Solutions 8

9.3.8

a) Yes. D is the set $r \leq 1$. It's clear from $\dot{r} = r(1 - r^2)$ that r either stays at 0 or approaches 1 as $t \to \infty$. So trajectories starting from D will never leave the region.

b) Yes. Any open subset of D can be an open set of initial conditions that are attracted to D (in fact they never leave D). The basin of attraction (we can speak of that even we don't know D is an attractor) of D is the whole plane.

c) No. The circle $x^2 + y^2 = 1$ is a proper subset of D. It's also invariant and attracts an open set of initial conditions (cf. part d).

d) Yes. First $r^* = 1$ is a stable fixed point of $\dot{r} = r(1 - r^2)$, so $x^2 + y^2 = 1$ is invariant and any initial condition with $r > 0$ will be attracted to $r = 1$. Secondly notice that that $\dot{\sigma} = 1$, trajectories starting on the circle will wind along it and never stop. So there can be no invariant proper subset of $x^2 + y^2 = 1$.

9.4.2

a) Because the graph of $f(x_n)$ looks like a tent.

b) The fixed points satisfy $x^* = f(x^*)$, Hence $x^* = 0$ or $x^* = \frac{2}{3}$. Since the multiplier is $\lambda = f'(x^*) = \pm 2$, both these fixed points are unstable.

c) We can solve for $p = f(q), q = f(p)$ where $[p, q]$ constitutes a 2-cycle. Let $p < \frac{1}{2}$ and $q > \frac{1}{2}$, then $2p = q, p = 2 - 2q$ gives $p = \frac{2}{5}, q = \frac{4}{5}$. Since the multiplier is $\lambda = [f(f(x))]'|_{x=p} = f'(p)f'(f(p)) = f'(p)f'(q) = -4$, the 2-cycle is unstable.

d) Similarily, we can solve for the period-3 and period-4 points by noting that only one of these points in the cycle is greater than $\frac{1}{2}$. Hence $[\frac{2}{9}, \frac{4}{9}, \frac{8}{9}]$ and $[\frac{2}{17}, \frac{4}{17}, \frac{8}{17}, \frac{16}{17}]$ are the period-3 and period-4 solutions. (In fact, the general period-n solutions has the form $[\frac{2^1}{2^n+1}, \frac{2^2}{2^n+1}, \ldots, \frac{2^n}{2^n+1}]$.) Since the multiplier for all a period-n orbit equal to $\prod_1^n f'(p_i)$, where p_i are the points on the orbit, the multiplier is always greater than 1 and all the periodic orbit are unstable.

10.1.2

The fixed points satisfy $x^* = (x^*)^3$. Hence $x^* = 0$ or $x^* = \pm 1$. The multiplier is $\lambda = f'(x^*) = 3(x^*)^2$. The fixed point $x^* = 0$ is stable since $|\lambda| = 0 < 1$, and $x^* = \pm 1$ is unstable since $|\lambda| = 3 > 1$. If we keep pressing the appropriate function key of a pocket calculater, unless we start at ± 1, eventually we get very close to 0.

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10.1.6

Line $y = x$ intersects $y = \tan x$ infinitely many times, so there are infinitely many fixed points. For any fixed point x^* where $x^* = \tan x^*$, the multiplier $\lambda = f'(x^*) = \sec^2(x^*)$ is always greater than 1 unless $x^* = 0$. So they're all unstable. Even in the marginal case $x^* = 0$, it's clear from the cobweb that x^* is also unstable since the positive part of the curve $y = \tan x$ containing $(0,0)$ is always above $y = x$ and the negative part below it. Therefore unless we start with a fixed point, we are not likely to get a pattern no matter how many times we press the button.

