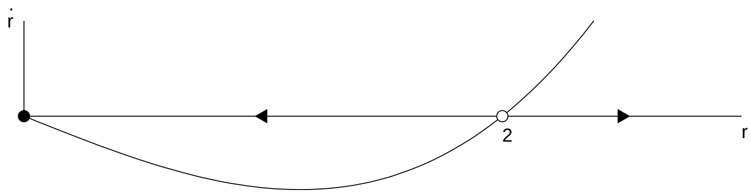
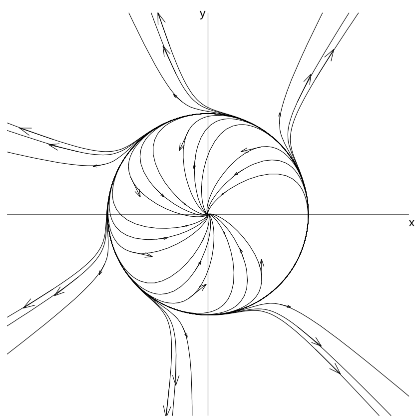
Solutions 6

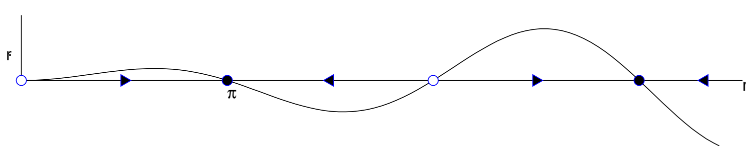
7.1.1 The radial and angular dynamics are uncoupled and so can be analyzed separately. Treating $\dot{r} = r^3 - 4r$ as a vector field on the line, we see that $r^* = 0$ is a stable fixed point and $r^* = 2$ is unstable.



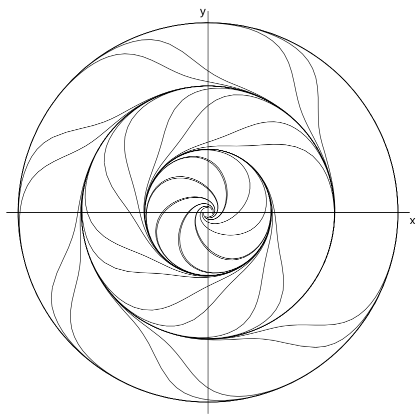
Hence, back in the phase plane, all trajectories inside the circle $r^* = 2$ approach the origin monotonically. All trajectories outside this circle approach the infinity. Since the motion in the θ-direction is simply rotation at constant angular velocity, we see that all trajectories spiral asymptotically away from a limit circle at $r = 2$.



7.1.4 Again, the radial and angular dynamics are uncoupled and so can be analyzed separately. Treating $\dot{r} = r\sin(r)$ as a vector field on the line, we see that $r^* = (2n-1)\pi$ are stable fixed points and $r^* = 2n\pi$ are unstable, where n is a positive integer.



Hence, back in the phase plane, since the motion in the θ-direction is simply rotation at constant angular velocity, all trajectories between the circles $r^* = 2n\pi$ and $r^* = 2(n+1)\pi$ spiral asymptotically towards the limit circles $r = (2n-1)\pi$.



7.2.6 a) First notice that the system satisfies

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x},$$

so we can find a V by integrating f and g. V should look like

$$V(x,y) = -\int_0^x (y^2 + y\cos x)\,dx + a(y) = -xy^2 - y\sin x + a(y),$$

where $a(y)$ satisfies

$$g(x,y) = -\frac{\partial V}{\partial y} = 2xy + \sin x - a'(y).$$

We can simply choose $a(y) \equiv 0$, and obtain the potential function

$$V(x,y) = -xy^2 - y\sin x.$$

b) Again we can find a V by integrating f and g. V should look like

$$V(x,y) = -\int_0^x (3x^2 - 1 - e^{2y})\,dx + a(y) = -x^3 + x + xe^{2y} + a(y),$$

where $a(y)$ satisfies

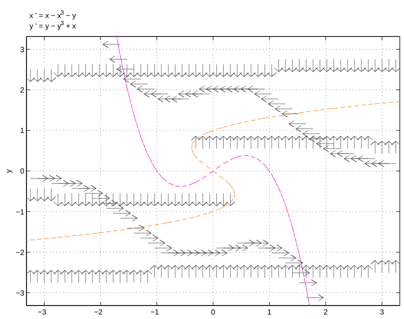
$$g(x,y) = -\frac{\partial V}{\partial y} = -2xe^{2y} - a'(y).$$

