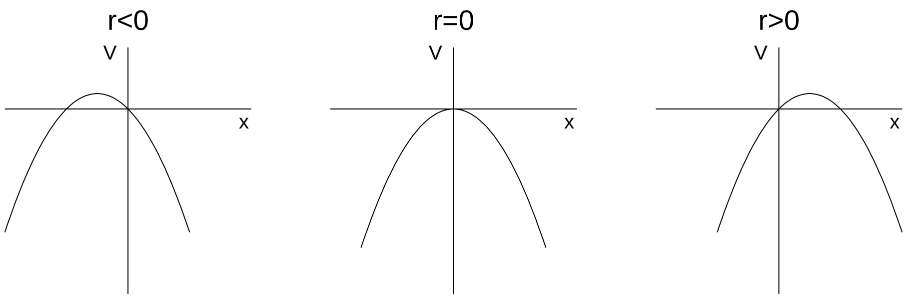
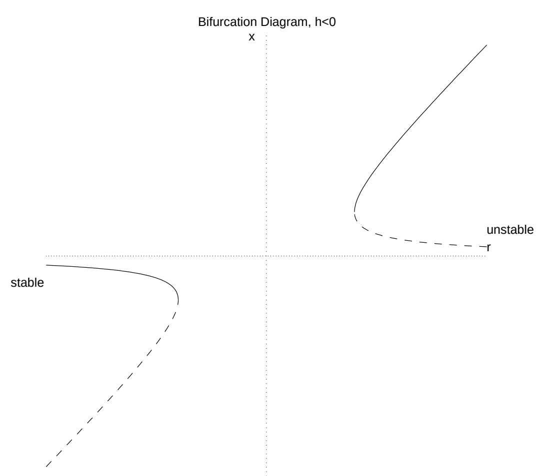
Solutions 3

3.6.2

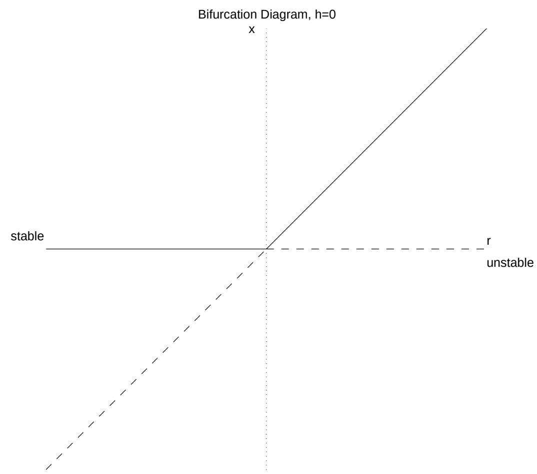
When $h = 0$ this system is the same as the one in Section 3.2. As h varies, the curves in the following pictures move vertically.



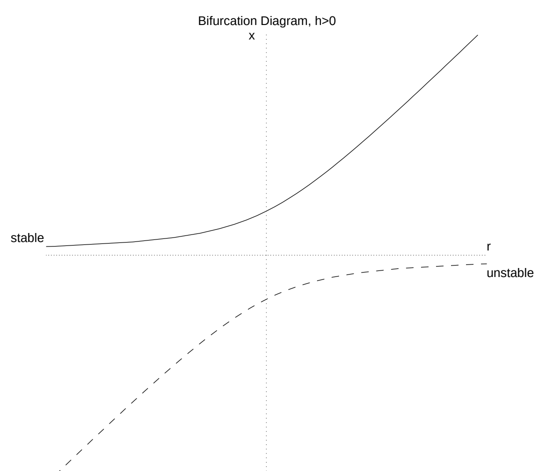
Part a) If $h < 0$, the above curves are shifted down. This could affect the number of fixed points when r is close to 0. In the bifurcation diagram below, the system has no fixed point in the middle.



