OPMT 5701 Term Project

Selected Answers

1. **Analyzing different types of utility functions**
Consider the case of a two-good world where both goods, x and y are consumed. Let the consumer, Beaufort, have the utility function $U = U(x, y)$. Beaufort has a fixed money budget of B and faces the money prices P_x and P_y. Therefore Myrtle's maximization problem is

Maximize

$$U = U(x, y)$$

Subject to

$$B = P_x x + P_y y$$

For EACH utility function listed below,

$$\begin{array}{cl} i & U = (xy)^2 \\ ii & U = (xy)^{1/2} \\ iii & U = 2x + \ln y \\ iv & U = (x+1)\,y \end{array}$$

do the following:

(a) Find the slope of the indifference curve (dy/dx) using the differential approach (i.e. $dU = U_x dx + U_y dy = 0$)

$$\begin{array}{clc} i & U = (xy)^2 & dy/dx = -\dfrac{y}{x} \\ ii & U = (xy)^{1/2} & dy/dx = -\dfrac{y}{x} \\ iii & U = 2x + \ln y & dy/dx = -2y \\ iv & U = (x+1)\,y & dy/dx = -\dfrac{y}{x+1} \end{array}$$

(b) Set up the lagrange equation and find the optimal x and y expressed as functions of only B, p_x and p_y

$$\begin{array}{clcc} i & U = (xy)^2 & x = \frac{B}{2P_x} & y = \frac{B}{2P_y} \\ ii & U = (xy)^{1/2} & x = \frac{B}{2P_x} & y = \frac{B}{2P_y} \\ iii & U = 2x + \ln y & x = \frac{2B - P_x}{2P_x} & y = \frac{P_x}{2P_y} \\ iv & U = (x+1)\,y & x = \frac{B - P_x}{2P_x} & y = \frac{B + P_x}{2P_y} \end{array}$$

(c) Suppose the budget is initially $B = 24$, and $P_x = 6, P_y = 6$. Find the optimal x and y, value for U

$$\begin{array}{clcc} i & U = (xy)^2 = 16 & x = 2 & y = 2 \\ ii & U = (xy)^{1/2} = 2 & x = 2 & y = 2 \\ iii & U = 2x + \ln y = 6.3 & x = \frac{7}{2} & y = \frac{1}{2} \\ iv & U = (x+1)\,y = 6.25 & x = \frac{3}{2} & y = \frac{5}{2} \end{array}$$

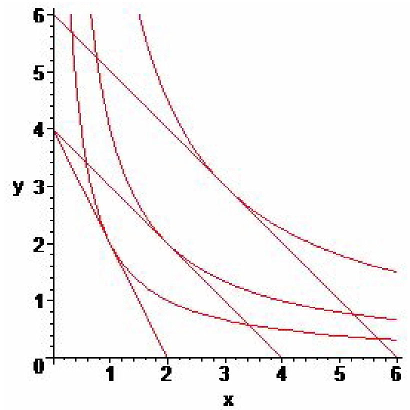
(d) Let B increase from 24 to 36. Find the NEW optimal x and y, value for U

$$\begin{array}{clcc} i & U = (xy)^2 = 81 & x = 3 & y = 3 \\ ii & U = (xy)^{1/2} = 3 & x = 3 & y = 3 \\ iii & U = 2x + \ln y = 10.3 & x = \frac{11}{2} & y = \frac{1}{2} \\ iv & U = (x+1)\,y = 12.25 & x = \frac{5}{2} & y = \frac{7}{2} \end{array}$$

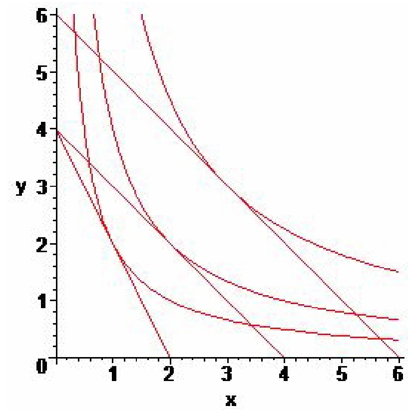
(e) Let B again be 24, but now let the price of x (p_x) rise from 6 to 12.Find the NEW optimal x and y , value for U. In any case is there an unusual results?

i	$U = (xy)^2 = 9$	$x = 1$	$y = 2$
ii	$U = (xy)^{1/2} = 3$	$x = 1$	$y = 2$
iii	$U = 2x + \ln y = 3$	$x = \frac{3}{2}$	$y = 1$
iv	$U = (x+1)\,y = 4.5$	$x = \frac{1}{2}$	$y = 3$

