Chapter 7

Rules of Differentiation and Their Use in Comparative Statics

The central problem of comparative-static analysis, that of finding a rate of change, can be identified with the problem of finding the derivative of some function $y = f(x)$, provided only an infinitesimal change in x is being considered. Even though the derivative dy/dx is defined as the limit of the difference quotient $q = g(v)$ as $v \to 0$, it is by no means necessary to undertake the process of limit-taking each time the derivative of a function is sought, for there exist various rules of differentiation (derivation) that will enable us to obtain the desired derivatives directly. Instead of going into comparative-static models immediately, therefore, let us begin by learning some rules of differentiation.

7.1 Rules of Differentiation for a Function of One Variable

First, let us discuss three rules that apply, respectively, to the following types of function of a single independent variable: $y = k$ (constant function), $y = x^n$, and $y = cx^n$ (power functions). All these have smooth, continuous graphs and are therefore differentiable everywhere.

Constant-Function Rule

The derivative of a constant function $y = k$, or $f(x) = k$, is identically zero, i.e., is zero for all values of x. Symbolically, this rule may be stated as: Given $y = f(x) = k$, the derivative is

$$\frac{dy}{dx} = \frac{dk}{dx} = 0 \qquad \text{or} \qquad f'(x) = 0$$

Alternatively, we may state the rule as: Given $y = f(x) = k$, the derivative is

$$\frac{d}{dx}y = \frac{d}{dx}f(x) = \frac{d}{dx}k = 0$$

where the derivative symbol has been separated into two parts, d/dx on the one hand, and y [or $f(x)$ or k] on the other. The first part, d/dx, is an *operator symbol,* which instructs us to perform a particular mathematical operation. Just as the operator symbol $\sqrt{}$ instructs us to take a square root, the symbol d/dx represents an instruction to take the derivative of, or to differentiate, (some function) with respect to the variable x. The function to be operated on (to be differentiated) is indicated in the second part; here it is $y = f(x) = k$.

The proof of the rule is as follows. Given $f(x) = k$, we have $f(N) = k$ for any value of N. Thus the value of $f'(N)$—the value of the derivative at $x = N$—as defined in (6.13) is

$$f'(N) = \lim_{x \to N} \frac{f(x) - f(N)}{x - N} = \lim_{x \to N} \frac{k - k}{x - N} = \lim_{x \to N} 0 = 0$$

Moreover, since N represents any value of x at all, the result $f'(N) = 0$ can be immediately generalized to $f'(x) = 0$. This proves the rule.

It is important to distinguish clearly between the statement $f'(x) = 0$ and the similar-looking but different statement $f'(x_0) = 0$. By $f'(x) = 0$, we mean that the derivative function f' has a zero value for *all* values of x; in writing $f'(x_0) = 0$, on the other hand, we are merely associating the zero value of the derivative with a particular value of x, namely, $x = x_0$.

As discussed before, the derivative of a function has its geometric counterpart in the slope of the curve. The graph of a constant function, say, a fixed-cost function $C_F = f(Q) = \$1200$, is a horizontal straight line with a zero slope throughout. Correspondingly, the derivative must also be zero for all values of Q:

$$\frac{d}{dQ} C_F = \frac{d}{dQ} 1200 = 0$$

Power-Function Rule

The derivative of a power function $y = f(x) = x^n$ is nx^{n-1}. Symbolically, this is expressed as

$$\frac{d}{dx} x^n = nx^{n-1} \qquad \text{or} \qquad f'(x) = nx^{n-1} \tag{7.1}$$

Example 1 The derivative of $y = x^3$ is $\frac{dy}{dx} = \frac{d}{dx} x^3 = 3x^2$.

Example 2 The derivative of $y = x^9$ is $\frac{d}{dx} x^9 = 9x^8$.

This rule is valid for any real-valued power of x; that is, the exponent can be any real number. But we shall prove it only for the case where n is some positive integer. In the simplest case, that of $n = 1$, the function is $f(x) = x$, and according to the rule, the derivative is

$$f'(x) = \frac{d}{dx} x = 1(x^0) = 1$$

The proof of this result follows easily from the definition of $f'(N)$ in (6.14′). Given $f(x) = x$, the derivative value at any value of x, say, $x = N$, is

$$f'(N) = \lim_{x \to N} \frac{f(x) - f(N)}{x - N} = \lim_{x \to N} \frac{x - N}{x - N} = \lim_{x \to N} 1 = 1$$

Since N represents any value of x, it is permissible to write $f'(x) = 1$. This proves the rule for the case of $n = 1$. As the graphical counterpart of this result, we see that the function $y = f(x) = x$ plots as a 45° line, and it has a slope of +1 throughout.

For the cases of larger integers, $n = 2, 3, \ldots,$ let us first note the following identities:

$$\frac{x^2 - N^2}{x - N} = x + N \qquad \text{[2 terms on the right]}$$

$$\frac{x^3 - N^3}{x - N} = x^2 + Nx + N^2 \qquad \text{[3 terms on the right]}$$

$$\vdots$$

$$\frac{x^n - N^n}{x - N} = x^{n-1} + Nx^{n-2} + N^2x^{n-3} + \cdots + N^{n-1} \qquad \text{[n terms on the right]} \qquad \textbf{(7.2)}$$

On the basis of (7.2), we can express the derivative of a power function $f(x) = x^n$ at $x = N$ as follows:

$$\begin{aligned} f'(N) &= \lim_{x \to N} \frac{f(x) - f(N)}{x - N} = \lim_{x \to N} \frac{x^n - N^n}{x - N} \\ &= \lim_{x \to N} (x^{n-1} + Nx^{n-2} + \cdots + N^{n-1}) && \text{[by (7.2)]} \\ &= \lim_{x \to N} x^{n-1} + \lim_{x \to N} Nx^{n-2} + \cdots + \lim_{x \to N} N^{n-1} && \text{[sum limit theorem]} \\ &= N^{n-1} + N^{n-1} + \cdots + N^{n-1} && \text{[a total of n terms]} \\ &= nN^{n-1} && \textbf{(7.3)} \end{aligned}$$

Again, N is any value of x; thus this last result can be generalized to

$$f'(x) = nx^{n-1}$$

which proves the rule for n, any positive integer.