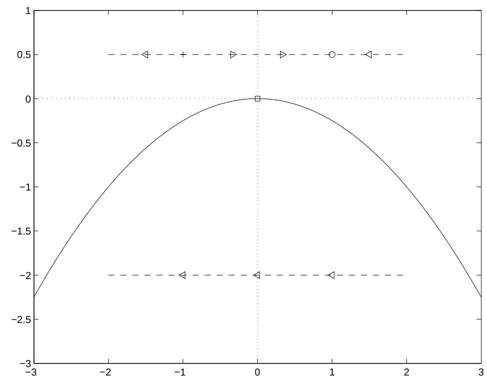
If $h = 0$, this system has a transcritical bifurcation, please see Section 3.2.



If $h > 0$, the curves are shifted up, there will always be two zeros. Notice that the curves crosse the x-axis from below at the smaller fixed point, so it's unstable.

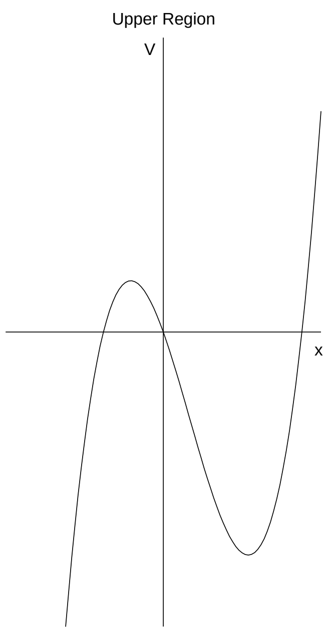


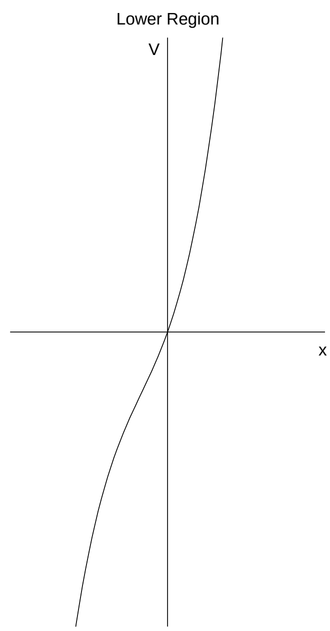
Part b) Bifurcation occurs precisely if $(h + rx - x^2)$ has only one solution. The quadratic has discriminant $(r^2 + \frac{1}{4}h)$, which distinguishes qualitatively different vector fields. When it's positive, the quadratic has two solutions, of which the smaller one is an unstable fixed point; when it's 0, bifurcation occurs; when it's negative, there's no fixed point.



Transcritical

Part c) There are two regions: the region above the curve $h = -\frac{1}{4}r^2$, and the region below it.





3.7.3

Part a) First let $x = \frac{N}{K}$, then the system becomes

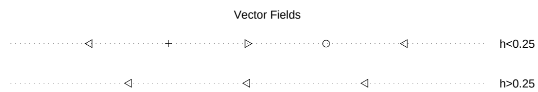
$$\frac{d(Kx)}{dt} = rKx(1-x) - H,$$

notice that $d(Kx) = Kdx$, (rK) is a constant,

$$\frac{1}{r}\frac{dx}{dt} = x(1-x) - \frac{H}{rK}.$$

Let $\tau = rt$, $h = \frac{H}{rK}$, we get the desired dimensionless form.

Part b) The maximum of $x(1-x)$ is $\frac{1}{4}$, thus $x(1-x)-h$ can be viewed as a parabola $y=x(1-x)$ intersecting $y=h$. The vector fields are



Part c) This is equivalent to solving $x(1-x)-h=0$, which can be done either by looking at the graph of $y=x(1-x)$, or by writing the above as

$$(x-\frac{1}{2})^2=\frac{1}{4}-h.$$

It's then clear that there are two fixed points when $h<\frac{1}{4}$ and no fixed points when $h>\frac{1}{4}$. The critical value $h_c=\frac{1}{4}$ is a saddle-node bifurcation.

Part d) If $h<h_c$, let x_u be the unstable fixed point and x_s the stable one. To avoid the model's silliness mentioned at the end of the problem, we assume $x_u>0$. $h<h_c$ means the amount of fishing is moderate. If initially the fish population is small ($x<x_u$), then eventually there will be no fish left. If $x>x_u$, the population will stabilize at x_s, i.e. both the fish and the fishermen are happy. If $h>h_c$, then the fishermen are asking too much, eventually there will be no fish left.

