OPMT 5701
Multivariable Optimization

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(Ch 11)

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1 Two Variable Maximization

Suppose we want to maximize the following function

$$z = f(x, y) = 10x + 10y + xy - x^2 - y^2$$

Note that there are two unknowns that must be solved for: x and y. This function is an example of a *three-dimensional dome.* (i.e. the roof of *BC Place*)

To solve this maximization problem we use **partial derivatives.** We take a partial derivative for each of the unknown choice variables and set them equal to zero

$$\begin{array}{ll} \frac{\partial z}{\partial x} = f_x = 10 + y - 2x = 0 & \textit{The slope in the "x" direction = 0} \\ \frac{\partial z}{\partial y} = f_y = 10 + x - 2y = 0 & \textit{The slope in the "y" direction = 0} \end{array}$$

This gives us a set of equations, one equation for each of the unknown variables. When you have the same number of independent equations as unknowns, you can solve for each of the unknowns.

rewrite each equation as

$$y = 2x - 10$$

$$x = 2y - 10$$

substitute one into the other

$$x = 2(2x - 10) - 10$$

$$x = 4x - 30$$

$$3x = 30$$

$$x = 10$$

similarly,

$$y = 10$$

REMEMBER: To maximize (minimize) a function of many variables you use the technique of partial differentiation. This produces a set of equations, one equation for each of the unknowns. You then solve the set of equations simulaneously to derive solutions for each of the unknowns.

2 Second order Conditions

To test for a maximum or minimum we need to check the second partial derivatives. Since we have two first partial derivative equations (f_x,f_y) and two variable in each equation, we will get four *second partials* ($f_{xx}, f_{yy}, f_{xy}, f_{yx}$)

Using our original first order equations and taking the partial derivatives for each of them (a second time) yields:

$$f_x = 10 + y - 2x = 0 \quad f_y = 10 + x - 2y = 0$$

$$\begin{array}{ll} f_{xx} = -2 & f_{yy} = -2 \\ f_{xy} = 1 & f_{yx} = 1 \end{array}$$

The two partials,f_{xx}, and f_{yy} are the direct effects of of a small change in x and y on the respective slopes in in the x and y direction. The partials, f_{xy} and f_{yx} are the indirect effects, or the cross effects of one variable on the slope in the other variable's direction. For both *Maximums and Minimums*, the direct effects must outweigh the cross effects

2.1 Rules for two variable Maximums and Minimums

1. Maximum

$$\begin{aligned} f_{xx} &< 0 \\ f_{yy} &< 0 \\ f_{yy}f_{xx} - f_{xy}f_{yx} &> 0 \end{aligned}$$

2. Minimum

$$\begin{aligned} f_{xx} &> 0 \\ f_{yy} &> 0 \\ f_{yy}f_{xx} - f_{xy}f_{yx} &> 0 \end{aligned}$$

3. Otherwise, we have a *Saddle Point*

From our second order conditions, above,

$$\begin{array}{ll} f_{xx} = -2 < 0 & f_{yy} = -2 < 0 \\ f_{xy} = 1 & f_{yx} = 1 \end{array}$$

and

$$f_{yy}f_{xx} - f_{xy}f_{yx} = (-2)(-2) - (1)(1) = 3 > 0$$

therefore we have a maximum.

2.2 Young's Theorem

For cross partial "effects" the order of differentiation is immaterial. Therefore:

$$f_{xy} = f_{yx}$$

As long as the cross partials are continuous. In the case of GENERAL FUNCTIONS , this will always be assumed to be true!