As mentioned previously, this rule applies even when the exponent n in the power expression x^n is not a positive integer. The following examples serve to illustrate its application to the latter cases.

Example 3 Find the derivative of $y = x^0$. Applying (7.1), we find

$$\frac{d}{dx} x^0 = 0(x^{-1}) = 0$$

Example 4 Find the derivative of $y = 1/x^3$. This involves the reciprocal of a power, but by rewriting the function as $y = x^{-3}$, we can again apply (7.1) to get the derivative:

$$\frac{d}{dx} x^{-3} = -3x^{-4} \qquad \left[= \frac{-3}{x^4}\right]$$

Example 5 Find the derivative of $y = \sqrt{x}$. A square root is involved in this case, but since $\sqrt{x} = x^{1/2}$, the derivative can be found as follows:

$$\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} \qquad \left[= \frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2x} \right]$$

Derivatives are themselves functions of the independent variable x. In Example 1, for instance, the derivative is $dy/dx = 3x^2$, or $f'(x) = 3x^2$, so that a different value of x will result in a different value of the derivative, such as

$$f'(1) = 3(1)^2 = 3 \qquad f'(2) = 3(2)^2 = 12$$

These specific values of the derivative can be expressed alternatively as

$$\left.\frac{dy}{dx}\right|_{x=1} = 3 \qquad \left.\frac{dy}{dx}\right|_{x=2} = 12$$

but the notations $f'(1)$ and $f'(2)$ are obviously preferable because of their simplicity.

It is of the utmost importance to realize that, to find the derivative values $f'(1)$, $f'(2)$, etc., we must *first* differentiate the function $f(x)$, to get the derivative function $f'(x)$, and *then* let x assume specific values in $f'(x)$. To substitute specific values of x into the primitive function $f(x)$ prior to differentiation is definitely not permissible. As an illustration, if we let $x = 1$ in the function of Example 1 before differentiation, the function will degenerate into $y = x = 1$—a constant function—which will yield a zero derivative rather than the correct answer of $f'(x) = 3x^2$.

Power-Function Rule Generalized

When a multiplicative constant c appears in the power function, so that $f(x) = cx^n$, its derivative is

$$\frac{d}{dx}cx^n = cnx^{n-1} \qquad \text{or} \qquad f'(x) = cnx^{n-1}$$

This result shows that, in differentiating cx^n, we can simply retain the multiplicative constant c intact and then differentiate the term x^n according to (7.1).

Example 6 Given $y = 2x$, we have $dy/dx = 2x^0 = 2$.

Example 7 Given $f(x) = 4x^3$, the derivative is $f'(x) = 12x^2$.

Example 8 The derivative of $f(x) = 3x^{-2}$ is $f'(x) = -6x^{-3}$.

For a proof of this new rule, consider the fact that for any value of x, say, $x = N$, the value of the derivative of $f(x) = cx^n$ is

$$\begin{aligned} f'(N) &= \lim_{x\to N} \frac{f(x) - f(N)}{x - N} = \lim_{x\to N} \frac{cx^n - cN^n}{x - N} = \lim_{x\to N} c\left(\frac{x^n - N^n}{x - N}\right) \\ &= \lim_{x\to N} c \lim_{x\to N} \frac{x^n - N^n}{x - N} && \text{[product limit theorem]} \\ &= c \lim_{x\to N} \frac{x^n - N^n}{x - N} && \text{[limit of a constant]} \\ &= cnN^{n-1} && \text{[from (7.3)]} \end{aligned}$$

In view that N is any value of x, this last result can be generalized immediately to $f'(x) = cnx^{n-1}$, which proves the rule.

EXERCISE 7.1

1. Find the derivative of each of the following functions:
   (a) $y = x^{12}$ (c) $y = 7x^5$ (e) $w = -4u^{1/2}$
   (b) $y = 63$ (d) $w = 3u^{-1}$ (f) $w = 4u^{1/4}$
2. Find the following:
   (a) $\frac{d}{dx}(-x^{-4})$ (c) $\frac{d}{dw}5w^4$ (e) $\frac{d}{du}au^b$
   (b) $\frac{d}{dx}9x^{1/3}$ (d) $\frac{d}{dx}cx^2$ (f) $\frac{d}{du} - au^{-b}$
3. Find $f'(1)$ and $f'(2)$ from the following functions:
   (a) $y = f(x) = 18x$ (c) $f(x) = -5x^{-2}$ (e) $f(w) = 6w^{1/3}$
   (b) $y = f(x) = cx^3$ (d) $f(x) = \frac{3}{4}x^{4/3}$ (f) $f(w) = -3w^{-1/6}$
4. Graph a function $f(x)$ that gives rise to the derivative function $f'(x) = 0$. Then graph a function $g(x)$ characterized by $f'(x_0) = 0$.

7.2 Rules of Differentiation Involving Two or More Functions of the Same Variable

The three rules presented in Sec. 7.1 are each concerned with a single given function $f(x)$. Now suppose that we have two *differentiable* functions of the same variable x, say, $f(x)$ and $g(x)$, and we want to differentiate the sum, difference, product, or quotient formed with these two functions. In such circumstances, are there appropriate rules that apply? More concretely, given two functions—say, $f(x) = 3x^2$ and $g(x) = 9x^{12}$—how do we get the derivative of, say, $3x^2 + 9x^{12}$, or the derivative of $(3x^2)(9x^{12})$?

