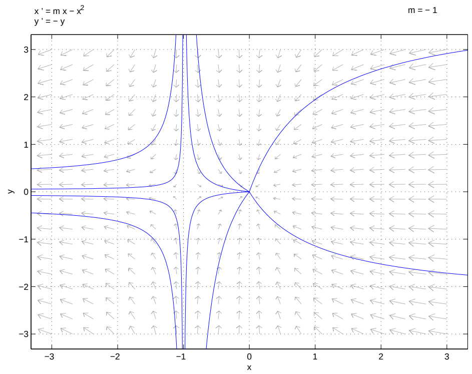
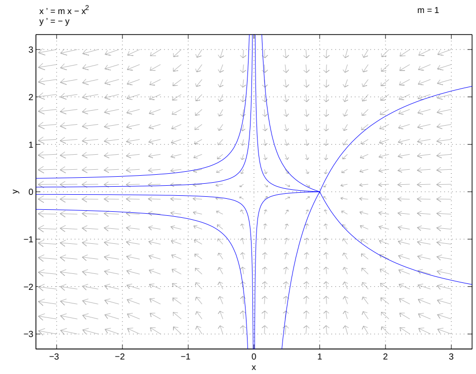
Solutions 7

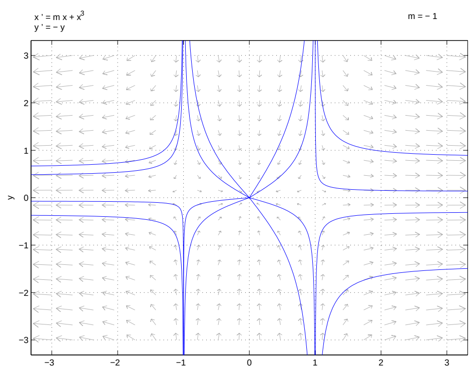
8.1.1

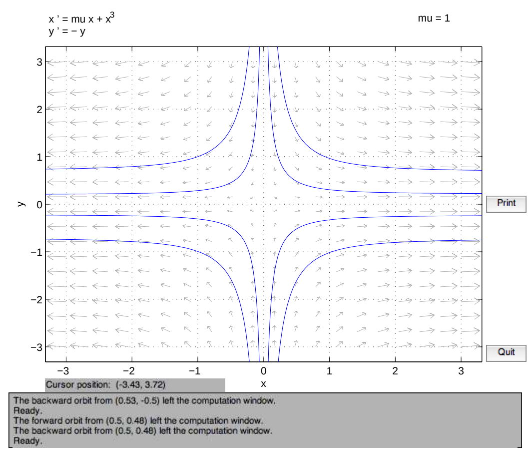
a) The stable fixed points are $(0,0)$ for $\mu \leq 0$ and $(\mu, 0)$ for $\mu > 0$. The eigenvalues are $-|\mu|, -1$. Fixed points exchange stability through the bifurcation.





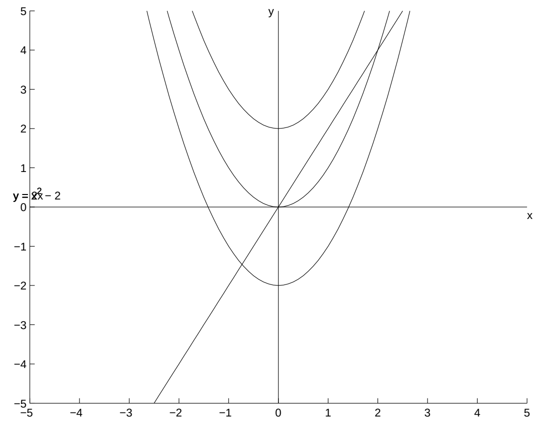
b) For $\mu < 0$, the origin $(0,0)$ is a stable fixed point; the other two (unstable) fixed points are $(\pm\sqrt{-\mu}, 0)$. For $\mu > 0$ the only fixed point is the origin, which is unstable. The eigenvalues at $(0,0)$ are $\mu, -1$, and the eigenvalues at $(\pm\sqrt{-\mu}, 0)$ are $-2\mu, -1$. The origin loses stability through the bifurcation.





