Solutions 5

6.2.2

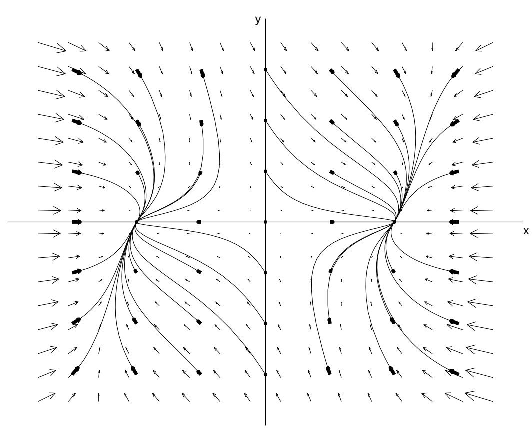
a) The components of the function $\mathbf{f}(x, y) = (y, -x + (1 - x^2 - y^2)y)$ are polynomials in x and y, and therefore are continuous with continuous partial derivatives. b) This is an immediate calculation; use $\cos^2(t) + \sin^2(t) = 1$. c) As a corollary to the existence and uniqueness theorem, we know that *different* trajectories do *not* intersect. Since the circle $x^2 + y^2 = 1$ is a trajectory (according to part b), any trajectory that starts inside it, such as the trajectory originating from $(1/2, 0)$, cannot intersect it and hence, remains inside the unit circle for all times.

6.3.4

First we find the fixed points by solving $\dot{x} = 0$, $\dot{y} = 0$ simultaneously. Hence we need $x = 0$ or $x = \pm 1$, and $y = 0$. The Jacobian matrix at a general point (x, y) is

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 - 3x^2 & 1 \\ 0 & -1 \end{pmatrix}.$$

Next we evaluate A at the fixed points. At $(0, 0)$, we find $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, so $(0, 0)$ is a saddle point. At $(\pm 1, 0)$, $A = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$, so both $(1, 0)$ and $(-1, 0)$ are stable nodes.



6.3.9

a) By solving $\dot{x} = 0$, $\dot{y} = 0$ simultaneously. we get $x = y = 0$ or $x = y = \pm 2$. The Jacobian matrix at a general point (x, y) is

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} -4 & 3y^2 \\ -3 & 3y^2 - 1 \end{pmatrix}.$$

Next we evaluate A at the fixed points. At $(0, 0)$, we find $A = \begin{pmatrix} -4 & 0 \\ -3 & -1 \end{pmatrix}$, so $(0, 0)$ is a stable node. At $(\pm 2, \pm 2)$, $A = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix}$, so both $(2, 2)$ and $(-2, -2)$ are saddles.

b) Let $u = x - y$, then

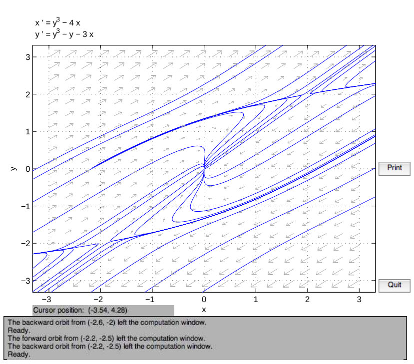
$$\dot{u} = \dot{x} - \dot{y} = (y^3 - 4x) - (y^3 - y - 3x) = y - x = -u.$$

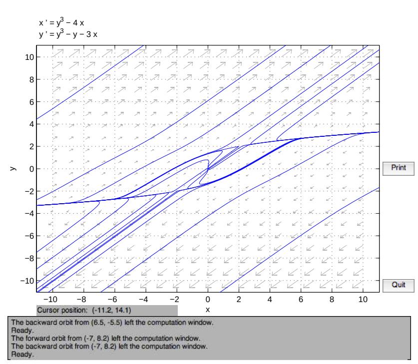
Any trajectory that starts on $x = y$ also starts on $u = 0$. But $u = 0$ is a fixed point of $\dot{u} = -u$, so the trajectory will stay at $u = 0$, i.e. $x = y$.

c) It can also be seen from $\dot{u} = -u$ that $u = 0$ is a stable fixed point. So $|u(t)| \to 0$ as $t \to \infty$.

d) To sketch the phase portrait, one should start locally at the fixed points $(0, 0)$, $(2, 2)$ and $(-2, -2)$, and connect some local flows to get a better idea what the phase portrait looks like. Also note that the system is invariant under the transformation $(x, y) \to (-x, -y)$.