10.1.11

a) Fixed points satisfy $x^* = f(x^*) = 3x^* - (x^*)^3$, therefore $x^* = 0, \pm\sqrt{2}$. The multiplier $\lambda = 3(1 - (x^*)^2)$ is always greater than 1, so all these fixed point are unstable.

b,c) Here are the cobweb graphs for $x_0 = 1.9$ (left) and $x_0 = 2.1$ (right).
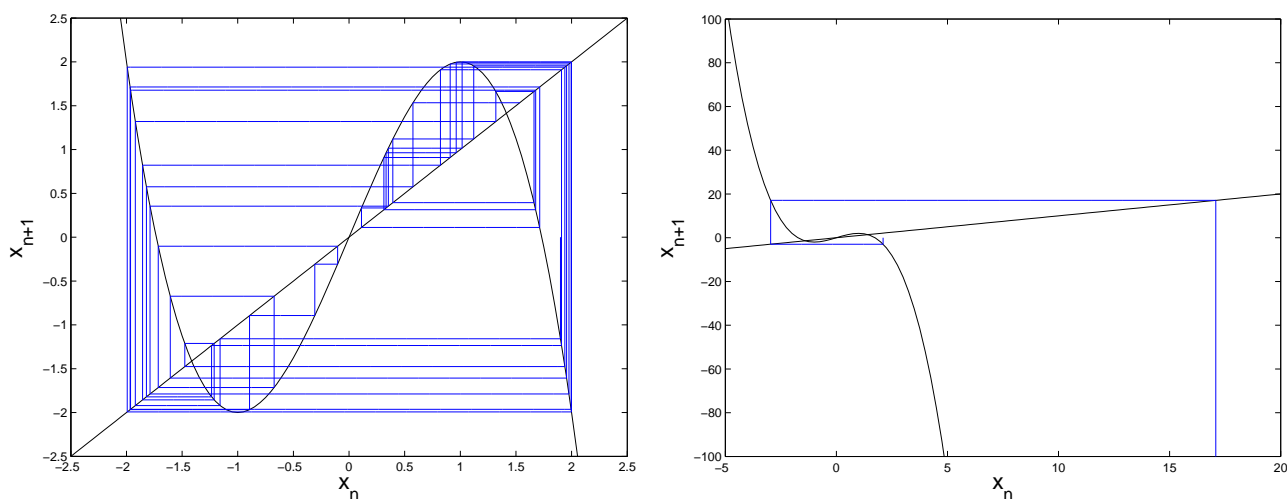
d) Note that $f(x)$ has local extrema equal to ± 2 at $x = \pm 1$. From the graphs above, we can see that if we start with an initial values x_0 where $|x_0| < 2$, the cobweb will stay inside the square with corners at $(\pm 2, \pm 2)$. However, if we start an initial value x_0 where $|x_0| > 2$, then after the cobweb misses the first peak/valley of $f(x)$, x_n will get larger and larger.

10.3.4

a) Fixed points satisfy $x^* = (x^*)^2 + c$, therefore

$$x^* = \frac{1 \pm \sqrt{1 - 4c}}{2},$$

where $c \leq \frac{1}{4}$. The multiplier $\lambda = 2x^* = 1 \pm \sqrt{1-4c}$ is always greater than 1 at the greater fixed point, denoted by x_1. So x_1 is unstable. At the other fixed point x_2 we have $-1 < \lambda < 1$ when $-\frac{3}{4} < c < \frac{1}{4}$, so x_2 is stable when $c > -\frac{3}{4}$, and unstable when $c < -\frac{3}{4}$.

b) It's clear from part a) that a saddle-node bifurcation occurs at $c = \frac{1}{4}$, where two fixed points are created, and a flip bifurcation occurs at $c = -\frac{3}{4}$, where x_2 loses its stability.

c) To get the 2-cycles we apply $f(x) = x^2 + c$ to itself and obtain the equation

$$x = f(f(x)) = (x^2 + c)^2 + c.$$

This is a quartic equation, but recall that all the fixed points should satisfy this equation and the fixed points are also roots of $x = x^2 + c$. So we write $x = f(f(x))$ as

$$(x^2 - x + c)(x^2 + x + c + 1) = 0,$$

and get the other two roots

$$p, q = \frac{-1 \pm \sqrt{-3 - 4c}}{2},$$

which are real for $c < -\frac{3}{4}$, thus a 2-cycle exists for all $c < -\frac{3}{4}$. The multiplier of the 2-cycle is

$$\lambda = \frac{d}{dx}(f(f(x)))_{x=p} = f'(p)f'(q) = 4pq.$$

Notice that p, q are roots of $x^2 + x + c + 1$, so $pq = c + 1$. Then $|\lambda| < 1$ if $-\frac{5}{4} < c < -\frac{3}{4}$, those are the values of c for which the 2-cycle is stable. The 2-cycle is superstable when $\lambda = 0$, i.e. $c = -1$.