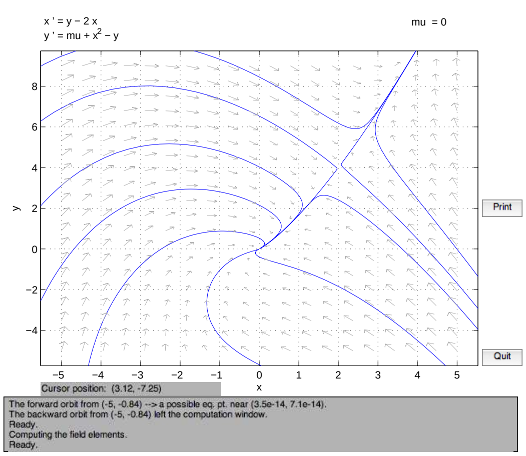
8.1.6

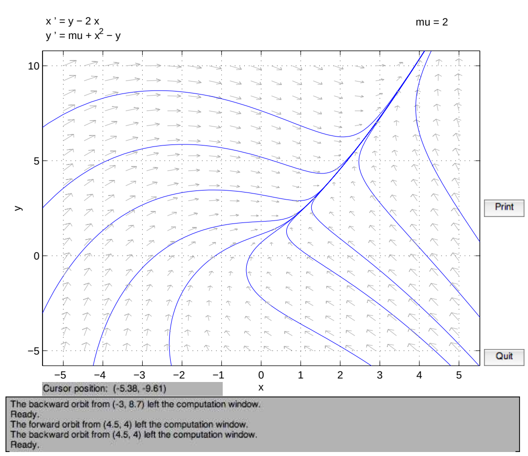
a) The nullclines are drawn for $\mu = -2$, 0, and 2.



b) The fixed points are the intersections of the nullclines. From the above picture it's clear that as μ increases, two fixed points first collide then disappear. The collision occurs when the parabola $y = x^2 + \mu$ is tangential to the line $y = 2x$. Since the slope of the tangent line of the parabola at point $(t, t^2 + \mu)$ is $2t$, if the tangent line is $y = 2x$, then $t = 1$. Therefore $y = x^2 + \mu$ is tangential to $y = 2x$ at point $(1, 1 + \mu)$, which implies that bifurcation occurs at $\mu = 1$. The motion of the fixed points described above also shows that this is a saddle-node bifurcation.

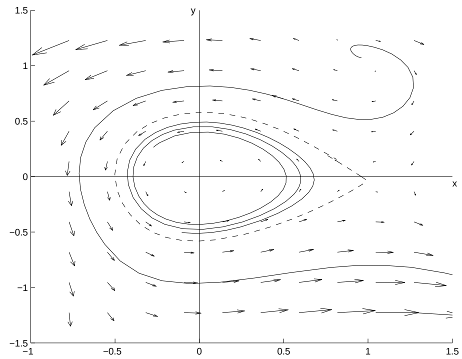
c) For $\mu < 1$ we have two fixed points with $x_* = 1 \pm \sqrt{1 - \mu}$, for $\mu > 1$ there are no fixed points. Sketched below are phase portraits for $\mu = 0 < 1$ and $\mu = 2 > 1$. On the left, the origin is a stable node, and $(2, 4)$ is a saddle.



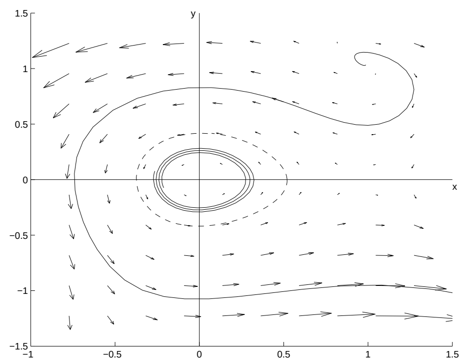


8.2.3

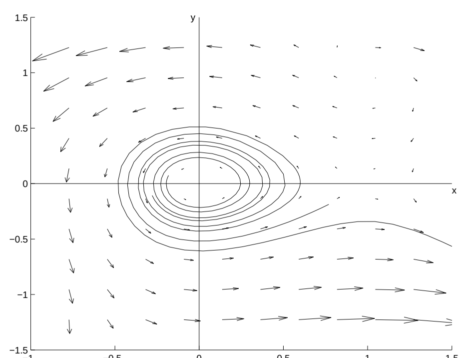
We start with a negative μ very close to 0. The figure below shows the phase portrait for $\mu = -0.04$, there is an unstable cycle around the origin:



As μ increases to -0.02, the cycle gets closer to the origin:



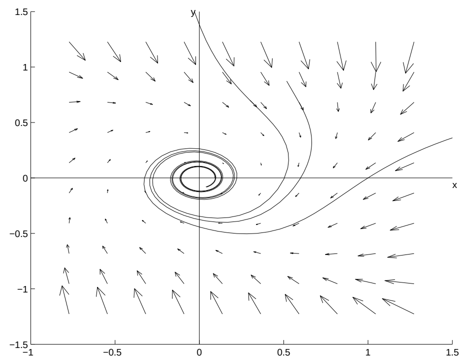
Now if μ becomes positive, the unstable limit cycle disappears and the origin becomes unstable:



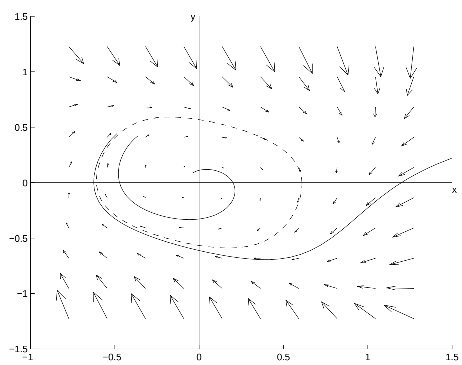
Therefore our computer experiment suggests that this Hopf bifurcation is subcritical.

8.2.6

When μ is negative, say $\mu = -0.04$, the phase portrait looks like

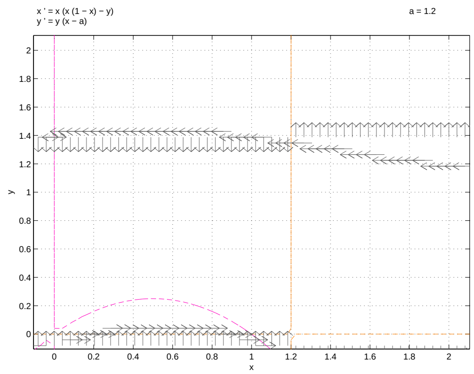


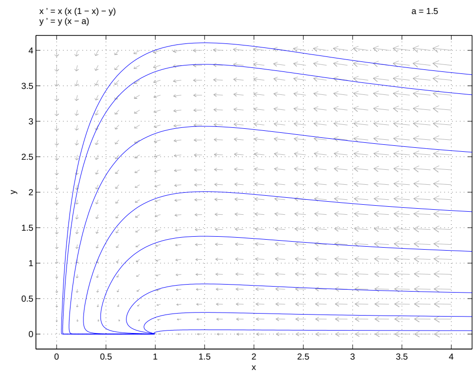
The origin is a stable spiral. Now let $\mu = 0.4$ be a small positive number, then a stable limit cycle appears, and the origin becomes an unstable spiral:



This experiment suggests that the Hopf bifurcation is supercritical.

8.2.8 a,c) We show below the nullclines in the first quadrant (left) and the phase portrait for $a = 1.2$ (right). We can see that all the predators go extinct ($y \to 0$).





b) The Jacobian $A = \begin{pmatrix} 2x - 3x^2 - y & -x \\ y & x - a \end{pmatrix}$ evaluates to $\begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}$ for the origin. The eigenvalues are $0, -a$ and it is a non-isolated fixed point.

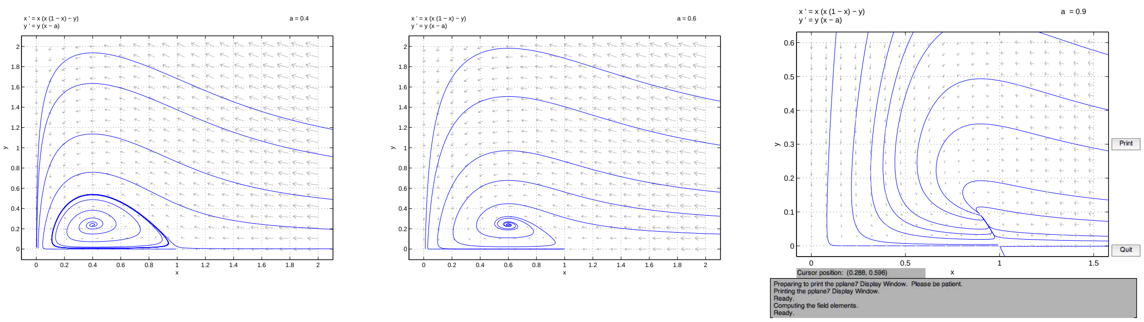
A evaluates to $\begin{pmatrix} -1 & -1 \\ 0 & 1-a \end{pmatrix}$ for $(1, 0)$. The eigenvalues are $-1, 1-a$. It is a stable node for $a > 1$ and a saddle for $a < 1$.