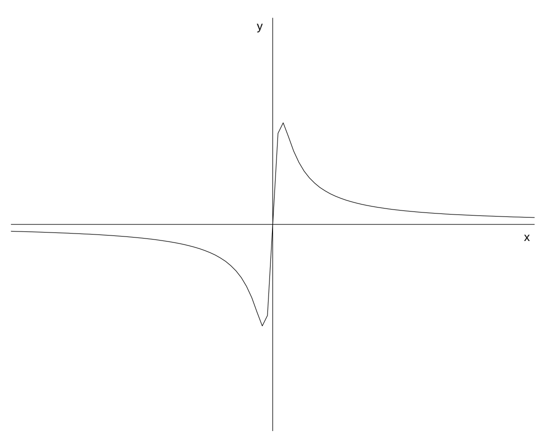
3.7.5

Part a) First take $g=k_4x$, the system becomes

$$k_4\dot{x}=k_1s_0-k_2k_4x+k_3\frac{x^2}{1+x^2},$$

then let $s=\frac{k_1s_0}{k_3}$, $r=\frac{k_2k_4}{k_3}$ and $\tau=\frac{k_3}{k_4}t$.

Part b) If $s=0$, the two fixed points correspond to the solutions of $\frac{x}{1+x^2}=r$. The graph of function $y=\frac{x}{1+x^2}$ looks like

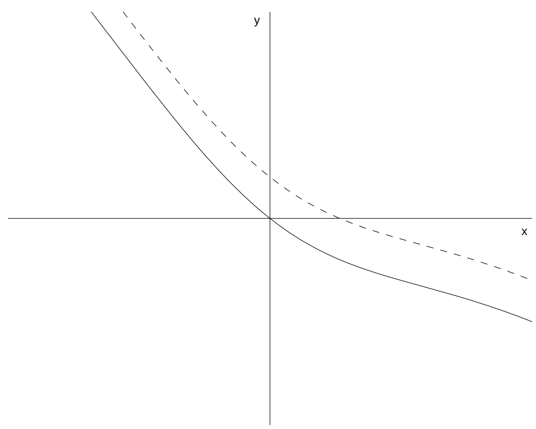


When $r > 0$ and r small, there are always two points on $y = r$. To determine r_c, we can look at the maximum of y. Since

$$y' = \frac{1 - x^2}{(1 + x^2)^2},$$

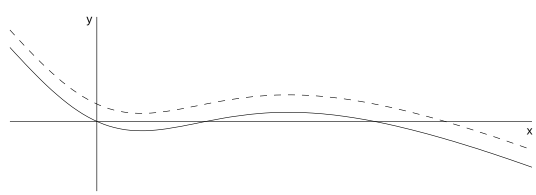
the maximum is achieved at $x = 1$, thus $r_c = \frac{1}{2}$.

Part c) If $r > r_c$,the graph of $y = -rx + \frac{x^2}{1+x^2}$ looks like the solid line in the picture below



When s increases, the graph becomes the dashed line. If we start at $g(0) = 0$, then g increases as t increases. Now let s go back to 0, then we go back to the solid line, and g will also return to 0 since it is the only fixed point and it's stable.

If $r < r_c$, we have the graphs



As s increases, the gene product g begins to accumulate. If s returns to 0 shortly after it takes off, then x has a value between 0 and the smaller positive fixed point, which is unstable. Therefore x will go back to 0 again, i.e. the gene will turn off. But if s stays positive long enough, so that x can accumulate until it exceeds the smaller fixed point, then even s goes back to 0, g will still be pushed to the larger fixed point, i.e. the gene is switched on.