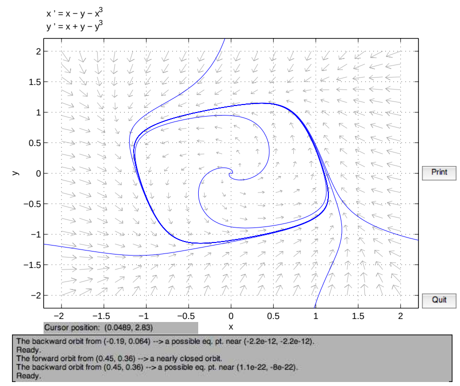
We can simply choose $a(y) \equiv 0$, and obtain the potential function

$$V(x,y) = -x^3 + x + xe^{2y}.$$

7.2.10 Consider $V(x,y) = ax^2 + by^2$. Then $\dot{V} = 2ax\dot{x} + 2by\dot{y} = 2ax(y - x^3) - 2by(x + y^3) = 2(a-b)xy - 2(ax^4 + by^4)$. If we choose $a = b > 0$, then $V > 0$ and $\dot{V} < 0$ for all $(x,y) \neq (0,0)$. Hence $V = ax^2 + by^2$ is a Liapunov function and so there are not closed orbits.

7.3.3 First we plot the nullclines (left plot):





We can find the trapping zone by doing the following. We start with point (a, a), where $a > 0$. It sits on the y-nullcline if $a + a - a^3 = 0$ or $a = \sqrt{2}$, and at this point, $\dot{x} < 0$. By symmetry, the point $(-\sqrt{2}, \sqrt{2})$ is sitting on the x-nullcline and $\dot{y} < 0$. We use the same argument for the point $(-\sqrt{2}, -\sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$ and combine with the figure above we find that the square which four corners are sitting on the points $(\pm\sqrt{2}, \pm\sqrt{2})$ is the trapping zone.

Now since the Jacobian $A = \begin{pmatrix} 1 - 3x^2 & -1 \\ 1 & 1 - 3y^2 \end{pmatrix}$ evaluates to $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ at the origin. The eigenvalues are $1 \pm i$ so it is an unstable spiral. Therefore the repelling fixed point drives all neighbouring trajectories into the trapping region and we can apply the Poincaré-Bendixson theorem here to show that there is a periodic solution. The limit cycle is shown in the right plot.

7.3.4

a) The Jacobian $A = \begin{pmatrix} 1 - 12x^2 - y^2 \frac{y}{2} & -2xy - \frac{1+x}{2} \\ -8xy - 2 - 4x & 1 - 4x^2 - 3y^2 \end{pmatrix}$ evaluates to $\begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & 1 \end{pmatrix}$ at the origin. The eigenvalues are $0, 2$ so it is an unstable fixed point.

b) Let $V = (1 - 4x^2 - y^2)^2$, then

$$\begin{aligned} \dot{V} &= -4(1 - 4x^2 - y^2)(4x\dot{x} + y\dot{y}) \\ &= -4(1 - 4x^2 - y^2)\{4x[x(1 - 4x^2 - y^2) - \frac{y(1+x)}{2}] + y[x(1 - 4x^2 - y^2) + 2x(1+x)]\} \\ &= -4(1 - 4x^2 - y^2)^2(4x^2 + y^2) \end{aligned}$$

Since $\dot{V} < 0$ unless $V = 0$ or $x = y = 0$. Since $(0, 0)$ is an repeller, all trajectories will approach the ellipse $4x^2 + y^2 = 1$ as $t \to \infty$.

Additional problem. Transforming to rectangular coordinates by $x = r\cos\theta$, $y = r\sin\theta$ we get

$$\dot{x} = x(1 - x^2 - y^2) + \mu x^2/\sqrt{x^2+y^2} - y$$
$$\dot{y} = y(1 - x^2 - y^2) + \mu xy/\sqrt{x^2+y^2} + x$$

A fixed point satisfies

$$x(1 - x^2 - y^2) + \mu x^2/\sqrt{x^2+y^2} - y = 0$$
$$y(1 - x^2 - y^2) + \mu xy/\sqrt{x^2+y^2} + x = 0$$

Multiply the first equation by $y \neq 0$ and the second equation by $x \neq 0$, then subtract the two equations, to get $x^2 + y^2 = 0$, which cannot occur if $x, y \neq 0$. Looking for fixed point with either $x = 0$ or $y = 0$ leads to the other coordinate being 0 as well.

The origin is a singular point for velocity expressed in rectangular coordinates. We can circumvent this by defining the vector field $(f(x,y), g(x,y))$ by

$$f(x,y) = \begin{cases} x(1 - x^2 - y^2) + \mu x^2/\sqrt{x^2+y^2} - y & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

$$g(x,y) = \begin{cases} y(1 - x^2 - y^2) + \mu xy/\sqrt{x^2+y^2} + x & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

With this definition, the system

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

has a fixed point at origin (compare to $r = 0$ for the original system in polar coordinates).