Sum-Difference Rule

The derivative of a sum (difference) of two functions is the sum (difference) of the derivatives of the two functions:

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) \pm g'(x)$$

The proof of this again involves the application of the definition of a derivative and of the various limit theorems. We shall omit the proof and, instead, merely verify its validity and illustrate its application.

Example 1 From the function $y = 14x^3$, we can obtain the derivative $dy/dx = 42x^2$. But $14x^3 = 5x^3 + 9x^3$, so that y may be regarded as the sum of two functions $f(x) = 5x^3$ and $g(x) = 9x^3$. According to the sum rule, we then have

$$\frac{dy}{dx} = \frac{d}{dx}(5x^3 + 9x^3) = \frac{d}{dx}5x^3 + \frac{d}{dx}9x^3 = 15x^2 + 27x^2 = 42x^2$$

which is identical with our earlier result.

This rule, which we stated in terms of two functions, can easily be extended to more functions. Thus, it is also valid to write

$$\frac{d}{dx}[f(x) \pm g(x) \pm h(x)] = f'(x) \pm g'(x) \pm h'(x)$$

Example 2 The function cited in Example 1, $y = 14x^3$, can be written as $y = 2x^3 + 13x^3 - x^3$. The derivative of the latter, according to the sum-difference rule, is

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3 + 13x^3 - x^3) = 6x^2 + 39x^2 - 3x^2 = 42x^2$$

which again checks with the previous answer.

This rule is of great practical importance. With it at our disposal, it is now possible to find the derivative of any polynomial function, since the latter is nothing but a sum of power functions.

Example 3

$$\frac{d}{dx}(ax^2 + bx + c) = 2ax + b$$

Example 4

$$\frac{d}{dx}(7x^4 + 2x^3 - 3x + 37) = 28x^3 + 6x^2 - 3 + 0 = 28x^3 + 6x^2 - 3$$

Note that in Examples 3 and 4 the constants c and 37 do not really produce any effect on the derivative, because the derivative of a constant term is zero. In contrast to the *multiplicative* constant, which is retained during differentiation, the *additive* constant drops out. This fact provides the mathematical explanation of the well-known economic principle that the fixed cost of a firm does not affect its marginal cost. Given a short-run total-cost function

$$C = Q^3 - 4Q^2 + 10Q + 75$$

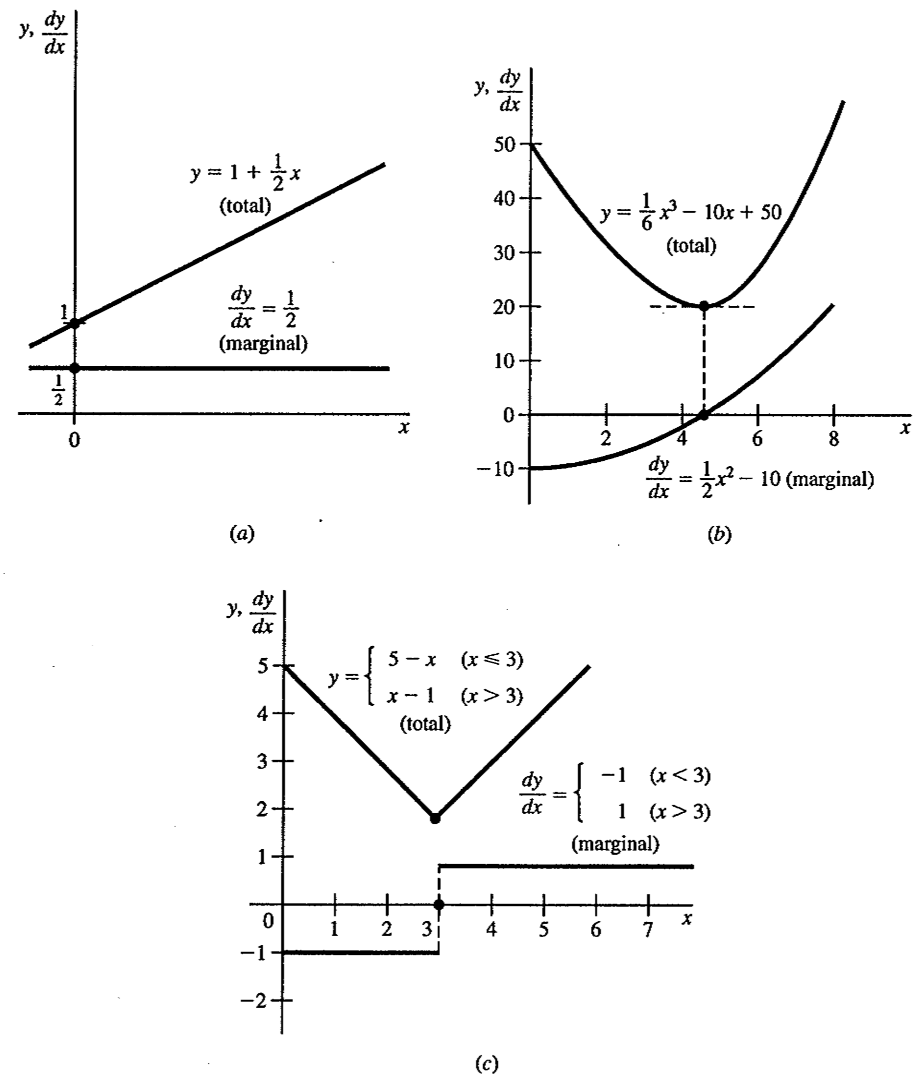
the marginal-cost function (for infinitesimal output change) is the limit of the quotient $\Delta C/\Delta Q$, or the derivative of the C function:

$$\frac{dC}{dQ} = 3Q^2 - 8Q + 10$$

whereas the fixed cost is represented by the additive constant 75. Since the latter drops out during the process of deriving dC/dQ, the magnitude of the fixed cost obviously cannot affect the marginal cost.