Example:

$$z = x^3 + 5xy = y^2$$

$$\underbrace{\begin{array}{cc} \underline{f_x = 3x^2 + 5y} & \underline{f_y = 5x - 2y} \\ f_{xx} = 6x & f_{yy} = -2 \\ f_{xy} = 5 & f_{yx} = 5 \end{array}}_{f_{xy} = f_{yx}}$$

2.3 Example: Two Market Monopoly with Joint Costs

A monopolist offers two different products, each having the following market demand functions

$$q_1 = 14 - \frac{1}{4}p_1$$
$$q_2 = 24 - \frac{1}{2}p_2$$

The monopolist's joint cost function is

$$C(q_1, q_2) = q_1^2 + 5q_1q_2 + q_2^2$$

The monopolist's profit function can be written as

$$\pi = p_1q_1 + p_2q_2 - C(q_1, q_2) = p_1q_1 + p_2q_2 - q_1^2 - 5q_1q_2 - q_2^2$$

which is the function of four variables: p_1, p_2, q_1,and q_2. Using the market demand functions, we can eliminate $p_1$and p_2 leaving us with a two variable maximization problem. First, rewrite the demand functions to get the inverse functions

$$p_1 = 56 - 4q_1$$
$$p_2 = 48 - 2q_2$$

Substitute the inverse functions into the profit function

$$\pi = (56 - 4q_1)q_1 + (48 - 2q_2)q_2 - q_1^2 - 5q_1q_2 - q_2^2$$

The first order conditions for profit maximization are

$$\frac{\partial\pi}{\partial q_1} = 56 - 10q_1 - 5q_2 = 0$$
$$\frac{\partial\pi}{\partial q_2} = 48 - 6q_2 - 5q_1 = 0$$

Solve the first order conditions

$$q_1^* = 2.75$$

$$q_2^* = .7$$

Using the inverse demand functions to find the respective prices, we get

$$p_1^* = 56 - 4(2.75) = 45$$
$$p_2^* = 48 - 2(5.7) = 36.6$$

From the profit function, the maximum profit is

$$\pi = 213.94$$

Next, check the second order conditions to verify that the profit is at a maximum. The various second derivatives for this problem are

$$\begin{array}{cc} \pi_{11} = -10 & \pi_{12} = -5 \\ \pi_{21} - 5 & \pi_{22} - 6 \end{array}$$

The sufficient conditions are

$$\pi_{11} = -10 < 0$$
$$\pi_{11}\pi_{22} - \pi_{12}\pi_{21} = (-10)(-6) - (-5)^2 = 35 > 0$$

2.4 Example: Cobb-Douglas production function and a competitive firm

Consider a competitive firm with the following profit function

$$\pi = TR - TC = PQ - wL - rK$$

where P is price, Q is output, L is labour and K is capital, and w and r are the input prices for L and K respectively. Since the firm operates in a competitive market, the exogenous variables are P,w and r. There are three endogenous variables, K, L and Q. However output, Q, is in turn a function of K and L via the production function

$$Q = f(K, L)$$

which in this case, is the Cobb-Douglas function

$$Q = L^a K^b$$

where a and b are positive parameters. If we further assume decreasing returns to scale, then a + b < 1. For simplicity, let's consider the symmetric case where $a = b = \frac{1}{4}$

$$Q = L^{\frac{1}{4}} K^{\frac{1}{4}}$$

Substituting Equation 3 into Equation 1 gives us

$$\pi(K, L) = PL^{\frac{1}{4}} K^{\frac{1}{4}} - wL - rK$$

The first order conditions are

$$\begin{array}{l} \frac{\partial \pi}{\partial L} = P\left(\frac{1}{4}\right) L^{-\frac{3}{4}} K^{\frac{1}{4}} - w = 0 \\ \frac{\partial \pi}{\partial K} = P\left(\frac{1}{4}\right) L^{\frac{1}{4}} K^{-\frac{3}{4}} - r = 0 \end{array}$$

This system of equations define the optimal L and K for profit maximization. But first, we need to check the second order conditions to verify that we have a maximum.

$$\begin{matrix} \pi_{LL} & \pi_{LK} \\ \pi_{KL} & \pi_{KK} \end{matrix} = \begin{matrix} P(-\frac{3}{16})L^{-\frac{7}{4}}K^{\frac{1}{4}} & P\left(\frac{1}{4}\right)^2 L^{-\frac{3}{4}}K^{-\frac{3}{4}} \\ P\left(\frac{1}{4}\right)^2 L^{-\frac{3}{4}}K^{-\frac{3}{4}} & P\left(-\frac{3}{16}\right) L^{\frac{1}{4}}K^{\frac{7}{4}} \end{matrix}$$

The second order conditions are satisfied.

