pi}{4k+1}$.

Part e) When r further decreases, two loci of fixed points will merge and vanish, which is clear if you stare the figure in part c for a while. These are also saddle-node bifurcations, shown below.