In general, if a primitive function $y = f(x)$ represents a *total* function, then the derivative function dy/dx is its *marginal* function. Both functions can, of course, be plotted against the variable x graphically; and because of the correspondence between the derivative of a function and the slope of its curve, for each value of x the marginal function should show the slope of the total function at that value of x. In Fig. 7.1a, a linear (constant-slope) total function is seen to have a constant marginal function. On the other hand, the nonlinear (varying-slope) total function in Fig. 7.1b gives rise to a curved marginal function, which lies below (above) the horizontal axis when the total function is negatively (positively) sloped. And, finally, the reader may note from Fig. 7.1c (cf. Fig. 6.5) that

FIGURE 7.1



"nonsmoothness" of a total function will result in a gap (discontinuity) in the marginal or derivative function. This is in sharp contrast to the everywhere-smooth total function in Fig. 7.1*b* which gives rise to a continuous marginal function. For this reason, the *smoothness* of a *primitive* function can be linked to the *continuity* of its *derivative* function. In particular, instead of saying that a certain function is smooth (and differentiable) everywhere, we may alternatively characterize it as a function with a continuous derivative function, and refer to it as a *continuously differentiable* function.

The following notations are often used to denote the continuity and the continuous differentiability of a function f:

$$f \in C^{(0)} \quad \text{or} \quad f \in C: \qquad f \text{ is continuous}$$

$$f \in C^{(1)} \quad \text{or} \quad f \in C': \qquad f \text{ is continuously differentiable}$$

where $C^{(0)}$, or simply C, is the symbol for the set of all continuous functions, and $C^{(1)}$, or C', is the symbol for the set of all continuously differentiable functions.

Product Rule

The derivative of the product of two (differentiable) functions is equal to the first function times the derivative of the second function plus the second function times the derivative of the first function:

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)] &= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) \\ &= f(x)g'(x) + g(x)f'(x) \qquad \textbf{(7.4)}\end{aligned}$$

It is also possible, of course, to rearrange the terms and express the rule as

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x) \qquad \textbf{(7.4')}$$

Example 5 Find the derivative of $y = (2x+3)(3x^2)$. Let $f(x) = 2x+3$ and $g(x) = 3x^2$. Then it follows that $f'(x) = 2$ and $g'(x) = 6x$, and according to (7.4) the desired derivative is

$$\frac{d}{dx}[(2x+3)(3x^2)] = (2x+3)(6x) + (3x^2)(2) = 18x^2 + 18x$$

This result can be checked by first multiplying out $f(x)g(x)$ and then taking the derivative of the product polynomial. The product polynomial is in this case $f(x)g(x) = (2x+3)(3x^2) = 6x^3 + 9x^2$, and direct differentiation does yield the same derivative, $18x^2 + 18x$.

The important point to remember is that the derivative of a product of two functions is *not* the simple product of the two separate derivatives. Instead, it is a weighted sum of $f'(x)$ and $g'(x)$, the weights being $g(x)$ and $f(x)$, respectively. Since this differs from what intuitive generalization leads one to expect, let us produce a proof for (7.4). According to (6.13), the value of the derivative of $f(x)g(x)$ when $x = N$ should be

$$\frac{d}{dx}[f(x)g(x)]\bigg|_{x=N} = \lim_{x\to N}\frac{f(x)g(x) - f(N)g(N)}{x-N} \qquad \textbf{(7.5)}$$

But, by adding *and* subtracting $f(x)g(N)$ in the numerator (thereby leaving the original magnitude unchanged), we can transform the quotient on the right of (7.5) as follows:

$$\begin{aligned}&\frac{f(x)g(x) - f(x)g(N) + f(x)g(N) - f(N)g(N)}{x-N} \\ &= f(x)\frac{g(x)-g(N)}{x-N} + g(N)\frac{f(x)-f(N)}{x-N}\end{aligned}$$

Substituting this for the quotient on the right of (7.5) and taking its limit, we then get

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)]\bigg|_{x=N} &= \lim_{x\to N} f(x)\lim_{x\to N}\frac{g(x)-g(N)}{x-N} \\ &\quad + \lim_{x\to N} g(N)\lim_{x\to N}\frac{f(x)-f(N)}{x-N} \qquad \textbf{(7.5')}\end{aligned}$$

The four limit expressions in (7.5′) are easily evaluated. The first one is $f(N)$, and the third is $g(N)$ (limit of a constant). The remaining two are, according to (6.13), respectively, $g'(N)$ and $f'(N)$. Thus (7.5′) reduces to

$$\frac{d}{dx}[f(x)g(x)]\bigg|_{x=N} = f(N)g'(N) + g(N)f'(N) \qquad \textbf{(7.5'')}$$

And, since N represents any value of x, (7.5″) remains valid if we replace every N symbol by x. This proves the rule.

As an extension of the rule to the case of *three* functions, we have

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x) \qquad \text{[cf. (7.4')]} \qquad \textbf{(7.6)}$$

In words, the derivative of the product of three functions is equal to the product of the second and third functions times the derivative of the first, plus the product of the first and third functions times the derivative of the second, plus the product of the first and second functions times the derivative of the third. This result can be derived by the repeated application of (7.4). First treat the product $g(x)h(x)$ as a single function, say, $\phi(x)$, so that the original product of three functions will become a product of *two* functions, $f(x)\phi(x)$. To this, (7.4) is applicable. After the derivative of $f(x)\phi(x)$ is obtained, we may reapply (7.4) to the product $g(x)h(x) \equiv \phi(x)$ to get $\phi'(x)$. Then (7.6) will follow. The details are left to you as an exercise.