Looking at $(0,0)$ more closely though, we notice that the velocity field $(f(x,y), g(x,y))$ is continuous there, but not differentiable. Hence, we cannot talk about the *linear* approximation at $(0,0)$.

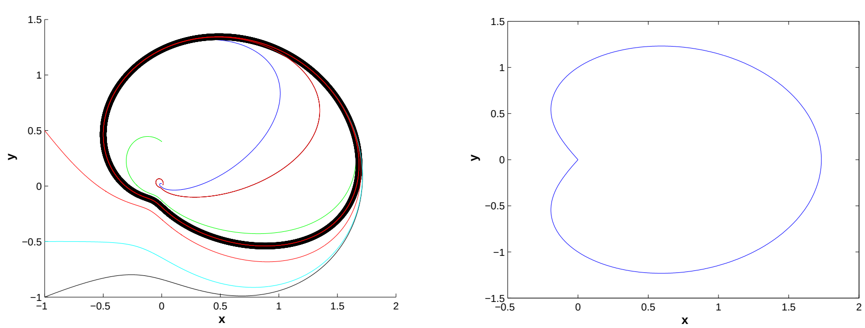
Let us return to the polar coordinate formulation and inspect the sign of $\dot{r}$. This was the approach taken for $\mu < 1$ in trying to identify a trapping region. Half of the calculation done for $\mu < 1$ works for $\mu > 1$ with no modifications. Indeed take any real number $R > 0$ with $R^2 > 1 + \mu$. Then, on the circle $r = R$, $\dot{r}$ has the value

$$R(1 - R^2) + \mu R\cos\theta \leq R(1 - R^2 + \mu) < 0.$$

Therefore, on the circle $r = R$ the velocity field is pointing toward the *inside* of the disk.

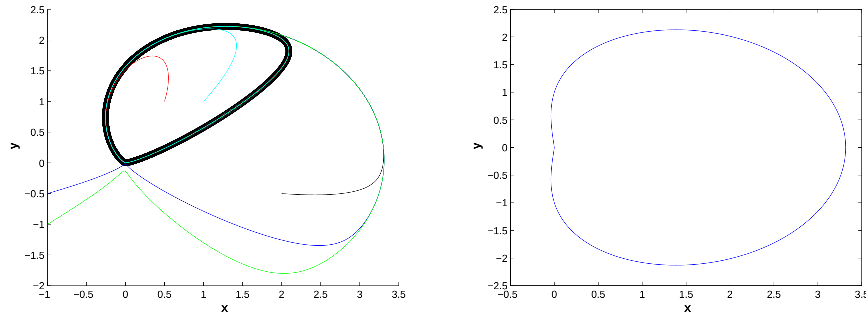
The issue is the other half of the estimate (the repelling region). The system has a fixed point at the origin which we have to exclude if we want to apply the Poincaré-Bendixson theorem. Are trajectories being repelled from within a region that contains the origin? Let's inspect the "nullcline" $\dot{r} = 0$ given by the curve $r^2 = 1 + \mu \cos\theta$. Since $r^2 \geq 0$, θ is restricted to the interval $[-\arccos(-1/\mu), \arccos(-1/\mu)]$.

The plot below (right) shows the nullcline for $\mu = 2$. Inside the curve we have $\dot{r} > 0$ and outside $\dot{r} < 0$. Based on this plot we make the important observation that trajectories approach the origin from outside the curve, while spiralling in the counter-clockwise direction, as $\theta(t) = t + \theta_0$. As such a trajectory approaches the origin, while rotating, it eventually enters inside the nullcline, where $\dot{r} > 0$ and then starts moving away from origin. This can be seen in the phase portrait on the left. The limit cycle is represented by black circles. Note that there are trajectories that pass arbitrarily close to the origin.



There are limitations in applying the Poincaré-Bendixson theorem here, as *any* circle around the origin, no matter how small, would have a side where it is *attracting*, instead of *repelling* trajectories. The reason trajectories are not sucked into the origin though, is because there is rotation at constant angular velocity $\dot{\theta} = 1$, which brings them into the $\dot{r} > 0$ region.

For larger μ, the attracting region on the left of the origin becomes larger. More trajectories "barely" miss the origin before approaching the limit cycle. The plots below correspond to $\mu = 10$. On the left it is the phase portrait and limit cycle, on the right it is the "nullcline" $r^2 = 1 + \mu \cos\theta$.



It requires to zoom in a small region near the origin to confirm that the origin is indeed *inside* the limit cycle. At the scale of the figures this is not obvious.