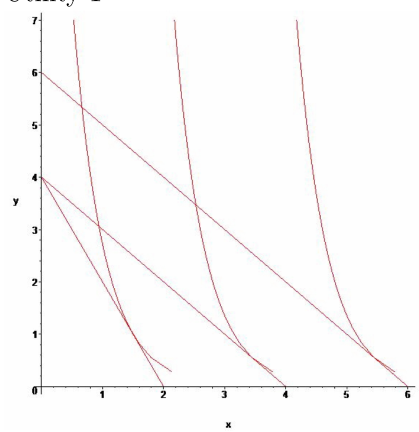
(f) GRAPH your answers for parts (b), (c), and (d). You will have one graph for EACH utility function. These need to be very well done. Accuracy is the Key. is there any case where the indifference curves have vertical or horizontal intercepts?



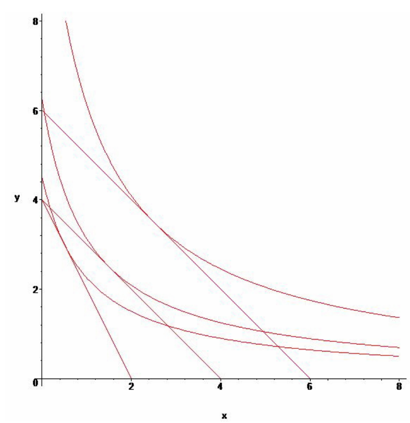
Utility 1



Utility 2



Utility 3



Utility 4

2. **Willingness to Pay versus Equivalent Compensation**
Skippy and Myrtle are friends who consume the same goods: yoga classes (X) and Timbits (Y). Skippy has the utility function $u = 10xy$ and faces the budget constraint: $M = p_x x + p_y y$. where M is income, and p_x, p_y are prices. Myrtle has the same budget constraint as Skippy but her utility function is $v = xy^2$.

(a) Use the Lagrange Method to show that Skippy and Myrtles' demand functions for x and y, which will be in the form:

$$\begin{aligned} L &= 10xy + \lambda(M - p_x x - p_y y) \\ L_x &= 10y - \lambda p_x = 0 \\ L_y &= 10x - \lambda p_y = 0 \\ L_\lambda &= M - p_x x - p_y y = 0 \end{aligned}$$

$$\begin{aligned} \frac{10y}{10x} &= \frac{\lambda p_x}{\lambda p_y} \\ \frac{y}{x} &= \frac{p_x}{p_y} \end{aligned}$$

Substitute into budget to find

$$x^s = \frac{M}{2p_x} \qquad y^s = \frac{M}{2p_y} \qquad u = \frac{10M^2}{4p_x p_y}$$

The expenditure function is

$$M^* = \sqrt{\frac{4p_x p_y u}{10}}$$

Then, for Myrtle,

$$\begin{aligned} L &= xy^2 + \lambda(M - p_x x - p_y y) \\ L_x &= y^2 - \lambda p_x = 0 \\ L_y &= 2xy - \lambda p_y = 0 \\ L_\lambda &= M - p_x x - p_y y = 0 \end{aligned}$$

$$\begin{aligned} \frac{y^2}{2xy} &= \frac{\lambda p_x}{\lambda p_y} \\ \frac{y}{2x} &= \frac{p_x}{p_y} \end{aligned}$$

$$\begin{aligned} x^m &= \frac{M}{3p_x} \qquad y^m = \frac{2M}{3p_y} \\ v &= \left(\frac{M}{3p_x}\right)\left(\frac{2M}{3p_y}\right)^2 = \frac{4M^3}{27p_x p_y^2} \end{aligned}$$

and Myrtle's expenditure function is

$$M^* = \sqrt[3]{\frac{27p_x p_y^2 v}{4}}$$

(b) Suppose $M = 120$, $P_y = 1$ and $P_x = 4$. What is Skippy and Myrtles' optimal x, y and utility number? If the price of x was lowered to \$2 what would be their x, y and utility numbers?
Using the formula's in part (a) At

$P_x = 4$	*Skippy*	*Myrtle*
x^*	15	10
y^*	60	80
U^*	9000	64000

Then

$P_x = 2$	$Skippy$	$Myrtle$
x^*	30	20
y^*	60	80
U^*	18000	128000

(c) If the price remains as \$4, how much additional income would each one require to get the same utility as they did when the price was \$2
Use each person's expenditure function $((M^*)$ shown in part (a)

