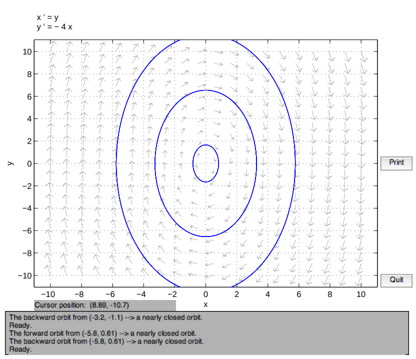
Solutions 4

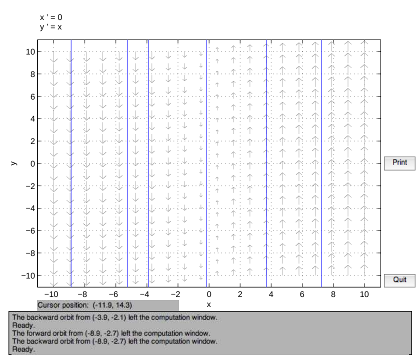
5.1.10

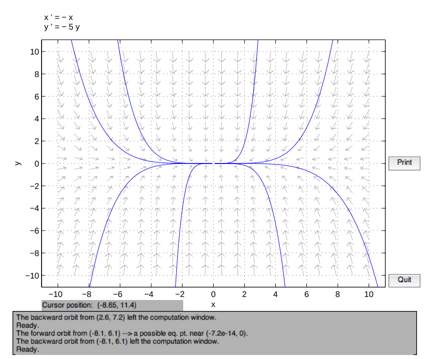
a) The coefficient matrix $\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$ has $\tau = 0$, so the origin is a centre and it is Liapunov stable. The origin is not asymptotically stable though, as it is not attracting.

c) In this system x is never changed. If we start at a point (x_0, y_0) where x_0 is positive, then the solution of $\dot{y} = x_0$ is $y = x_0 t$, which goes arbitrarily large in the long run (it approaches either ∞ of $-\infty$ depending on the sign of x_0). So in this case the origin is unstable.

e) Notice that the system is already decoupled. Since 0 is a stable fixed point for both systems of x and y, an arbitrary flow starting at any point always gets closer to the origin as time evolves, so it is asymptotically stable (both attracting and Liapunov stable).

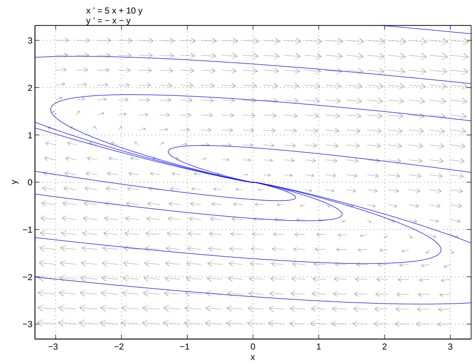






5.2.4

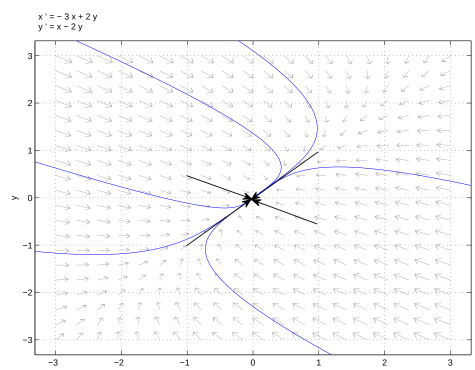
The coefficients matrix is $\begin{pmatrix} 5 & 10 \\ -1 & -1 \end{pmatrix}$, which has $\tau = 4$, $\Delta = 5$. The characteristic equation is $\lambda^2 - 4\lambda + 5 = 0$, hence $\lambda_1 = 2 + \mathrm{i}, \lambda_2 = 2 - \mathrm{i}$.. The corresponding eigenvectors are $\mathbf{v}_1 = \binom{-3-\mathrm{i}}{1}, \mathbf{v}_2 = \binom{-3+\mathrm{i}}{1}$.



The origin is an unstable spiral.

5.2.6

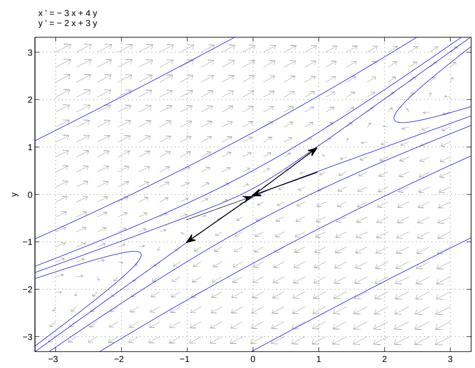
The coefficients matrix is $\begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$, which has $\tau = -5$, $\Delta = 4$. The characteristic equation is $\lambda^2 + 5\lambda + 4 = 0$, hence $\lambda_1 = -4, \lambda_2 = -1$. The corresponding eigenvectors are $\mathbf{v}_1 = \binom{-2}{1}, \mathbf{v}_2 = \binom{1}{1}$.



The origin is a stable node. The two eigendirections are indicated in black. The trajectories approach the origin ($t \to \infty$) tangential to the slow eigendirection $\mathbf{v}_2$. Also, as time goes backwards ($t \to -\infty$), the trajectory asymptotes with the fast eigendirection $\mathbf{v}_1$.