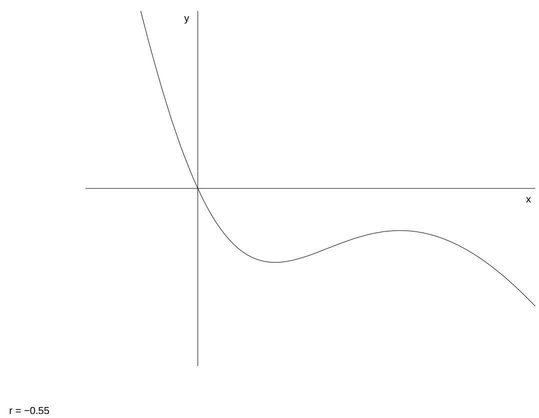
Part d) If $r >> r_c$, there will always be one stable fixed point, no bifurcation occurs. If $r < r_c$, notice that in the above graph, $f(x) = -rx + \frac{x^2}{1+x^2}$ has a minimum $f_{\min}$, a saddle-node bifurcation will occur at $s = -f_{\min}$. To get the parametric form of r and s, recall that

f achieves its minimum when

$$f'(x) = -\frac{r + 2rx^2 + rx^4 - 2x}{(1+x^2)^2} = 0.$$

Therefore $r = \frac{2x}{(1+x^2)^2}$, and $s = -f_{\min} = \frac{x^2(1-x^2)}{(1+x^2)^2}$.

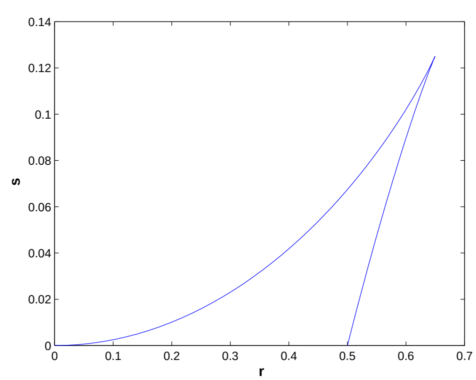
The interesting case is when $r \geq r_c$ but not too large, which corresponds to the figure below



At $s = 0$ there is only one fixed point $x = 0$, but as s increases, there will be three fixed points.

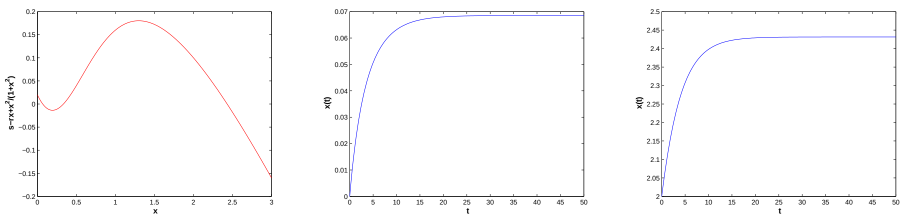
All the bifurcations in this part are saddle-node bifurcations.

Part e) The graph of (r, s) looks as in the figure below:



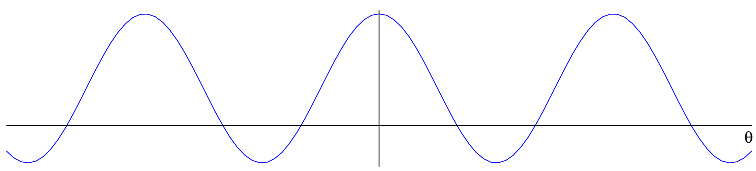
The curve parameter x ranges from $x = 0$ (corresponding to point $(0, 0)$ on the curve) to $x = 1$ (that corresponds to $(0.5, 0)$). The cusp occurs at $x = 1/\sqrt{3}$, where both $r'(x)$ and $s'(x)$ vanish. Both r and s increase on $(0, 1/\sqrt{3})$ and decrease on $(1/\sqrt{3}, 1)$.

For the additional computational part, below are some plots for $r = 0.36$ and $s = 0.02$, which correspond to the region in the parameter space that has 3 fixed points. The right-hand-side of the equation (the velocity) is plotted on the left. Depending on the initial value $x(0)$ the solution approaches the fixed point near 0 (centre plot) or the fixed point near 2.4 (right plot).