$Skippy$	$Myrtle$
$u = 18000$	$v = 128000$
$M^* = \sqrt{\frac{4p_xp_yu}{10}}$	$M^* = \sqrt[3]{\frac{27p_xp_y^2v}{4}}$
$M = 169.7$	$M = 151.2$

(d) Draw a graph for Skippy and another for Myrtle. In each graph show the budget constraints, indifference curves and equilibrium values of x and y for both $p_x = 4$ and $p_x = 2$. Be ACCURATE and NEAT. (minimum 1/2 page for each graph)

(e) Suppose that the Yoga Studio offers two options: a drop-fee of \$4, or a membership of \$30 that lets the member take Yoga classes for \$2. What will be the Utility of Skippy and Myrtle if they each buy a membership? Given the options (drop-in or member) which will each one choose? Carefully add the new budget constraint, indifference curve and equilibrium x and y to each girl's graph.

	$Skippy$	$Myrtle$
$Drop-in$	9000	64000
$Member$	10125	54000

Skippy will join, Myrtle will not join

(f) What is the membership fee that would make Skippy indifferent between drop at \$4 and membership that charges \$2 per yoga class? Add this to Skippy's graph.

$$Membership = 120 - 84.5 = 35.5$$

(g) What is the membership fee that would make Myrtle indifferent between drop at \$4 and membership that charges \$2 per yoga class? Add this to Myrtle's graph.

$$Membership = 120 - 95.4 = 24.6$$

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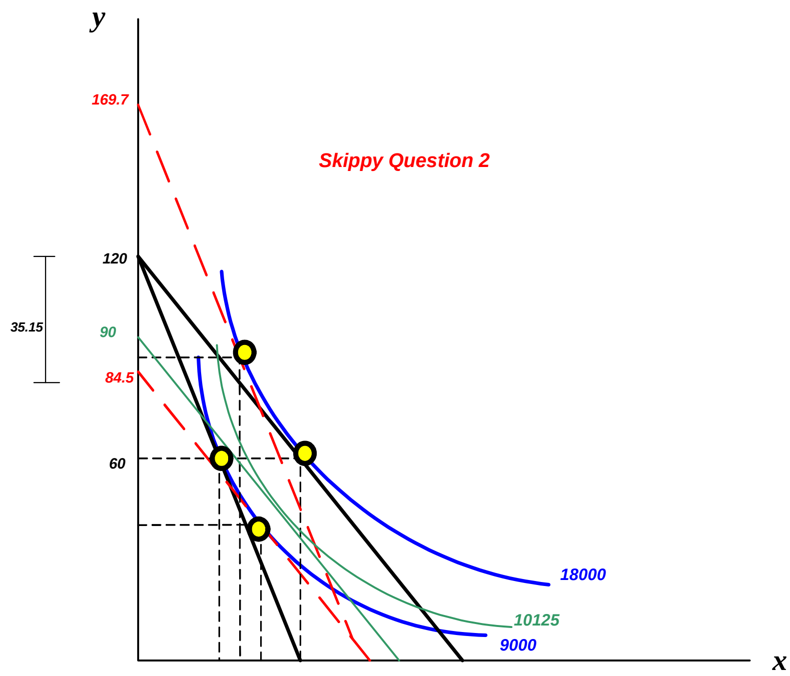
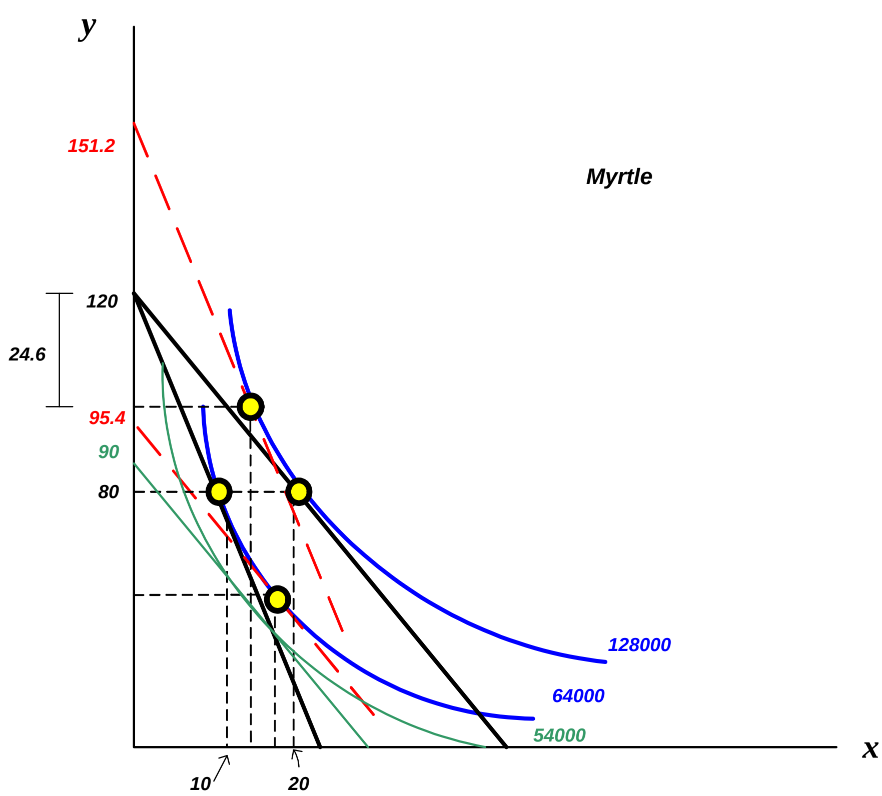
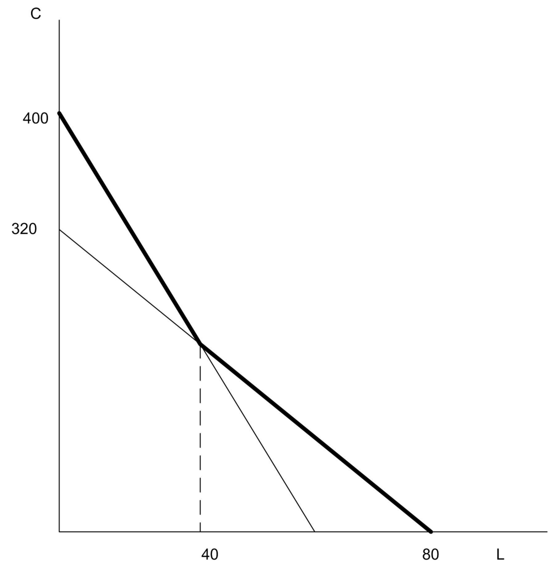