Finally, A evaluates to $\begin{pmatrix} a - 2a^2 & -a \\ a - a^2 & 0 \end{pmatrix}$ for $(a, a-a^2)$. The eigenvalues are $\frac{a(1-2a\pm\sqrt{4a^2-3})}{2}$ and the fixed point is an unstable spiral when $0 \le a < \frac{1}{2}$, a stable spiral when $\frac{1}{2} < a < \frac{\sqrt{3}}{2}$, a stable node for $\frac{\sqrt{3}}{2} < a < 1$ and a saddle when $a > 1$.

d) A supercritical Hopf bifurcation occurs at $a_c = \frac{1}{2}$, when the real parts of the eigenvalues simultaneously cross the imaginary axis into the right half plane. See the phase portraits in part f).

e) Near $a_c = \frac{1}{2}$, the frequency of the limited cycle oscillation is about $a_c\frac{\sqrt{3-4a_c^2}}{2} = \frac{1}{2\sqrt{2}}$.

f) We show the phase portraits for $a = 0.4$ (left), $a = 0.6$ (middle), and $a = 0.9$ (right). The qualitatively different phase portraits correspond to the cases when the fixed point $(a, a - a^2)$ is (a) an unstable spiral surrounded by a stable limit cycle, (b) a stable spiral, and (c) a stable node.



8.4.2

The radial and angular dynamics are uncoupled and so can be analyzed separately. Treating $\dot{r} = r(\mu - \sin r)$ as a vector field on the line, we see that for $|\mu| > 1, r^* = 0$ is the only fixed point and it is stable for $\mu < -1$ and unstable for $\mu > 1$. At $\mu = -1$, the system undergoes a saddle node bifurcation of cycles at $r^* = (2n + \frac{3}{2})\pi$. For $-1 < \mu < 0, r^* = 0, (2n-1)\pi - \arcsin\mu, 2n\pi + \arcsin\mu$. The stabilities of fixed points alternate with increasing r^*, starting with $r^* = 0$ being stable. At $\mu = 0$ the zero fixed point undergoes a supercritical Hopf bifurcation and a new cycle $r^* = \arcsin\mu$ emerges. For $0 < \mu < 1, r^* = 0, \arcsin\mu, (2n-1)\pi - \arcsin\mu, 2n\pi + \arcsin\mu$. The stabilities of fixed points alternate with increasing r^*, starting with $r^* = 0$ being unstable. Subsequently, the system undergoes another saddle node bifurcation of cycles when $\mu = 1$ in which pairs of adjacent fixed points $[\arcsin\mu, \pi - \arcsin\mu], [2\pi + \arcsin\mu, 3\pi - \arcsin\mu]\ldots$ collide at $\frac{\pi}{2}, 2\pi + \frac{\pi}{2}, \ldots$. Since the motion in the θ-direction is simply rotation at constant angular velocity, we see that all trajectories spiral asymptotically to or from limit circles at $r = r^*$.

9.2.1

a) Recall that the fixed points C^+ and C^- are

$$(x^*, y^*, z^*) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1),$$

where $r > 1$. In the following we'll write (x, y, z) instead of (x^*, y^*, z^*).

The Jacobian $A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$ has characteristic polynomial

$$\det(\lambda I - A) = \lambda^3 + (\sigma + 1 + b)\lambda^2 + \\ [b(\sigma+1) + x^2 + \sigma(z+1-r)]\lambda + [b\sigma(z+1-r) + \sigma(x^2 + xy)].$$

Notice that at C^+ and C^-, we have $x^2 = xy = b(r-1)$, $z = r-1$, therefore the eigenvalues satisfy

$$\lambda^3 + (\sigma + 1 + b)\lambda^2 + b(r+\sigma)\lambda + 2\sigma b(r-1) = 0.$$

b) Hopf bifurcation occurs when two eigenvalues are pure imaginary (cf. Figure 8.2.4). In this case $\lambda = i\omega$, where ω is real and nonzero. Thus the characteristic equation becomes

$$-i\omega^3 - (\sigma + 1 + b)\omega^2 + ib(r+\sigma)\omega + 2\sigma b(r-1) = 0.$$

We then seperate the real part and the imaginary part of the above and obtain that

$$\omega^2 = b(r+\sigma) = \frac{2\sigma b(r-1)}{\sigma + 1 + b}.$$

Solving for r, we get

$$r = r_H = \sigma\left(\frac{\sigma + b + 3}{\sigma - b - 1}\right).$$

It is required that r must be positive, so Hopf bifurcation can only occur if r_H is positive, i.e. $\sigma > b + 1$.

c) If $r = r_H$, then the two imaginary roots are

$$\lambda_{1,2} = \pm i\sqrt{b(r_H + \sigma)}.$$

It's well known that all 3 roots should add up to $-(\sigma + b + 1)$, since the two imaginary ones cancel, we have $\lambda_3 = -(\sigma + b + 1)$.