The picture on the left uses a localized domain near the origin: $-3 \le x, y \le 3$. We can see trajectories being sucked at the stable node or passing by the saddles. Picture on the right uses the larger domain $-10 \le x, y \le 10$ and it gives a more global picture. The stable and unstable manifolds of the two saddles become evident in this figure. The very interesting feature that seems to occur here (see also part e below) is that the stable manifolds of the two saddles come very close together near $x = \pm 6$ and remain close for large x.



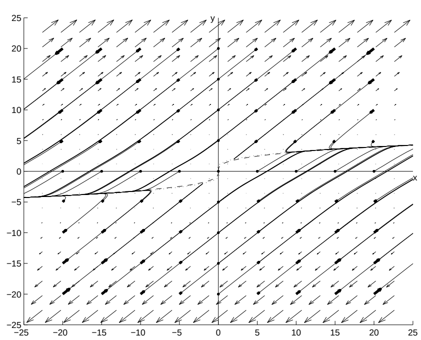


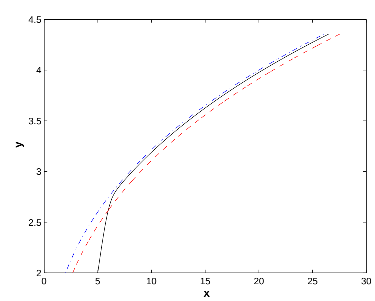
e) In the bigger domain $-20 \leq x, y \leq 20$ the trajectories seem to approach a curve as $t \to -\infty$ (left figure). Given the discussion above we expect that this curve is part of the stable manifold(s) of the two saddles. These stable manifolds are not known exactly, we only know their local behaviour near the saddles (tangent to the stable eigendirections). We now attempt to guess them. In the figure on the left two curves are also drawn: $y^3 - y = 3x$ (one of the nullclines) and $y^3 = 3x$. You can only see one curve (the dashed one) there because they are barely distinguishable. You can use either one as an approximation of the curve where trajectories asymptote as $t \to -\infty$.

In the figure on the right the black solid curve represents a time-backward trajectory starting at $(5, 0)$, that is, the solution of

$$\begin{aligned}\dot{x} &= -(y^3 - 4x)\\ \dot{y} &= -(y^3 - y - 3x),\end{aligned}$$

initialized at $(5, 0)$. Note the negative sign in front of the vector field that results from the change of variable $t \to -t$. The time-backward system is solved numerically by the 2D Euler Matlab code provided with the homework. Also, in dashed red we plot the curve $y^3 = 3x$ and in dashed-dot blue the curve $y^3 - y = 3x$. The time-backward trajectory seems to lie between the two curves, which are very close to each other as x increases. A more detailed analysis would include a proof of this fact.



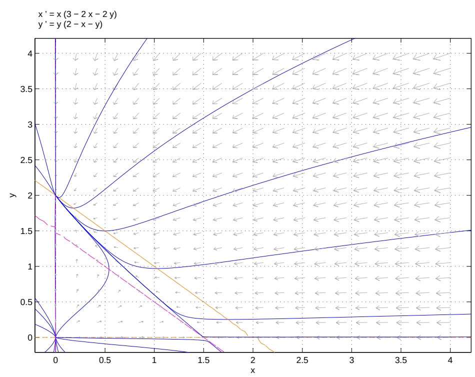


6.4.3

The fixed points of the system are $(x^*, y^*) = (0, 0), (0, 2)$ and $(\frac{3}{2}, 0)$. We compute the Jacobian:

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 4x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$

Next we evaluate A at the fixed points. At $(0,0)$, we find $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, so $(0,0)$ is unstable node. At $(0,2)$, $A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$, so $(0,2)$ is a stable node. At $(\frac{3}{2},0)$, $A = \begin{pmatrix} -3 & 3 \\ 0 & \frac{1}{2} \end{pmatrix}$, so $(\frac{3}{2},0)$ is a saddle point. The basin of attraction is $x > 0, y \geq 0$.