y
169.7
Skippy Question 2
120
35.15
90
84.5
60
18000
10125
9000
x


y
151.2
Myrtle
120
24.6
95.4
90
80
128000
64000
54000
10
20
x



3. Labour supply

(a) There are 168 hours in one week. Subtracting 48 hours for the weekend and 8 hours per night for sleep leaves 80 hours for work and leisure. Sparky works at a job that pays w per hour for the first 40 hours, then $1.5w$ for each hour after that. Let h be the hours of work such that $h = 80 - L$. Sparky consumes a composite commodity C which costs p. Sparky maximizes her utility $u(C, L)$ subject to the following budget constraints:

$$\begin{aligned} L &= 80 - h \\ 80w &= pC + wL \quad if \quad 0 \leq h \leq 40 \\ 100w &= pC + 1.5wL \quad if \quad h > 40 \end{aligned}$$

(b) If $p = 1$ and $w = 4$, graph both constraints. Verify that the two constraints cross at 40 hours of work. Given the restrictions on l, use a hi-lite pen to show Sparky's "Effective Budget Constraint".

(c) If Sparky's utility function is $u = CL$, find her optimal L and C. (hint: do the Lagrangian twice, once with each constraint; then check which solution satisfies all the conditions).

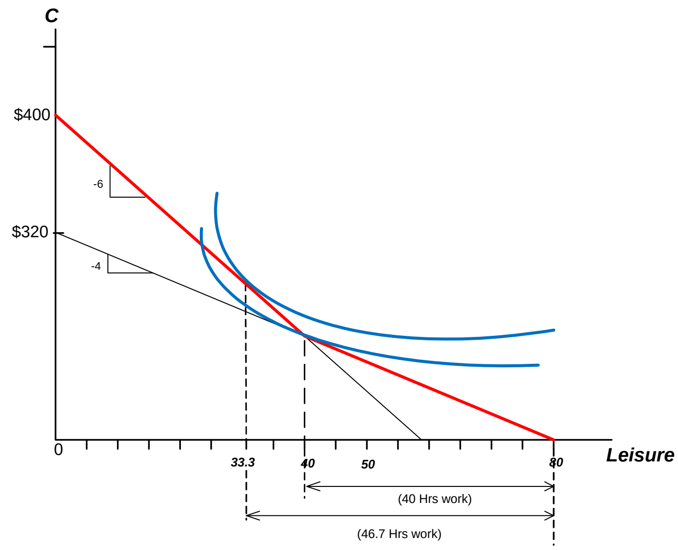
$$\begin{aligned} Z &= CL + \lambda(320 - C - 4L) \\ Z_L &= C - 4\lambda = 0 \\ Z_C &= L - \lambda = 0 \\ Z_\lambda &= 320 - C - 4L = 0 \\ L &= 40, C = 160, U = 6400 \\ h &= 40 \end{aligned}$$