We can now return to the first order conditions to solve for the optimal K and L. Rewriting the first equation in Equation 5 to isolate K

$$P\left(\tfrac{1}{4}\right) L^{-\frac{3}{4}}K^{\frac{1}{4}} = w$$
$$K = (\tfrac{4w}{p}L^{\frac{3}{4}})^4$$

Substituting into the second equation of Equation 5

$$\tfrac{P}{4}L^{\frac{1}{4}}K^{-\frac{3}{4}} = \left(\tfrac{P}{4}\right) L^{\frac{1}{4}} \left[\left(\tfrac{4w}{p}L^{\frac{3}{4}}\right)^4\right]^{-\frac{3}{4}} = r$$
$$= P^4 \left(\tfrac{1}{4}\right)^4 w^{-3}L^{-2} = r$$

Re-arranging to get L by itself gives us

$$L^* = (\frac{P}{4}w^{-\frac{3}{4}}r^{-\frac{1}{4}})^2$$

Taking advantage of the symmetry of the model, we can quickly find the optimal K

$$K^* = (\frac{P}{4}r^{-\frac{3}{4}}w^{-\frac{1}{4}})^2$$

L* and K^* are the firm's factor demand equations.

2.5 Economic Interpretation of the 2nd Derivatives

2.5.1 Example: Output Maximization

Let

$$Q = 10L + 10K + LK - L^2 - K^2$$

F.O.C.'s

$$\begin{bmatrix} \frac{\partial Q}{\partial L} = 10 + K - 2L = 0 \\ \frac{\partial Q}{\partial K} = 10 + L - 2K = 0 \end{bmatrix} \text{ OR } \left\{ \begin{matrix} 2L - K = 10 \\ -L + 2K = 10 \end{matrix} \right\}$$

2 Equations with 2 unknowns from FOC. :

$$\begin{aligned} L &= 10 \\ K &= 10 \end{aligned}$$

Now check 2nd order conditions from F.O.C

$$\begin{matrix} Q_{LL} = -2 & Q_{LK} = 1 \\ Q_{KL} = 1 & Q_{KK} = -2 \end{matrix}$$

$$\underbrace{Q_{LL} = -2 < 0 \quad Q_{LL}Q_{KK} - Q_{KL}^2 = (-2)(-2) - (1) > 0}_{\underline{\text{Therefore Q is Max at K=10, L=10}}}$$

Given the production function

$$Q = Q(K, L)$$

$$Q_L > 0, Q_{LL} < 0 \text{ implies the "Law of Diminishing Returns"}$$

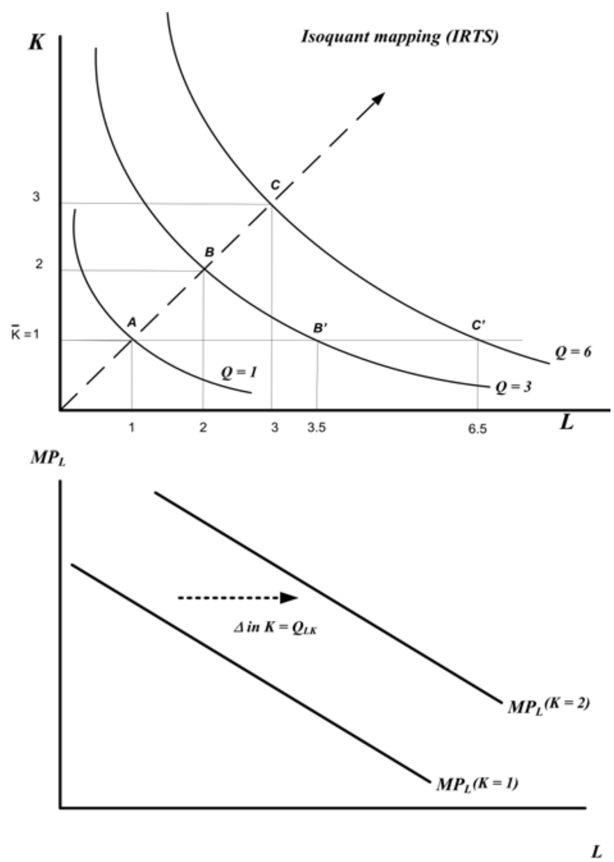
The condition

$$Q_{LL}Q_{KK} - Q_{KL}^2 > 0$$

1. (a) says that for a Maximum the direct effects (Q_{LL}, Q_{KK}) must outweigh the indirect effects (Q_{KL}, Q_{LK})