6.5.1

a) This can be rewritten as the vector field

$$\dot{x} = y$$
$$\dot{y} = x^3 - x$$

where y represents the particle's velocity. Equilibrium points occur where $(\dot{x},\dot{y}) = (0,0)$. Hence the equilibria are $(x^*,y^*) = (0,0)$ and $(\pm 1, 0)$. To classify fixed points we compute the Jacobian:

$$A = \begin{pmatrix} 0 & 1 \\ 3x^2 - 1 & 0 \end{pmatrix}.$$

At $(0,0)$, we have $\tau = 0$, $\Delta = 1$, so the origin is a center. But when $(x^*,y^*) = (\pm 1, 0)$, we find $\Delta = -2$; hence these equilibria are saddles.

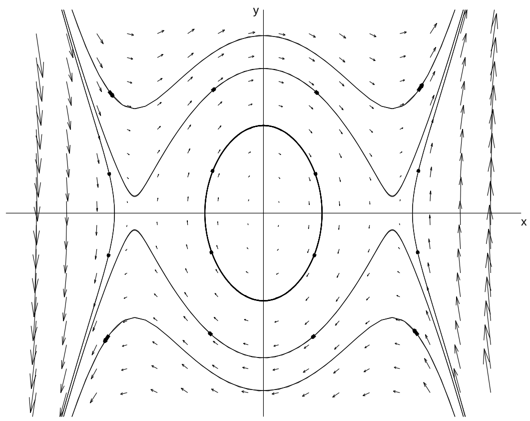
b) Any antiderivative of $x - x^3$ induces a conserved quantity, for example

$$V(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4$$

gives us a conserved quantity

$$E(x) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4.$$

c) Since this system is conservative, the *isolated* fixed point which we predicted to be a center by the linear approximation should indeed be a center. The phase portrait looks as follows:



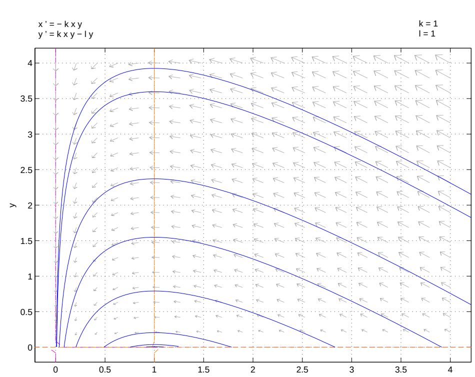
6.5.6

a) The fixed points are $(x^*, y^*) = (\hat{x}, 0), \hat{x} > 0$. The Jacobian matrix at a general point (x, y) is

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} -ky & -kx \\ ky & kx - l \end{pmatrix}.$$

Next we evaluate A at the fixed points. At $(\hat{x}, 0)$, we find $A = \begin{pmatrix} 0 & -k\hat{x} \\ 0 & k\hat{x} - l \end{pmatrix}$, so $(\hat{x}, 0)$ are non-isolated fixed points.

b,d) The dashed lines indicate the nullclines. As $t \to \infty$, $y(t) \to 0$ and $x(t) \to x_\infty \in [0, \frac{l}{k})$.