Overtime:

$$\begin{aligned} Z &= CL + \lambda(400 - C - 6L) \\ Z_L &= C - 6\lambda = 0 \\ Z_C &= L - \lambda = 0 \\ Z_\lambda &= 400 - C - 6L = 0 \\ L &= 33.3, C = 200, U = 6666 \\ h &= 46.7 \end{aligned}$$

Therefore Sparky prefers overtime

(d) Calculate the utility values for both solutions in (b) illustrate both equilibrium points and indifference curves in your graph drawn for (a). Does the incentive of an overtime premium induce Sparky to work more than 40 hours?



(e) Sparky's cousin Otis gets hired for the same job. However Otis's utility function is $u = CL^2$. Re-do parts (a)-(c) from above except using Otis's utility function.

$$\begin{aligned} Z &= CL^2 + \lambda(320 - C - 4L) \\ Z_L &= 2CL - 4\lambda = 0 \\ Z_C &= L^2 - \lambda = 0 \\ Z_\lambda &= 320 - C - 4L = 0 \\ L &= 53.3, C = 106.6, U = 303407 \\ h &= 26.7 \end{aligned}$$

Overtime:

$$\begin{aligned} Z &= CL^2 + \lambda(400 - C - 6L) \\ Z_L &= 2CL - 6\lambda = 0 \\ Z_C &= L^2 - \lambda = 0 \\ Z_\lambda &= 400 - C - 6L = 0 \\ L &= 44.4, C = 133.3, U = 263374 \\ h &= 35.6 \end{aligned}$$

Therefore Otis will NOT work any overtime

4. Skippy lives on an island where she produces two goods, x *and* y, according the the production possibility frontier $400 \geq x^2 + y^2$, and she consumes all the goods herself. Skippy also faces and environmental constraint on her total output of both goods. The environmental constraint is given by $x + y \leq 28$. Her utility function is

$$u = x^{1/2}y^{1/2}$$

(a) Write down the Kuhn Tucker first order conditions.

$$Z = x^{1/2}y^{1/2} + \lambda_1(400 - x^2 - y^2) + \lambda_2(28 - x - y)$$

The Kuhn-Tucker conditions are now

$$\begin{array}{ll} Z_x = \frac{1}{2}x^{-1/2}y^{1/2} - 2x\lambda_1 - \lambda_2 = 0 & \\ Z_y = \frac{1}{2}x^{1/2}y^{-1/2} - 2y\lambda_1 - \lambda_2 = 0 & \\ Z_{\lambda_1} = 400 - x^2 - y^2 \geq 0 & \lambda_1 \geq 0 \quad \lambda_1 \cdot Z_{\lambda_1} = 0 \\ Z_{\lambda_2} = 28 - x - y \geq 0 & \lambda_2 \geq 0 \quad \lambda_2 \cdot Z_{\lambda_2} = 0 \end{array}$$

(b) Find Skippy's optimal x *and* y. Identify which constaints are binding.**From the first two equations, we get:**

$$\begin{aligned} \frac{\frac{1}{2}x^{-1/2}y^{1/2}}{\frac{1}{2}x^{1/2}y^{-1/2}} &= \frac{2x\lambda_1+\lambda_2}{2y\lambda_1+\lambda_2} \\ & \quad or \\ \frac{y}{x} &= \frac{2x\lambda_1+\lambda_2}{2y\lambda_1+\lambda_2} \end{aligned}$$

try $\lambda_2 = 0$. then we get:

$$\begin{aligned} \frac{y}{x} &= \frac{2x\lambda_1}{2y\lambda_1} \\ y^2 &= x^2 \end{aligned}$$

Sub into the PPF constraint $(400 - x^2 - y^2 = 0)$ to get,

$$\begin{aligned} 400 &= 2x^2 \\ x^2 &= 200 \\ x &= 14.14 \\ y &= 14.14 \end{aligned}$$

But this violates the other constraint $(x + y \leq 28)$ so try $\lambda_1 = 0$

$$\begin{aligned} \frac{y}{x} &= \frac{\lambda_2}{\lambda_2} = \frac{1}{1} \\ x &= y \\ x &= 14, y = 14 \end{aligned}$$

Environmental Constraint is Binding

(c) Graph your results.