The validity of a rule is one thing; its serviceability is something else. Why do we need the product rule when we can resort to the alternative procedure of multiplying out the two functions $f(x)$ and $g(x)$ and then taking the derivative of the product directly? One answer to this question is that the alternative procedure is applicable only to *specific* (numerical or parametric) functions, whereas the product rule is applicable even when the functions are given in the *general* form. Let us illustrate with an economic example.

Finding Marginal-Revenue Function from Average-Revenue Function

If we are given an average-revenue (AR) function in specific form,

$$\text{AR} = 15 - Q$$

the marginal-revenue (MR) function can be found by first multiplying AR by Q to get the total-revenue (R) function:

$$R \equiv \text{AR} \cdot Q = (15 - Q)Q = 15Q - Q^2$$

and then differentiating R:

$$\text{MR} \equiv \frac{dR}{dQ} = 15 - 2Q$$

But if the AR function is given in the general form $\text{AR} = f(Q)$, then the total-revenue function will also be in a general form:

$$R \equiv \text{AR} \cdot Q = f(Q) \cdot Q$$

and therefore the "multiply out" approach will be to no avail. However, because R is a product of two functions of Q, namely, $f(Q)$ and Q itself, the product rule can be put to work. Thus we can differentiate R to get the MR function as follows:

$$\text{MR} \equiv \frac{dR}{dQ} = f(Q)\cdot 1 + Q\cdot f'(Q) = f(Q) + Qf'(Q) \qquad \textbf{(7.7)}$$

However, can such a general result tell us anything significant about the MR? Indeed it can. Recalling that $f(Q)$ denotes the AR function, let us rearrange (7.7) and write

$$\text{MR} - \text{AR} = \text{MR} - f(Q) = Qf'(Q) \qquad \textbf{(7.7')}$$

This gives us an important relationship between MR and AR: namely, they will always differ by the amount $Qf'(Q)$.

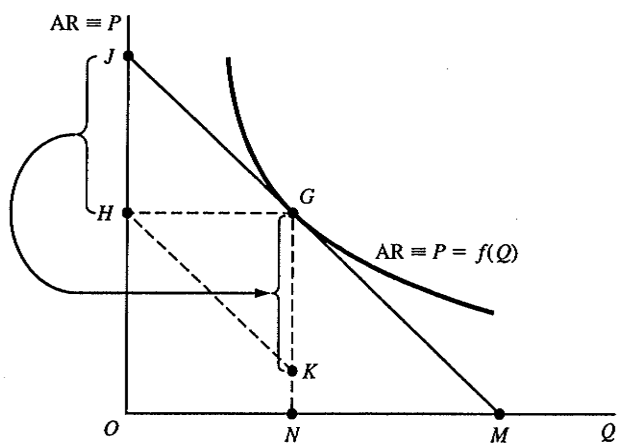
It remains to examine the expression $Qf'(Q)$. Its first component Q denotes output and is always nonnegative. The other component, $f'(Q)$, represents the slope of the AR curve plotted against Q. Since "average revenue" and "price" are but different names for the same thing:

$$\text{AR} \equiv \frac{R}{Q} \equiv \frac{PQ}{Q} \equiv P$$

the AR curve can also be regarded as a curve relating price P to output Q: $P = f(Q)$. Viewed in this light, the AR curve is simply the *inverse* of the demand curve for the product of the firm, i.e., the demand curve plotted after the P and Q axes are reversed. Under pure competition, the AR curve is a horizontal straight line, so that $f'(Q) = 0$ and, from (7.7′), $\text{MR} - \text{AR} = 0$ for all possible values of Q. Thus the MR curve and the AR curve must coincide. Under imperfect competition, on the other hand, the AR curve is normally downward-sloping, as in Fig. 7.2, so that $f'(Q) < 0$ and, from (7.7′), $\text{MR} - \text{AR} < 0$ for all positive levels of output. In this case, the MR curve must lie below the AR curve.

The conclusion just stated is *qualitative* in nature; it concerns only the relative positions of the two curves. But (7.7′) also furnishes the *quantitative* information that the MR curve will fall short of the AR curve at any output level Q by precisely the amount $Qf'(Q)$. Let us look at Fig. 7.2 again and consider the particular output level N. For that output, the

FIGURE 7.2



expression $Qf'(Q)$ specifically becomes $Nf'(N)$; if we can find the magnitude of $Nf'(N)$ in the diagram, we shall know how far below the average-revenue point G the corresponding marginal-revenue point must lie.

The magnitude of N is already specified. And $f'(N)$ is simply the slope of the AR curve at point G (where $Q = N$), that is, the slope of the tangent line JM measured by the ratio of two distances OJ/OM. However, we see that $OJ/OM = HJ/HG$; besides, distance HG is precisely the amount of output under consideration, N. Thus the distance $Nf'(N)$, by which the MR curve must lie below the AR curve at output N, is

$$Nf'(N) = HG\frac{HJ}{HG} = HJ$$

Accordingly, if we mark a vertical distance $KG = HJ$ directly below point G, then point K must be a point on the MR curve. (A simple way of accurately plotting KG is to draw a straight line passing through point H and parallel to JG; point K is where that line intersects the vertical line NG.)