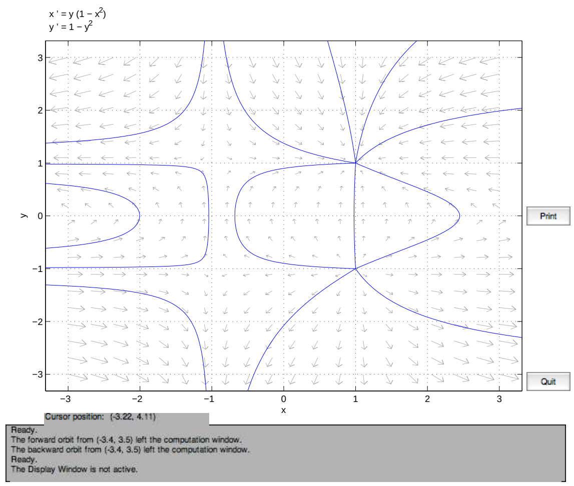


c) Note that $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{kxy - ly}{-kxy} = -1 + \frac{l}{kx}$. Separate the variables and integrate on both sides, we have $g(x, y) = y + x - \frac{l}{k}\log x$ as a conserved quantity for the system.

e) We can see from the phase portrait that an epidemic occurs when $x_0 > \frac{l}{k}$.

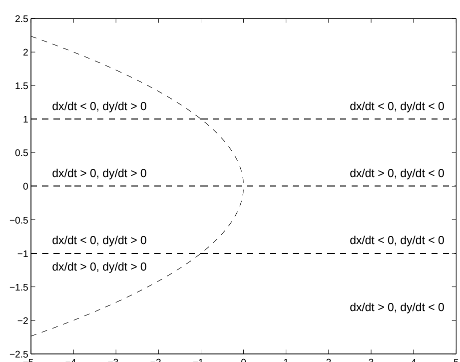
6.6.1

The system is invariant under the change of variables $t \to -t$ and $y \to -y$. Hence the system is reversible.



6.6.6

a, b) For $\dot{x}$, it is positive when $0 < y < 1$ or $y < -1$ and negative when $-1 < y < 0$ or $y > 1$. For $\dot{y}$, it is positive when $x < -y^2$ and negative when $x > -y^2$ (see the figure below, where the dashed lines are nullclines).



c) The Jacobian is

$$A = \begin{pmatrix} 0 & 1 - 3y^2 \\ -1 & -2y \end{pmatrix}.$$

At $(-1, \pm 1)$, we have $\tau = \mp 2, \Delta = -2$. The eigenvalues are $\lambda_1 = \mp 1 + \sqrt{3}$ and $\lambda_2 = \mp 1 - \sqrt{3}$. The corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ \pm\frac{1}{2} - \frac{\sqrt{3}}{2} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ \pm\frac{1}{2} + \frac{\sqrt{3}}{2} \end{pmatrix}$.

d) Consider the unstable manifold of the saddle point $(-1, -1)$. This manifold leaves the saddle point along the vector $\mathbf{v} = \begin{pmatrix} -1 \\ \frac{1}{2} + \frac{\sqrt{3}}{2} \end{pmatrix}$. Hence part of the unstable manifold lies in the region where $\dot{x} < 0, \dot{y} > 0$. A phase point along the manifold will move up and to the left and eventually will cut the negative x-axis. Since the system is reversible under the transformation $t \to -t, y \to -y$. By reversibility, there must be a twin trajectory with the same endpoint on the negative x-axis from $(1, 1)$ but with arrow reversed. Together the two trajectories form a heteroclinic orbit.

e) Consider the unstable manifold of the saddle point $(-1, 1)$. This manifold leaves the saddle point along the vector $\mathbf{v} = \begin{pmatrix} 1 \\ \frac{1}{2} - \frac{\sqrt{3}}{2} \end{pmatrix}$. Hence part of the unstable manifold lies in the region where $\dot{x} > 0, \dot{y} < 0$ (Note that $\frac{1}{2} - \frac{\sqrt{3}}{2} > -2$). A phase point along the manifold will move down and to the right and eventually will cut the positive x-axis (the origin is a centre). Since the system is reversible under the transformation $t \to -t, y \to -y$. By reversibility, there must be a twin trajectory with the same endpoint on the positive x-axis from $(1, -1)$ but with arrow reversed. Together the two trajectories form another heteroclinic orbit.