5.2.8

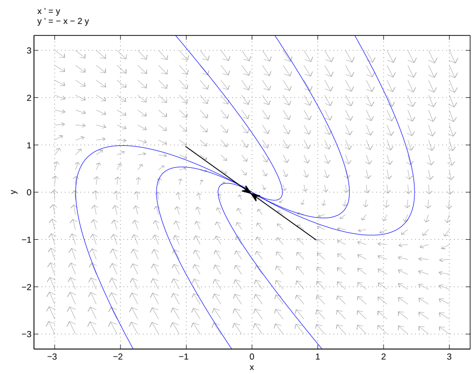
The coefficients matrix is $\begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}$, which has $\tau = 0$, $\Delta = -1$. The characteristic equation is $\lambda^2 - 1 = 0$, hence $\lambda_1 = 1$, $\quad \lambda_2 = -1$. The corresponding eigenvectors are $\mathbf{v}_1 = \binom{1}{1}, \mathbf{v}_2 = \binom{2}{1}$.



The origin is a saddle point. The two eigendirections are shown in black. Arrows indicate stable vs unstable manifolds.

5.2.10

The coefficients matrix is $\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$, which has $\tau = -2$, $\Delta = 1$. The characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$, hence $\lambda_1 = \lambda_2 = -1$. An eigenvector $\mathbf{v} = (v_1, v_2)$ satisfies $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \binom{v_1}{v_2} = \binom{0}{0}$, which has a nontrivial solution $(v_1, v_2) = (-1, 1)$. Since there is only one eigenvector, the origin is a degenerate node. Trajectories asymptote the eigendirection as $t \to \pm\infty$.



5.2.13

The coefficients matrix is $\begin{pmatrix} 0 & 1 \\ -k/m & -b/m \end{pmatrix}$, which has $\tau = -b/m < 0$, $\Delta = k/m > 0$, $\tau^2 - 4\Delta = b^2/m^2 - 4k/m$. We distinguish the cases:

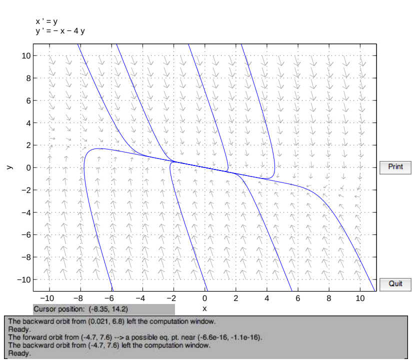
i) $\Delta > 0$ or $b > \sqrt{4km}$: real negative eigenvalues. The origin is a stable node.

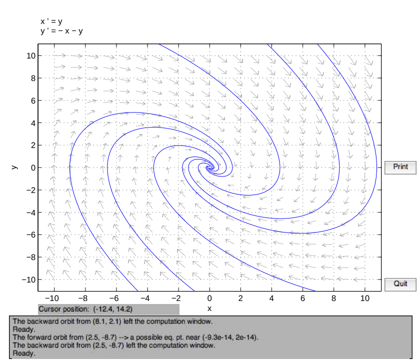
ii) $\Delta < 0$ or $b < \sqrt{4km}$: imaginary eigenvalues with negative real part. The origin is a stable spiral.

iii) $\Delta = 0$ or $b = \sqrt{4km}$: double negative eigenvalue. The origin is a degenerate stable node.

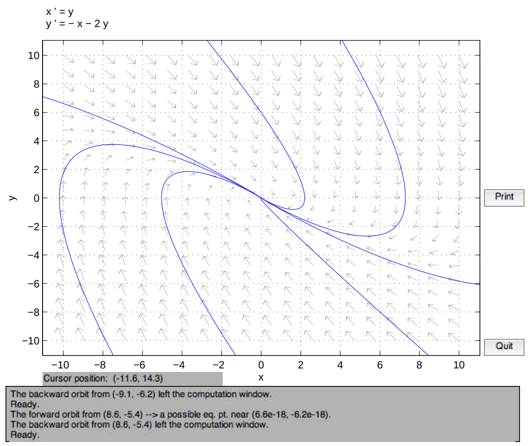
In plots below: i) $m = 1$, $k = 1$, $b = 4$. Eigenvalues $\lambda_1 = -2 + \sqrt{3}$, $\lambda_2 = -2 - \sqrt{3}$. Slow eigendirection $(-2 - \sqrt{3}, 1)$, fast eigendirection $(-2 + \sqrt{3}, 1)$.

ii) $m = 1$, $k = 1$, $b = 1$





In the borderline case iii) illustrated below: $m = 1$, $k = 1$, $b = 2$. Double eigenvalue $\lambda = -1$ with eigendirection $(-1, 1)$. Trajectories approach the eigendirection as $t \to \pm\infty$.



Physically, the different behaviours are due to the size of the friction measured by the parameter b. Case i) corresponds to overdamping, ii) to underdamping and case iii) to critical damping.

5.3.4

The coefficients matrix is $\begin{pmatrix} a & b \\ -b & -a \end{pmatrix}$, which has $\tau = 0$, $\Delta = -a^2 + b^2$. The characteristic equation is $\lambda^2 - a^2 + b^2 = 0$, hence $\lambda_1 = \sqrt{a^2 - b^2}, \lambda_2 = -\sqrt{a^2 - b^2}$. The corresponding eigenvectors are $\mathbf{v}_1 = \left(\begin{smallmatrix} 1 \\ \frac{\sqrt{a^2-b^2}-a}{b} \end{smallmatrix}\right), \mathbf{v}_2 = \left(\begin{smallmatrix} 0 \\ -\frac{\sqrt{a^2-b^2}+a}{b} \end{smallmatrix}\right)$. If $a^2 - b^2 > 0$ then the origin is a saddle point and the relationship will be explosive. Their feelings are opposite, since $\frac{\sqrt{a^2-b^2}-a}{b} < 0$ (see the left figure for $a = 2, b = 1$). If $a^2 - b^2 < 0$ then the origin is a centre and the relationship will be cyclical (see the right figure for $a = 1, b = 2$).