The same procedure can be used to locate other points on the MR curve. All we must do, for any chosen point G' on the curve, is first to draw a tangent to the AR curve at G' that will meet the vertical axis at some point J'. Then draw a horizontal line from G' to the vertical axis, and label the intersection with the axis as H'. If we mark a vertical distance $K'G' = H'J'$ directly below point G', then the point K' will be a point on the MR curve. This is the graphical way of deriving an MR curve from a given AR curve. Strictly speaking, the accurate drawing of a tangent line requires a knowledge of the value of the derivative at the relevant output, that is, $f'(N)$; hence the graphical method just outlined cannot quite exist by itself. An important exception is the case of a linear AR curve, where the tangent to any point on the curve is simply the given line itself, so that there is in effect no need to draw any tangent at all. Then the graphical method will apply in a straightforward way.

Quotient Rule

The derivative of the quotient of two functions, $f(x)/g(x)$, is

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

In the numerator of the right-hand expression, we find two product terms, each involving the derivative of only one of the two original functions. Note that $f'(x)$ appears in the positive term, and $g'(x)$ in the negative term. The denominator consists of the square of the function $g(x)$; that is, $g^2(x) \equiv [g(x)]^2$.

Example 6

$$\frac{d}{dx}\left(\frac{2x-3}{x+1}\right) = \frac{2(x+1) - (2x-3)(1)}{(x+1)^2} = \frac{5}{(x+1)^2}$$

Example 7

$$\frac{d}{dx}\left(\frac{5x}{x^2+1}\right) = \frac{5(x^2+1) - 5x(2x)}{(x^2+1)^2} = \frac{5(1-x^2)}{(x^2+1)^2}$$

Example 8

$$\frac{d}{dx}\left(\frac{ax^2+b}{cx}\right) = \frac{2ax(cx) - (ax^2+b)(c)}{(cx)^2} = \frac{c(ax^2-b)}{(cx)^2} = \frac{ax^2-b}{cx^2}$$

This rule can be proved as follows. For any value of $x = N$, we have

$$\left.\frac{d}{dx}\frac{f(x)}{g(x)}\right|_{x=N} = \lim_{x\to N}\frac{f(x)/g(x) - f(N)/g(N)}{x - N} \tag{7.8}$$

The quotient expression following the limit sign can be rewritten in the form

$$\frac{f(x)g(N) - f(N)g(x)}{g(x)g(N)}\,\frac{1}{x - N}$$

By adding *and* subtracting $f(N)g(N)$ in the numerator and rearranging, we can further transform the expression to

$$\frac{1}{g(x)g(N)}\left[\frac{f(x)g(N) - f(N)g(N) + f(N)g(N) - f(N)g(x)}{x - N}\right]$$
$$= \frac{1}{g(x)g(N)}\left[g(N)\,\frac{f(x) - f(N)}{x - N} - f(N)\frac{g(x) - g(N)}{x - N}\right]$$

Substituting this result into (7.8) and taking the limit, we then have

$$\begin{aligned}\left.\frac{d}{dx}\frac{f(x)}{g(x)}\right|_{x=N} &= \lim_{x\to N}\frac{1}{g(x)g(N)}\left[\lim_{x\to N} g(N)\lim_{x\to N}\frac{f(x) - f(N)}{x - N}\right.\\ &\qquad \left. - \lim_{x\to N} f(N)\lim_{x\to N}\frac{g(x) - g(N)}{x - N}\right]\\ &= \frac{1}{g^2(N)}[g(N)f'(N) - f(N)g'(N)] \qquad \text{[by (6.13)]}\end{aligned}$$

which can be generalized by replacing the symbol N with x, because N represents any value of x. This proves the quotient rule.

Relationship Between Marginal-Cost and Average-Cost Functions

As an economic application of the quotient rule, let us consider the rate of change of average cost when output varies.

Given a total-cost function $C = C(Q)$, the average-cost (AC) function is a quotient of two functions of Q, since $\text{AC} \equiv C(Q)/Q$, defined as long as $Q > 0$. Therefore, the rate of change of AC with respect to Q can be found by differentiating AC:

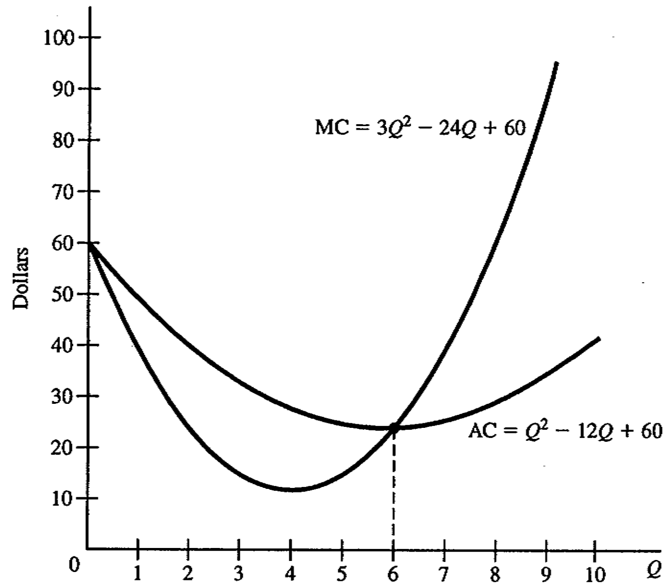
$$\frac{d}{dQ}\frac{C(Q)}{Q} = \frac{[C'(Q)\cdot Q - C(Q)\cdot 1]}{Q^2} = \frac{1}{Q}\left[C'(Q) - \frac{C(Q)}{Q}\right] \tag{7.9}$$

From this it follows that, for $Q > 0$,

$$\frac{d}{dQ}\frac{C(Q)}{Q} \gtreqless 0 \quad \text{iff} \quad C'(Q) \gtreqless \frac{C(Q)}{Q} \tag{7.10}$$