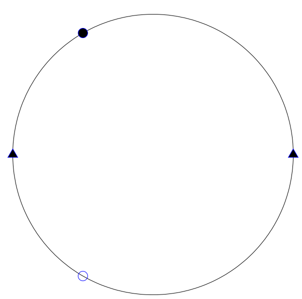


4.1.2

The graph of $\dot{\theta} = 1 + 2\cos(\theta)$ looks like



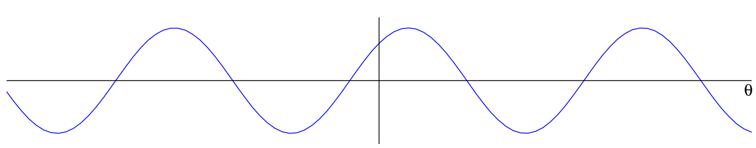
We can then draw its phase portrait



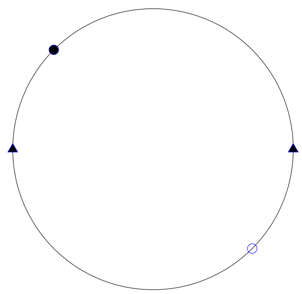
The stable fixed point is $\theta^* = \frac{2\pi}{3}$ while the unstable fixed point is $\theta^* = \frac{4\pi}{3}$.

4.1.5

The graph of $\dot{\theta} = \sin(\theta) + \cos(\theta)$ looks like



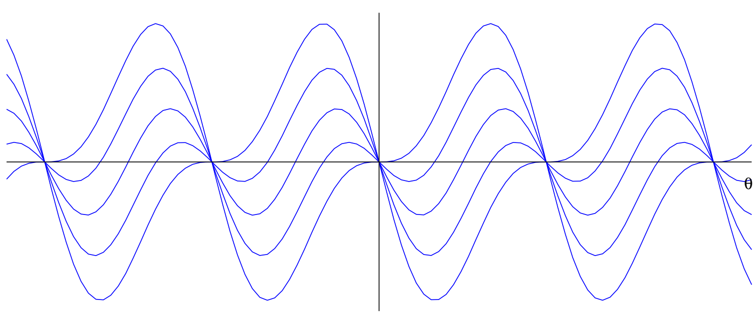
We can then draw its phase portrait



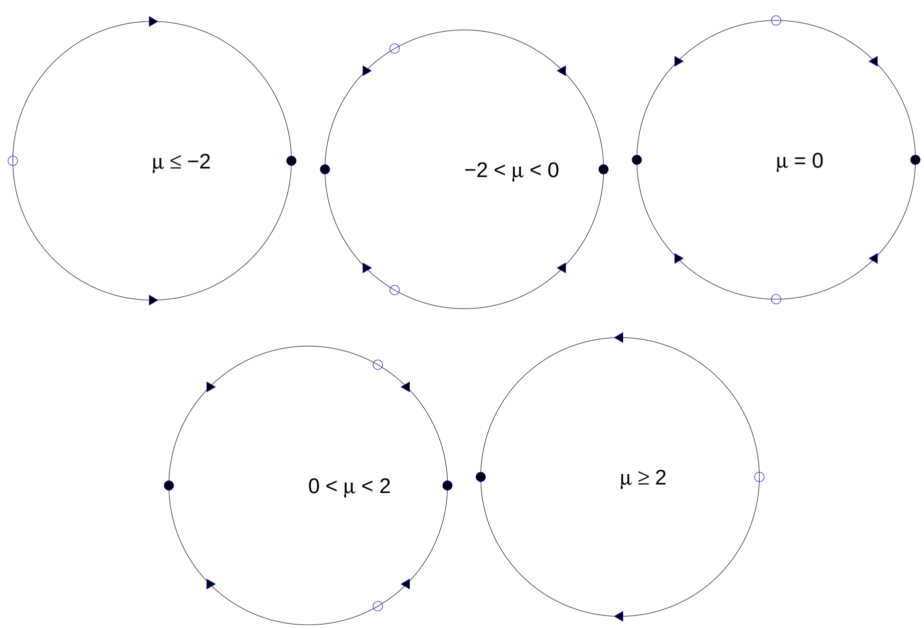
The stable fixed point is $\theta^* = \frac{3\pi}{4}$ while the unstable fixed point is $\theta^* = \frac{7\pi}{4}$.