(d) On the next island lives Sparky who has all the same constraints as Skippy but Sparky's utility function is $u = \ln x + 3 \ln y$. Redo a, b, and c for Sparky. The only difference is that from the first order conditions

$$\frac{y}{3x} = \frac{2x\lambda_1+\lambda_2}{2y\lambda_1+\lambda_2}$$

try $\lambda_2 = 0$. then we get:

$$\begin{aligned} \frac{y}{3x} &= \frac{2x\lambda_1}{2y\lambda_1} \\ y^2 &= 3x^2 \end{aligned}$$

Sub into the PPF constraint $(400 - x^2 - y^2 = 0)$ to get,

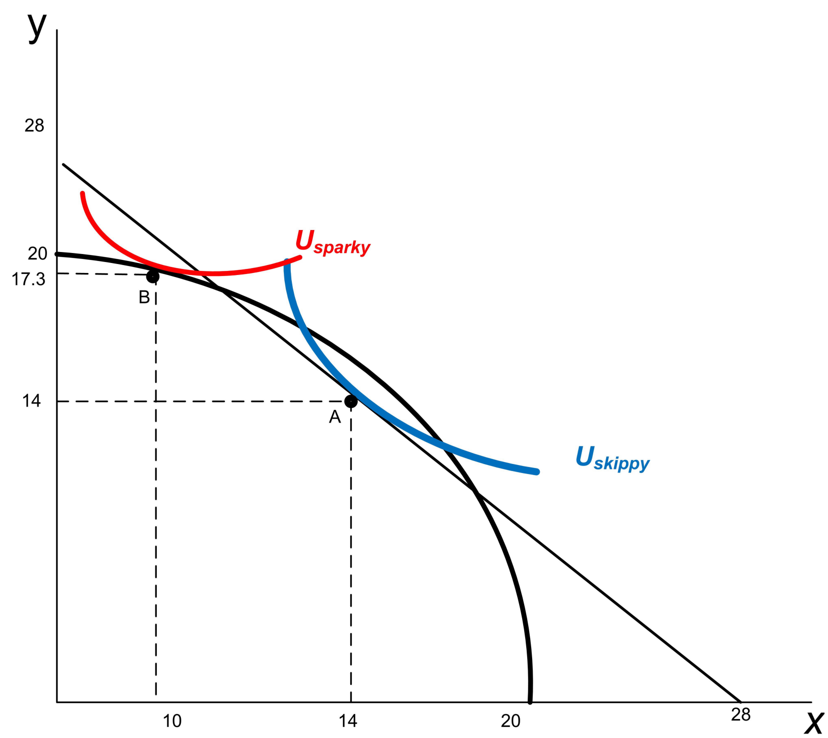
$$\begin{aligned} 400 &= 4x^2 \\ x^2 &= 100 \\ x &= 10 \end{aligned}$$