Since the derivative $C'(Q)$ represents the marginal-cost (MC) function, and $C(Q)/Q$ represents the AC function, the economic meaning of (7.10) is: The slope of the AC

FIGURE 7.3



curve will be positive, zero, or negative if and only if the marginal-cost curve lies above, intersects, or lies below the AC curve. This is illustrated in Fig. 7.3, where the MC and AC functions plotted are based on the specific total-cost function

$$C = Q^3 - 12Q^2 + 60Q$$

To the left of $Q = 6$, AC is declining, and thus MC lies below it; to the right, the opposite is true. At $Q = 6$, AC has a slope of zero, and MC and AC have the same value.†

The qualitative conclusion in (7.10) is stated explicitly in terms of cost functions. However, its validity remains unaffected if we interpret $C(Q)$ as *any other* differentiable total function, with $C(Q)/Q$ and $C'(Q)$ as its corresponding average and marginal functions. Thus this result gives us a *general* marginal-average relationship. In particular, we may point out, the fact that MR lies below AR when AR is downward-sloping, as discussed in connection with Fig. 7.2, is nothing but a special case of the general result in (7.10).

† Note that (7.10) does *not* state that, when AC is negatively sloped, MC must also be negatively sloped; it merely says that AC must exceed MC in that circumstance. At $Q = 5$ in Fig. 7.3, for instance, AC is declining but MC is rising, so that their slopes will have opposite signs.

EXERCISE 7.2

1. Given the total-cost function $C = Q^3 - 5Q^2 + 12Q + 75$, write out a variable-cost (VC) function. Find the derivative of the VC function, and interpret the economic meaning of that derivative.
2. Given the average-cost function $AC = Q^2 - 4Q + 174$, find the MC function. Is the given function more appropriate as a long-run or a short-run function? Why?

3. Differentiate the following by using the product rule:
   (*a*) $(9x^2 - 2)(3x + 1)$ (*c*) $x^2(4x + 6)$ (*e*) $(2 - 3x)(1 + x)(x + 2)$
   (*b*) $(3x + 10)(6x^2 - 7x)$ (*d*) $(ax - b)(cx^2)$ (*f*) $(x^2 + 3)x^{-1}$
4. (*a*) Given $AR = 60 - 3Q$, plot the average-revenue curve, and then find the MR curve by the method used in Fig. 7.2.
   (*b*) Find the total-revenue function and the marginal-revenue function mathematically from the given AR function.
   (*c*) Does the graphically derived MR curve in (*a*) check with the mathematically derived MR function in (*b*)?
   (*d*) Comparing the AR and MR functions, what can you conclude about their relative slopes?
5. Provide a mathematical proof for the general result that, given a *linear* average curve, the corresponding marginal curve must have the same vertical intercept but will be twice as steep as the average curve.
6. Prove the result in (7.6) by first treating $g(x)h(x)$ as a single function, $g(x)h(x) \equiv \phi(x)$, and then applying the product rule (7.4).
7. Find the derivatives of:
   (*a*) $(x^2 + 3)/x$ (*c*) $6x/(x + 5)$
   (*b*) $(x + 9)/x$ (*d*) $(ax^2 + b)/(cx + d)$
8. Given the function $f(x) = ax + b$, find the derivatives of:
   (*a*) $f(x)$ (*b*) $xf(x)$ (*c*) $1/f(x)$ (*d*) $f(x)/x$
9. (*a*) Is it true that $f \in C' \Rightarrow f \in C$?
   (*b*) Is it true that $f \in C \Rightarrow f \in C'$?
10. Find the marginal and average functions for the following total functions and graph the results.
    Total-cost function:
    (*a*) $C = 3Q^2 + 7Q + 12$
    Total-revenue function:
    (*b*) $R = 10Q - Q^2$
    Total-product function:
    (*c*) $Q = aL + bL^2 - cL^3 \quad (a, b, c > 0)$

7.3 Rules of Differentiation Involving Functions of Different Variables

In Sec. 7.2, we discussed the rules of differentiation of a sum, difference, product, or quotient of two (or more) differentiable functions of the same variable. Now we shall consider cases where there are two or more differentiable functions, each of which has a *distinct* independent variable.

Chain Rule

If we have a differentiable function $z = f(y)$, where y is in turn a differentiable function of another variable x, say, $y = g(x)$, then the derivative of z with respect to x is equal to the derivative of z with respect to y, times the derivative of y with respect to x. Expressed symbolically,

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = f'(y)g'(x) \qquad \textbf{(7.11)}$$

This rule, known as the *chain rule*, appeals easily to intuition. Given a Δx, there must result a corresponding Δy via the function $y = g(x)$, but this Δy will in turn bring about a Δz via the function $z = f(y)$. Thus there is a "chain reaction" as follows:

$$\Delta x \xrightarrow{\text{via } g} \Delta y \xrightarrow{\text{via } f} \Delta z$$

The two links in this chain entail two difference quotients, $\Delta y/\Delta x$ and $\Delta z/\Delta y$, but when they are multiplied, the Δy will cancel itself out, and we end up with

$$\frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x} = \frac{\Delta z}{\Delta x}$$

a difference quotient that relates Δz to Δx. If we take the limit of these difference quotients as $\Delta x \to 0$ (which implies $\Delta y \to 0$), each difference quotient will turn into a derivative; i.e., we shall have $(dz/dy)(dy/dx) = dz/dx$. This is precisely the result in (7.11).