   (b) a production function can have the properties of "the law of diminishing returns" and "increasing returns to scale" <u>at the same tim</u>e

   (c) Therefore Q(K,L) has no unconstrained maximum



2.6 Example: Profit Maximization

Suppose we have the following production

$$q = f(K, L) = L^{\frac{1}{2}} + K^{\frac{1}{2}} \qquad \left\{ \begin{array}{c} q = output \\ L = labour \\ K = capital \end{array} \right\}$$

Then the profit function for a competitive firm is

$$\begin{aligned} \pi &= Pq - wL - rK \qquad \left\{ \begin{array}{c} P = market\ price \\ w = wage\ rate \\ r = rental\ rate \end{array} \right\} \\ or\ \ \pi &= PL^{\frac{1}{2}} + PK^{\frac{1}{2}} - wL - rK \end{aligned}$$

First Order Conditions

$$\begin{aligned} (1) \quad \frac{\partial \pi}{\partial L} &= \frac{P}{2} L^{\frac{1}{2}} - w = 0 \qquad \overbrace{\{Pf_L - w = 0\}}^{\text{General Form}} \\ (2) \quad \frac{\partial \pi}{\partial K} &= \frac{P}{2} K^{\frac{1}{2}} - r = 0 \qquad \{Pf_K - r = 0\} \end{aligned}$$

Solving (1) and (2) we get

$$L^* = \left(\tfrac{P}{2w}\right)^2 \qquad K^* = \left(\tfrac{P}{2r}\right)^2$$

n

$$\begin{aligned} &\begin{pmatrix} Pf_{LL} & Pf_{LK} \\ Pf_{KL} & Pf_{KK} \end{pmatrix} \begin{pmatrix} dL \\ dK \end{pmatrix} \\ |H_1| &= Pf_{LL} < 0 \\ |H_2| &= P\left[f_{LL}f_{KK} - (f_{KK})^2\right] > 0 \end{aligned}$$

Specific

$$
\begin{aligned}
-\frac{P}{4}L^{\frac{-3}{2}}dL + (0)dK &= 0 \\
-\frac{P}{4}K^{\frac{-3}{2}}dL + (0)dL &= 0
\end{aligned}
$$

Hessian

$$
\begin{array}{ll}
\pi_{LL} = -\frac{P}{4}L^{\frac{-3}{2}} & \pi_{LK} = 0 \\
\pi_{KL} = 0 & \pi_{KK} = -\frac{P}{4}K^{\frac{-3}{2}}
\end{array}
$$

$$
\left(-\frac{P}{4}L^{\frac{-3}{2}}\right)\left(-\frac{P}{4}K^{\frac{-3}{2}}\right) - 0 > 0
$$

therefore Profit Max

From the FOC's we know:

$$
L^* = \left(\tfrac{P}{2w}\right)^2 \qquad K^* = \left(\tfrac{P}{2r}\right)^2
$$

by subbing K* and L* into the profit function, we get:

$$
\begin{aligned}
\pi^* &= PL^{\frac{1}{2}} + PK^{\frac{1}{2}} - wL - rK \\
\pi^* &= P\left[\left(\frac{P}{2w}\right)^2\right]^{\frac{1}{2}} + P\left[\left(\frac{P}{2r}\right)^2\right]^{\frac{1}{2}} - w\left(\frac{P}{2w}\right)^2 - r\left(\frac{P}{2r}\right)^2 \\
\pi^* &= \frac{P^2}{2w} + \frac{P^2}{2r} - \frac{P^2}{4w} - \frac{P^2}{4r}
\end{aligned}
$$