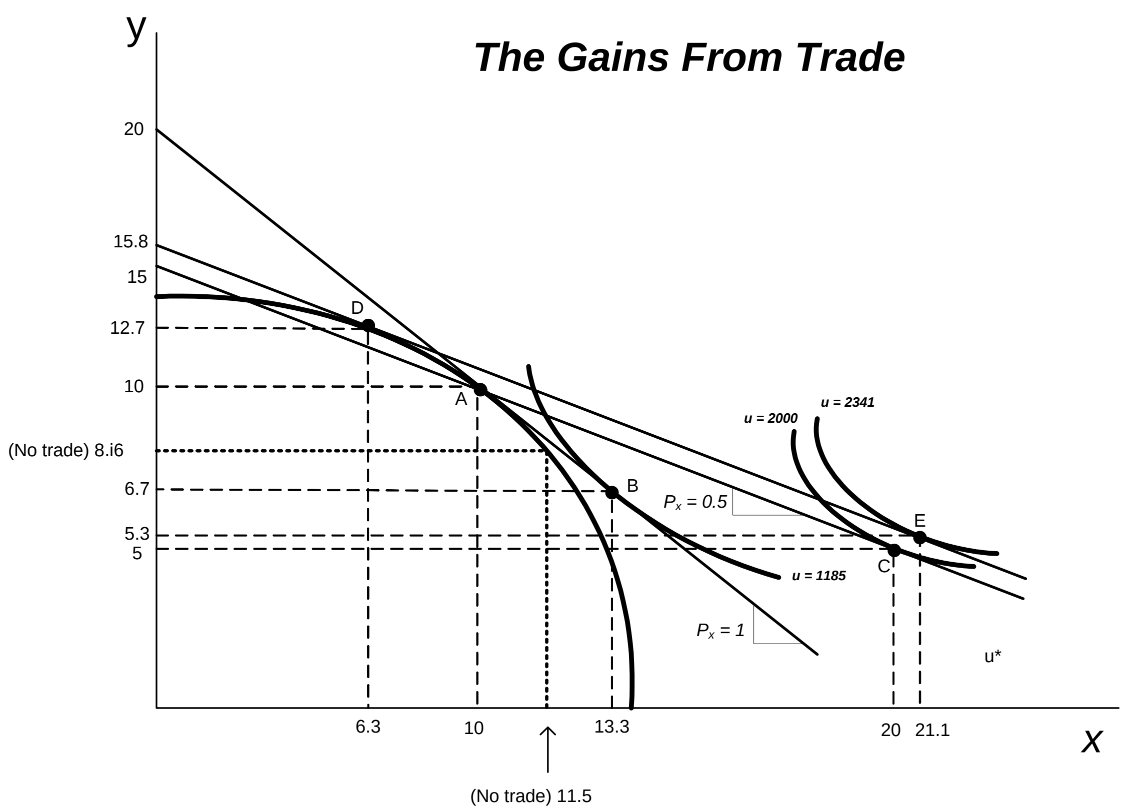
and

$$\begin{aligned} y^2 &= 300 \\ y &= 17.3 \end{aligned}$$

which does NOt violate the other constraint $(x + y \leq 28)$

(c)





5. International Trade: Solving the problem of Production and Exchange.
   The following graph is found in most economics texts and it illustrates how an economy can get outside its PPF through free trade. Solving this problem is a straight forward application of Lagrange. Figure 1:

   (a) Skippy lives on an island full of resources that lets her produce two goods: x and y. Her production possibility frontier (constraint) is given by

   $$x^2 + y^2 = 200$$

   Skippy works alone to maximize her utility which is given by $u = x^2 y$

   (b) Set up the Lagrangian for Skippy and solve for x^*, y^*.Use your solution to calculate u^*.

   $$\begin{aligned} L &= x^2 y + \lambda \left(200 - x^2 - y^2\right) \\ x &= 11.5, y = 8.16, u = 1089 \end{aligned}$$

   (c) Carefully graph and label the PPF and Indifference Curve at the optimum.

   (d) Now Skippy has the opportunity of to trade with Myrtle on a neighboring island. Skippy can trade whatever x and y she has at prices p_x and p_y. To maximize her gains from trade, Skippy needs to follow a 2-step process.

   $$\begin{aligned} L^E &= p_x x + p_y y + \lambda(200 - x^2 - y^2) \\ L^U &= x^2 y + \lambda \left(p_x X_E + p_y Y_E - p_x x - p_y y\right) \end{aligned}$$

   (e) If $p_x = p_y = 1$, solve the Lagrangian that maximizes her endowments. Then use the endowment values and prices given in the trading constraint and maximize her utility by the Lagrangian method. Calculate her utility number.

$$
\begin{aligned}
L &= x + y + \lambda(200 - x^2 - y^2) \\
x &= y = 10 \\
L &= x^2 y + \lambda(20 - x - y) \\
x &= 13.3, y = 6.7, u = 1185
\end{aligned}
$$

(f) CAREFULLY construct a graph containing her PPF, Trading Line, and indifference curve at the optimum. Is her utility number greater than in part one?

(g) Now suppose the trading prices change such that $p_x = 0.5$. Using your endowment values (X_E, Y_E) from before, calculate Skippy's new utility maximizing bundle. Draw the new trading line and indifference curve on your graph in (b) above.

$$
\begin{aligned}
x &= y = 10 \\
L &= x^2 y + \lambda(15 - 0.5x - y) \\
x &= 20, y = 5, u = 2000
\end{aligned}
$$

(h) Since the trading prices have changed, Skippy needs to adjust her production decision. Re-do (a) and (b) with the new prices.

$$
\begin{aligned}
L &= 0.5x + y + \lambda(200 - x^2 - y^2) \\
x &= 6.3, y = 12.7 \\
L &= x^2 y + \lambda(15.8 - 0.5x - y) \\
x &= 21.1, y = 5.3, u = 2341
\end{aligned}
$$

6. Skippy has the following utility function $U = \sqrt{W}$, where W is her wealth. She has a probability P she has a bad day (State 1) loses an amount L.Her expected utility is:

$$EU = (1-P)W^{1/2} + P(W-L)^{1/2}$$

Suppose her initial wealth is \$2500, the loss, $L = \$2100$ and P = 30%

(a) What is Skippy's expected utility? What is the certainty equivalent wealth? Graph your answer.