4.3.3

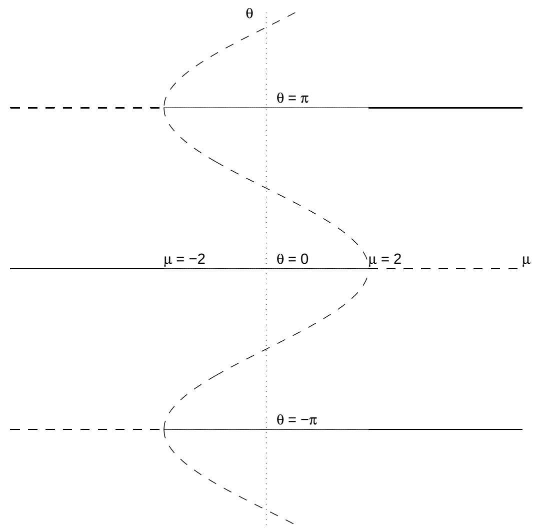
The graphs of $\dot{\theta} = \mu \sin(\theta) - \sin(2\theta)$ look like



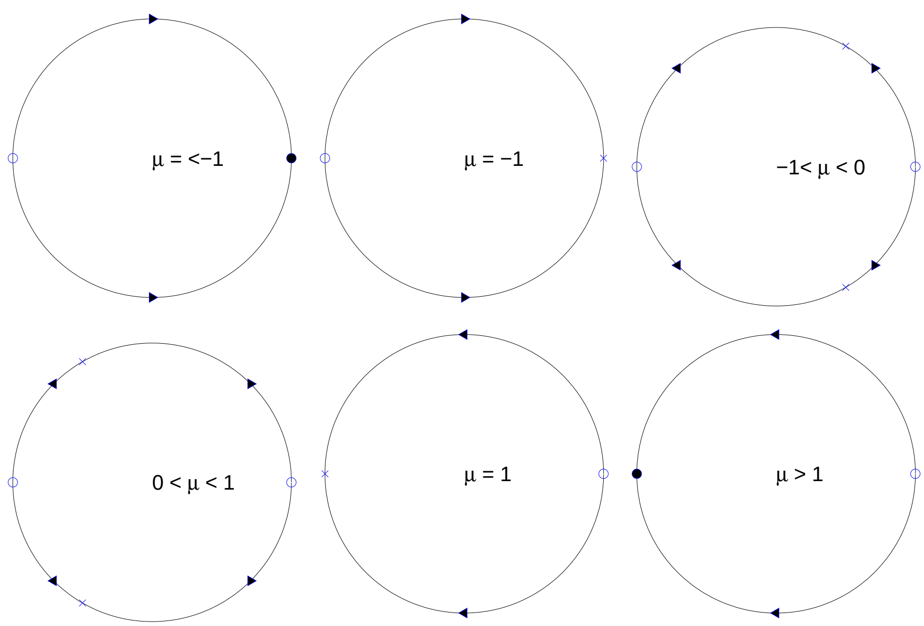
where, in ascending order, $\mu = -2, -1, 0, 1, 2$. We can draw the phase portraits as follow:



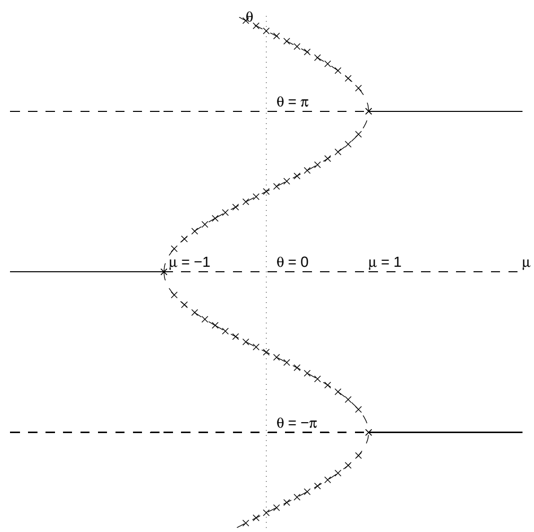
Hence we can see that for $\mu = -2$, the system undergoes a supercritical pitchfork bifurcation at $\theta^* = \pi$ while for $\mu = 2$, the system undergoes a subcritical pitchfork bifurcation at $\theta^* = 0$ (see the bifurcation diagram below).