9.2.2

Let $C(t) = rx(t)^2 + \sigma y(t)^2 + \sigma(z(t) - 2r)^2$ be the value of C at time t. Then

$$\begin{aligned}
C'(t) &= 2rxx' + 2\sigma yy' + 2\sigma(z-2r)z' \\
&= 2rx\sigma(y-x) + 2\sigma y(rx - y - xz) + 2\sigma(z-2r)(xy - bz) \\
&= -2\sigma(rx^2 + y^2 + bz^2 - 2brz) \\
&= -2\sigma(rx^2 + y^2 + b(z-r)^2 - r^2 b) \\
&= -\frac{2\sigma}{r^2 b}\left(\frac{x^2}{br} + \frac{y^2}{br^2} + \frac{(z-r)^2}{r^2} - 1\right).
\end{aligned}$$

It is then clear that if at time t, (x, y, z) is outside the ellipsoid

$$K: \quad \frac{x^2}{br} + \frac{y^2}{br^2} + \frac{(z-r)^2}{r^2} \leq 1,$$

then $C(t)$ will decrease. We can pick C so large that the above ellipsoid is contained in

$$E: \quad rx^2 + \sigma y^2 + \sigma(z-2r)^2 \leq C,$$

which is another ellipsoid, then eventually all trajectories will enter E.

The smallest possible value of C is obtained when the ellipsoids E and K are tangent. The picture is that one shrinks E by decreasing C until the surface of E touches K. This is equivalent to the following problem:

Given condition
$$\frac{x^2}{br} + \frac{y^2}{br^2} + \frac{(z-r)^2}{r^2} = 1,$$
find the maximum of
$$rx^2 + \sigma y^2 + \sigma(z-2r)^2.$$
The maximum is the smallest possible C. This problem can be solved using Langrange multipliers. In practice, given a pair of parameters r and σ, one can use a computer to handle the extreme value problem.

9.2.6

a) Let $\mathbf{f} = (-\nu x + zy, -\nu y + (z-a)x, 1 - xy)$ be the instantaneous velocity, then
$$\begin{aligned}\nabla \cdot \mathbf{f} &= \frac{\partial}{\partial x}(-\nu x + zy) + \frac{\partial}{\partial y}(-\nu y + (z-a)x) + \frac{\partial}{\partial z}(1-xy) \\ &= -\nu - \nu + 0 = -2\nu < 0,\end{aligned}$$
therefore the system is dissipative (cf. Figure 9.2.1).

b) A fixed points (x, y, z) satisfies that

(1) $$zy = \nu x$$

(2) $$(z-a)x = \nu y$$

(3) $$1 = xy.$$

From (1) and (3) we have $z = \nu x^2$, then use (2)
$$(\nu x^2 - a)x^2 = \nu xy = \nu.$$
Since $xy = 1$, $x \neq 0$, so
$$a = \nu x^2 - \nu x^{-2} = \nu(k^2 - k^{-2}),$$
where $x^2 = k^2$, $xy = 1$, and $z = \nu x^2$. This is exactly the parametric form described in the problem.

c) The Jacobian $A = \begin{pmatrix} -\nu & z & y \\ z-a & -\nu & x \\ -y & -x & 0 \end{pmatrix}$ evaluates to $\begin{pmatrix} -\nu & \nu k^2 & k^{-1} \\ \nu k^2 - a & -\nu & k \\ -k^{-1} & -k & 0 \end{pmatrix}$ at fixed point $(k, k^{-1}, \nu k^2)$. The characteristic polynomial is
$$f(\lambda) = \lambda^3 + 2\nu\lambda^2 + [\nu^2 - \nu k^2(\nu k^2 - a) + k^2 + k^{-2})\lambda + 2\nu(k^2 + k^{-2}).$$

Using the relation $a = \nu(k^2 - k^{-2})$ to eliminate a, it follows that

$$f(\lambda) = \lambda^3 + 2\nu\lambda^2 + (k^2 + k^{-2})\lambda + 2\nu(k^2 + k^{-2}) = (\lambda + 2\nu)(\lambda^2 + k^2 + k^{-2}).$$

Therefore the eigenvalues are -2ν and $\pm\sqrt{k^2 + k^{-2}}\,i$. Since there are two imaginary eigenvalues, the fixed points are centers.