d) In the bifurcation diagram below, solid lines indicate stable fixed points, dashed lines indicate unstable fixed points, the pair of dashed-dot lines indicate stable 2-cycles. The diagram is drawn according to the results in shown part a, b and c. Notice that the stable part is also part of the orbit diagram.
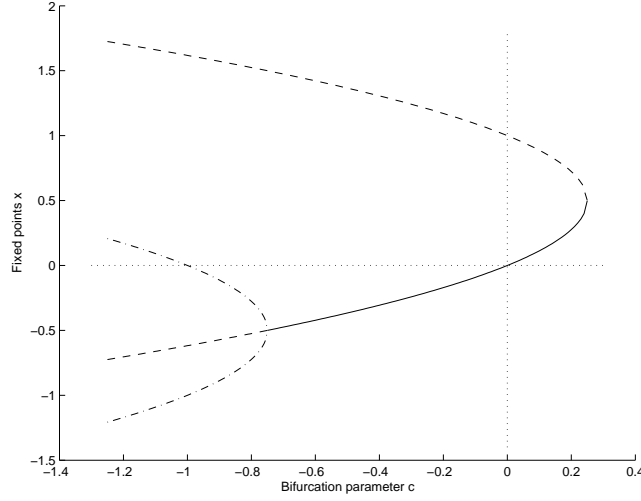
10.3.6

a) Fixed points satisfy $x^* = rx^* - (x^*)^3$, therefore

$$x^* = 0, \pm\sqrt{r-1},$$

where $r \geq 1$. The multiplier $\lambda = r - 3(x^*)^2$ which equal to r for $x^* = 0$ and $3 - 2r$ for $x^* = \pm\sqrt{r-1}$. For $|r| < 1$, zero fixed point is stable, while for $1 < r < 2$ the fixed point $x^* = \pm\sqrt{r-1}$ is stable.

b) Suppose $f(p) = q$ and $f(q) = p$ and let $s = q^2 - r$ then we have

$$r(rq - q^3) - (rq - q^3)^3 = q$$
$$q[r(r - q^2) - q^2(r - q^2)^3 - 1] = 0$$
$$q[-rs + (s + r)s^3 - 1] = 0$$
$$q[s^4 + rs^3 - rs - 1] = 0$$
$$q(s - 1)(s + 1)(s^2 + rs + 1) = 0$$
$$q(q^2 - r - 1)(q^2 - r + 1)(q^4 - rq^2 + 1) = 0$$

Solve for q and we have $q = 0, \pm\sqrt{r-1}, \pm\sqrt{r+1}, \pm\sqrt{\frac{r \pm \sqrt{r^2-4}}{2}}$. Note that the first three solutions are the fixed point. Therefore the 2-cycles are $[+\sqrt{r+1}, -\sqrt{r+1}]$ and $[\pm\sqrt{\frac{r+\sqrt{r^2-4}}{2}}, \pm\sqrt{\frac{r-\sqrt{r^2-4}}{2}}]$ for $r > 2$.

c) For the 2-cycle $[+\sqrt{r+1}, -\sqrt{r+1}]$, the multiplier $\lambda = f'(p)f'(q) = (r - 3(r + 1))^2 = (3 + 2r)^2$ which is always larger than 1 for $r > -1$. Therefore this 2-cycle is always unstable.

For $[\pm\sqrt{\frac{r+\sqrt{r^2-4}}{2}}, \pm\sqrt{\frac{r-\sqrt{r^2-4}}{2}}$, the multiplier $\lambda = f'(p)f'(q) = 9 - 2r^2$ which sits between -1 and 1 for $2 < r < \sqrt{5}$. Therefore these 2-cycles are stable for this range of values of r.

d) In the bifurcation diagram below, solid lines indicate stable fixed points, the pair of dashed-dot lines indicate stable 2-cycles.