Finally:

$$
\pi^* = \pi^*(w, r, P) = \frac{P^2}{4w} + \frac{P^2}{4r}
$$

where $\pi^*(w, r, P)$ is "Maximum profits as a function of w,r, and P"

2.7 Iso-Profit Curves (Level Curves)

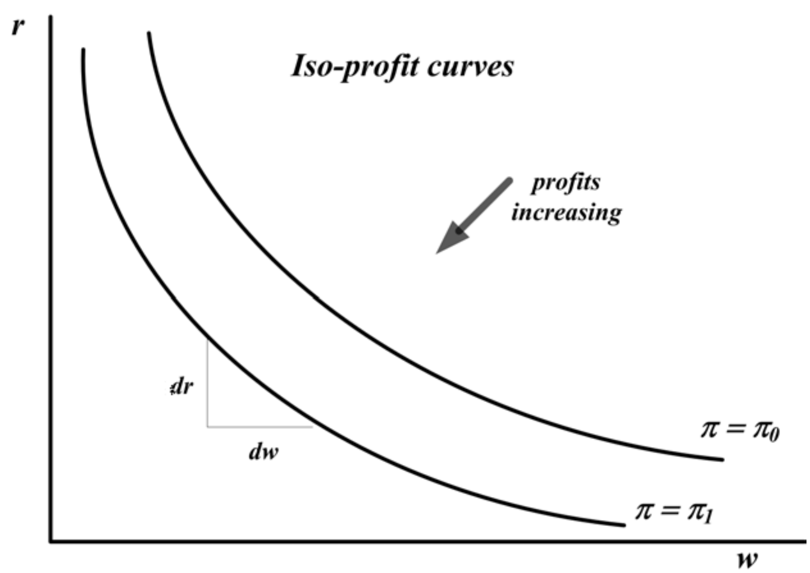
Take the total differential of $\pi^*(w, r, P)$; let $\mathrm{d}\pi^* = 0$

$$\begin{aligned} d\pi^* &= -\frac{P^2}{4w^2}dw + -\frac{P^2}{4r^2}dr = 0 \\ \frac{dr}{dw} &= -\frac{\frac{P^2}{4w^2}}{\frac{P^2}{4r^2}} = -\frac{r^2}{w^2} < 0 \ \ \text{(slope of Iso-Profit Curve)} \end{aligned}$$

Concave or Convex?

$$\frac{d}{dw}\left(\frac{dr}{dw}\right) = -\left(-2\frac{r^2}{w^3}\right) = 2\frac{r^2}{w^3} > 0$$

Therefore the slope of the Iso-Profit curve is negative $\left(\frac{dr}{dw}\right)$ but the slope is becoming less negative: $\left(\frac{d^2r}{dw^2}\right) > 0$ Therefore: Convex



3 Price Discrimination and Game Theory

3.0.1 Example

Let

$$P_1 = 100 - q_1 \qquad P_2 = 150 - 2q_2 \qquad \text{Mkt. AR Functions}$$

Let

$$\begin{aligned} TC &= 100 + (q_1 + q_2)^2 \\ \pi &= P_1 q_1 + P_2 q_2 - 100 - (q_1 + q_2)^2 \\ \pi &= 100q_1 - q_1^2 + 150q_2 - 2q_2^2 - 100 - (q_1 + q_2)^2 \end{aligned}$$

FOC's

$$\begin{aligned} \pi_1 &= 100 - 2q_1 - 2(q_1 + q_2) = 100 - 4q_1 - 2q_2 = 0 \\ \pi_2 &= 150 - 4q_2 - 2((q_1 + q_2) = 150 - 2q_1 - 6q_2 = 0 \end{aligned}$$

$$\begin{aligned} q_1 &= 15 \\ q_2 &= 20 \end{aligned}$$

$$P_1^* = 85 \qquad P_2^* = 110$$

SOC's

$$\begin{array}{ll} \pi_{11} = -4 & \pi_{12} = -2 \\ \pi_{21} = -2 & \pi_{22} = -6 \end{array} \qquad \pi_{11}\pi_{22} - \pi_{12}\pi_{21} = 20 > 0$$