4.3.4 We can draw the phase portraits for $\dot{\theta} = \frac{\sin(\theta)}{\mu+\cos(\theta)}$ as follow:

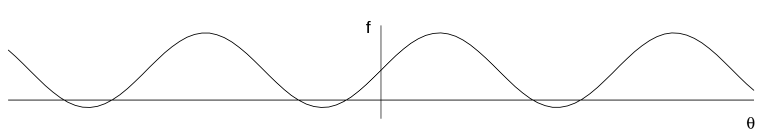


For $-1 < \mu < 1$, there exists two angles θ_1, θ_2 (denoted as crosses in the above figures) such that $\dot{\theta}|_{\theta=\theta_{1,2}}$ are not defined. They are called *attractors* or *finite time singularity* since the flow is toward them but NOT fixed points (as the dynamics is not well defined at those points). As $\mu \to -1^-$, the stable fixed point at $\theta^* = 0$ undergoes a supercritical bifurcation at $\mu = -1$ and produce the two attractors. On the other hand, as $\mu \to 1^-$, two attractors move toward $\theta^* = \pi$, indicated by the lines with crosses. It undergoes a subcritical bifurcation at $\mu = 1$ and produce a stable fixed point at $\mu = 1$. Notice that at these critical values $\mu = \pm$, $\theta^* = \pi, 0$ are not longer fixed points but attractors themselves (see the bifurcation diagram below).

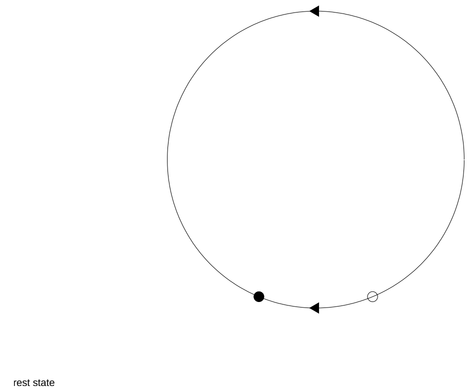


4.5.3

Part a) When μ is slightly less than 1, the graph of $f(\theta) = \mu + \theta$ looks like

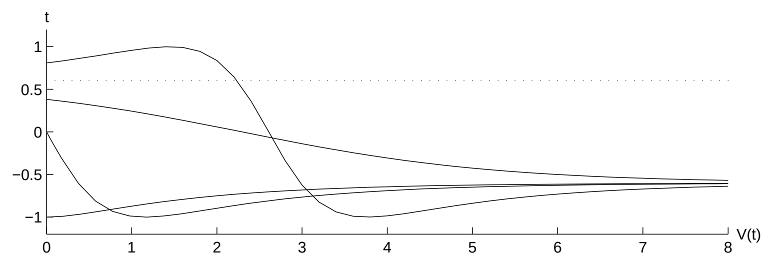


We can then draw its phase portrait



The stable fixed point is the globally attracting rest state, when θ passes the unstable fixed point, i.e. the "threshold", the system will go almost all the way around the circle before it returns to the "rest state".

Part b) The dotted line in the figure below corresponds to the threshold. If initally θ is on the right of the threshold, where V is above the dotted line, then V will reach 1 before it returns to the rest state.



For an illustration of a bottleneck computation ($\mu > 1$) see the graph below. It corresponds to the solution for $\mu = 1.001$. Starting at $\theta(0) = -\pi$ the solution gets into a bottleneck at $\theta = -\pi/2$. The time spent in the bottleneck scales with $\pi\sqrt{2}/\sqrt{\mu - 1}$. For $\mu = 1.001$, this is approximately 140.5, which is consistent with the plot.