(b) Skippy can buy units of insurance at a price of a_1 on good days that will give her units of coverage a_2 on bad days. If the insurance is offered at fair odds, then

$$\begin{aligned} (1-P)a_1 - Pa_2 &= 0 \\ 0.7a_1 - 0.3a_2 &= 0 \end{aligned}$$

which is the "budget constraint" Skippy faces. If she buys insurance her expected utility will be

$$EU = (1-P)\left(W - a_1\right)^{1/2} + P(W - L + a_2)^{1/2}$$

Set up the Lagrangian for this problem. Show that Skippy will fully insure *(i.e. her net wealth on both good and bad days will be equal).*

$$\begin{aligned} L &= 0.7(2500 - a_1)^{1/2} + 0.3(400 + a2)^{1/2} + \lambda\left(0.7a_1 - 0.3a_2\right) \\ L_1 &= \frac{0.7}{2}(2500 - a_1)^{-1/2}(-1) + 0.7\lambda = 0 \\ L_2 &= \frac{0.3}{2}(400 + a_2)^{-1/2} - 0.3\lambda = 0 \\ L_\lambda &= 0.7a_1 - 0.3a_2 = 0 \end{aligned}$$

from equations (1) and (2)

$$\begin{aligned} \frac{\frac{0.7}{2}(2500 - a_1)^{-1/2}}{\frac{0.3}{2}(400 + a_2)^{-1/2}} &= \frac{0.7\lambda}{0.3\lambda} \\ \frac{(2500 - a_1)^{-1/2}}{(400 + a_2)^{-1/2}} &= \frac{1}{1} \\ (2500 - a_1)^{-1/2} &= (400 + a_2)^{-1/2} \\ 2500 - a_1 &= 400 + a_2 \\ 2100 &= a_1 + a_2 \end{aligned}$$

Use the budget constraint, re-written as $a_1 = \frac{3}{7}a_2$

$$\begin{aligned} 2100 &= \frac{3}{7}a_2 + a_2 = \frac{10}{7}a_2 \\ a_2 &= \frac{7}{10}(2100) = 1470 \\ a_1 &= 630 \end{aligned}$$

Skippy's wealth in each state will be

$$\begin{aligned} &\textit{good} \quad W = 2500 - 630 = 1870 \\ &\textit{Bad} \quad W = 400 + 1470 = 1870 \end{aligned}$$

(c) Myrtle is the same as Skippy in every way except she has a 20% chance of losing \$2100. However the insurance company cannot tell the two apart, so they offer fair insurance based on the average probability (0.25).How much insurance will Skippy and Myrtle each buy?
For Myrtle, the Lagrange is

$$\begin{aligned} L &= 0.8(2500 - a_1)^{1/2} + 0.2(400 + a2)^{1/2} + \lambda\left(0.75a_1 - 0.25a_2\right) \\ L_1 &= \frac{0.8}{2}(2500 - a_1)^{-1/2}(-1) + 0.75\lambda = 0 \\ L_2 &= \frac{0.2}{2}(400 + a_2)^{-1/2} - 0.25\lambda = 0 \\ L_\lambda &= 0.75a_1 - 0.25a_2 = 0 \end{aligned}$$

The solution is $a_1 = 282.5$ and $a_2 = 847.4$
For Skippy

$$\begin{aligned}
L &= 0.7(2500 - a_1)^{1/2} + 0.3(400 + a2)^{1/2} + \lambda\,(0.75a_1 - 0.25a_2) \\
L_1 &= \frac{0.7}{2}(2500 - a_1)^{-1/2}(-1) + 0.75\lambda = 0 \\
L_2 &= \frac{0.3}{2}(400 + a_2)^{-1/2} - 0.25\lambda = 0 \\
L_\lambda &= 0.75a_1 - 0.25a_2 = 0
\end{aligned}$$

The solution is $a_1 = 802.2$ and $a_2 = 2406.5$