Therefore a Max

3.1 Limit Output Model

Suppose a monopolist faces the following demand curve

$$p = a - q \qquad \text{a is a constant} > 0$$

His cost function is

$$TC = k + cq \qquad \text{where K = set up costs, cq = variable costs}$$

Therefore

$$ATC = \frac{k}{q} + c \qquad \{= AFC + AVC\}$$

The profit function is

$$\pi = pq - (K + cq)$$

Maximize

$$\frac{\partial \pi}{\partial q} = a - 2q - c = 0 \qquad \longrightarrow \qquad q = \frac{a-c}{2}$$

$$p = a - 1 = a - \left(\frac{a-c}{2}\right) = \frac{a+c}{2}$$

Set $MR = MC$

$$\begin{aligned} a - 2q &= c \\ q &= \frac{a-c}{2} \end{aligned}$$

Now consider a potential entrant to the monopolist's market

Assumption: Entrant takes monopolist's output as given

Let

$$\begin{aligned} q_e &= Entrant's\ Output \\ q_m &= Monopolist's\ Output \end{aligned}$$

If entrant does enter, market price will be:

$$p = a - (q_m - q_e)$$

Entrant's profits

$$\begin{aligned} \pi &= pq_e - k - cq_e \\ \pi_e &= (a - q_e - q_m)q_e - k - cq_e \\ \frac{\partial \pi_e}{\partial q_e} &= a - q_m - 2q_e - c = 0 \\ q_e &= \frac{a - c - q_m}{2} \end{aligned}$$

Entrant's output as a function of the monopolist's output.

Entrant's output

$$q_e = \frac{a - c - q_m}{2}$$

Sub into profit function

$$\begin{aligned} \pi_e &= (a - q_e - q_m)q_e - k - cq_e \\ \pi_e &= (a - q_m)\left(\frac{a - c - q_m}{2}\right) - \left(\frac{a - c - q_m}{2}\right)^2 - k - c\left(\frac{a - c - q_m}{2}\right) \end{aligned}$$

Entrant's profit function is a function of a, c, k, and q_m

He will enter if: $\pi_e > 0$ OR if: $(a - q_m - q_e)q_e - cq_e > k$

Which says: If an entrant's profits (gross) can cover fixed costs (k) then he will enter the market of the monopolist.

The monopolist knows that

$$q_e^* = \frac{a - c - q_m}{2}$$

or generally$q_e^* = f(q_m)$Therefore the monopolist can effect the entrant's choice q_e^*

The monopolist can choose q$_m$ such that when the entrant chooses the optimal q_e^* he will not earn any profits

Therefore the monopolists maximization problem is:

MAX:

$$\pi_m = (a - q_m) - q_m - k - cq_m$$

Subject to:

$$\pi_e = (a - q_m - q_e)q_e - cq_e \leq k$$

Substitute

$$q_e = \frac{a - c - q_m}{2}$$

into the monopolist's max problem, Max

$$aq_m - q_m^2 - cq_m - k$$

subject to

$$(a - q_m)\left[\frac{a - c - q_m}{2}\right] - \left[\frac{a - c - q_m}{2}\right]^2 - c\left[\frac{a - c - q_m}{2}\right] = K$$

Notice that there is now only one choice variable, q_m.

There q_m^* is determined by the constant

Without differentiating solve the constraint for q_m^*

Answer:

$$q_m^* = a - c - \sqrt[2]{k}$$

3.2 Cournot Duopoly

Suppose the monopolist decides to allow entry. The result: Duopoly

Assumption: Each firm takes the other firms output as exongenous and chooses the output to maximize its own profits

Market Demand:

$$\begin{aligned} P &= a - bq \\ \text{or } P &= a - b(q_1 + q_2) \qquad \{q_1 + q_2 = q\} \end{aligned}$$

where q_i is firm i's output $\{i = 1, 2\}$

Each firm faces the same cost function

$$TC = K + cq_i \qquad \{i = 1, 2\}$$

Each firm's profit function is:

$$\pi_i = pq_i - cq_i - K$$

Firm 1:

$$\begin{aligned} \pi_1 &= pq_1 - cq_1 - K \\ \pi_1 &= (a - bq_1 - bq_2)q_1 - cq_1 - K \end{aligned}$$

Max π_1, treating q_2 as a constant

$$\begin{aligned} \frac{\partial \pi_1}{\partial q_1} &= a - bq_2 - 2bq_1 - c = 0 \\ 2bq_1 &= a - c - bq_2 \\ q_1 &= \frac{a-c}{2b} - \frac{q_2}{2} \qquad \longrightarrow \qquad \text{"Best Response Function"} \end{aligned}$$

Best Response Function: Tells firm 1 the profit maximizing q_1 for any level of q_2

For Firm 2:

$$\pi_2 = (a - bq_1 - bq_2)q_2 - cq_2 - K$$

Max π_2 (treating q_1 as a constant) gives

$$q_2 = \frac{a-c}{2b} - \frac{q_1}{2} \quad \text{Firm 2's Best Response Function}$$

The two "Best Response" Functions

$$\begin{array}{ll} (1) & q_1 = \frac{a-c}{2b} - \frac{q_2}{2} \\ (2) & q_2 = \frac{a-c}{2b} - \frac{q_1}{2} \end{array}$$

gives us two equations and two unknowns.

The solution to this system of equations is the equilibrium to the "Cournot Duopoly" game

Using Cramer's Rule:

$$\begin{array}{ll} (1) & q_1^* = \frac{a-c}{3b} \\ (2) & q_2^* = \frac{a-c}{3b} \end{array}$$

$$\text{Market Output:} \qquad q_1^* + q_2^* = \frac{2(a-c)}{3b}$$

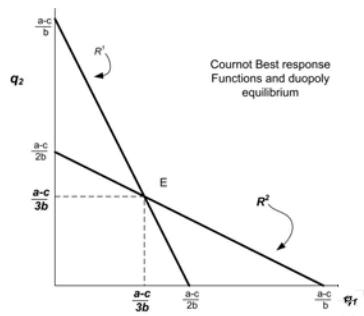
Best Response Functions Graphically

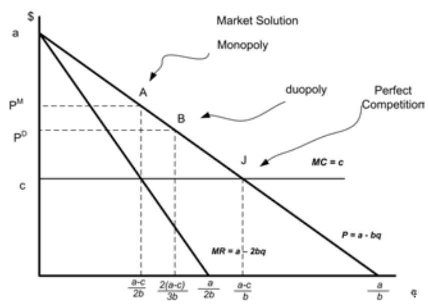
3.3 Stackelberg Duopoly

In the Cournot Duopoly , 2 firms picked output simultaneously. Suppose firm 1 was able to choose output first, knowing how firm 2‘s output would vary with firm 1‘s output.

3.3.1 Firm 1‘s Max Problem

$$Max\ q_1 : (a - bq_1 - bq_2)q_1 - cq_1 - K$$





Subject to:

$$q_2 = \frac{a-c}{2b} - \frac{q_1}{2} \quad \{\text{2's Response Function}\}$$

Sub in for q_2

$$\begin{aligned} Max\ q_1 \quad &: \quad aq_1 - bq_1^2 - bq_1\left(\frac{a-c}{2b} - \frac{q_1}{2}\right) - cq_1 - K \\ \frac{\partial \pi_1}{\partial q_1} &= a - 2bq_1 - \left(\frac{a-c}{2b}\right) + bq_1 - c = 0 \\ q_1^* &= \frac{a-c}{2b} \end{aligned}$$

Firm 2:

$$q_2 = \frac{a-c}{2b} - \frac{q_1}{2}$$

Sub in

$$
\begin{aligned}
q_1 &= \frac{a-c}{2b} \\
q_2^* &= \frac{a-c}{2b} - \frac{1}{2}\left(\frac{a-c}{2b}\right) = \frac{a-c}{4b}
\end{aligned}
$$

Graphically: Stackelberg and Cournot Equilibrium