In view of the function $y = g(x)$, we can express the function $z = f(y)$ as $z = f[g(x)]$, where the contiguous appearance of the two function symbols f and g indicates that this is a *composite function* (function of a function). It is for this reason that the chain rule is also referred to as the *composite-function rule* or *function-of-a-function rule.*

The extension of the chain rule to three or more functions is straightforward. If we have $z = f(y)$, $y = g(x)$, and $x = h(w)$, then

$$\frac{dz}{dw} = \frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dw} = f'(y)g'(x)h'(w)$$

and similarly for cases in which more functions are involved.

Example 1 If $z = 3y^2$, where $y = 2x + 5$, then

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = 6y(2) = 12y = 12(2x + 5)$$

Example 2 If $z = y - 3$, where $y = x^3$, then

$$\frac{dz}{dx} = 1(3x^2) = 3x^2$$

Example 3 The usefulness of this rule can best be appreciated when we must differentiate a function such as $z = (x^2 + 3x - 2)^{17}$. Without the chain rule at our disposal, dz/dx can be found only via the laborious route of first multiplying out the 17th-power expression. With the chain rule, however, we can take a shortcut by defining a new, *intermediate* variable $y = x^2 + 3x - 2$, so that we get in effect two functions linked in a chain:

$$z = y^{17} \qquad \text{and} \qquad y = x^2 + 3x - 2$$

The derivative dz/dx can then be found as follows:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = 17y^{16}(2x + 3) = 17(x^2 + 3x - 2)^{16}(2x + 3)$$

Example 4 Given a total-revenue function of a firm $R = f(Q)$, where output Q is a function of labor input L, or $Q = g(L)$, find dR/dL. By the chain rule, we have

$$\frac{dR}{dL} = \frac{dR}{dQ}\frac{dQ}{dL} = f'(Q)g'(L)$$

Translated into economic terms, dR/dQ is the MR function and dQ/dL is the marginal-physical-product-of-labor (MPP_L) function. Similarly, dR/dL has the connotation of the marginal-revenue-product-of-labor (MRP_L) function. Thus the result shown constitutes the mathematical statement of the well-known result in economics that $MRP_L = MR \cdot MPP_L$.

Inverse-Function Rule

If the function $y = f(x)$ represents a one-to-one mapping, i.e., if the function is such that each value of y is associated with a unique value of x, the function f will have an *inverse function* $x = f^{-1}(y)$ (read: "x is an inverse function of y"). Here, the symbol f^{-1} is a function symbol which, like the derivative-function symbol f', signifies a function related to the function f; it does *not* mean the reciprocal of the function $f(x)$.

What the existence of an inverse function essentially means is that, in this case, not only will a given value of x yield a unique value of y [that is, $y = f(x)$], but also a given value of y will yield a unique value of x. To take a nonnumerical instance, we may exemplify the one-to-one mapping by the mapping from the set of all husbands to the set of all wives in a monogamous society. Each husband has a unique wife, and each wife has a unique husband. In contrast, the mapping from the set of all fathers to the set of all sons is not one-to-one, because a father may have more than one son, albeit each son has a unique father.

When x and y refer specifically to numbers, the property of one-to-one mapping is seen to be unique to the class of functions known as *strictly monotonic* (or *monotone*) *functions.* Given a function $f(x)$, if successively larger values of the independent variable x *always* lead to successively larger values of $f(x)$, that is, if

$$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$$

then the function f is said to be a *strictly increasing* function. If successive increases in x *always* lead to successive decreases in $f(x)$, that is, if

$$x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$$

on the other hand, the function is said to be a *strictly decreasing* function. In either of these cases, an inverse function f^{-1} exists.[†]

A practical way of ascertaining the strict monotonicity of a given function $y = f(x)$ is to check whether the derivative $f'(x)$ always adheres to the same algebraic sign (not zero) for all values of x. Geometrically, this means that its slope is either always upward or always

[†] By omitting the adverb *strictly,* we can define *monotonic* (or *monotone*) functions as follows: An *increasing function* is a function with the property that

$$x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2) \qquad \text{[with the weak inequality } \geq\text{]}$$

and a *decreasing function* is one with the property that

$$x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2) \qquad \text{[with the weak inequality } \leq\text{]}$$

Note that, under this definition, an ascending (descending) step function qualifies as an increasing (decreasing) function, despite the fact that its graph contains horizontal segments. Since such functions do not have a one-to-one mapping, they do not have inverse functions.

downward. Thus a firm's demand curve $Q = f(P)$ that has a negative slope throughout is strictly decreasing. As such, it has an inverse function $P = f^{-1}(Q)$, which, as mentioned previously, gives the average-revenue curve of the firm, since $P \equiv \text{AR}$.

Example 5 The function

$$y = 5x + 25$$

has the derivative $dy/dx = 5$, which is positive regardless of the value of x; thus the function is strictly increasing. It follows that an inverse function exists. In the present case, the inverse function is easily found by solving the given equation $y = 5x + 25$ for x. The result is the function

$$x = \tfrac{1}{5}y - 5$$

It is interesting to note that this inverse function is also strictly increasing, because $dx/dy = \frac{1}{5} > 0$ for all values of y.

Generally speaking, if an inverse function exists, the original and the inverse functions must both be strictly monotonic. Moreover, if f^{-1} is the inverse function of f, then f must be the inverse function of f^{-1}; that is, f and f^{-1} must be inverse functions of each other.

It is easy to verify that the graph of $y = f(x)$ and that of $x = f^{-1}(y)$ are one and the same, only with the axes reversed. If one lays the x axis of the f^{-1} graph over the x axis of the f graph (and similarly for the y axis), the two curves will coincide. On the other hand, if the x axis of the f^{-1} graph is laid over the y axis of the f graph (and vice versa), the two curves will become *mirror images* of each other with reference to the 45° line drawn through the origin. This mirror-image relationship provides us with an easy way of graphing the inverse function f^{-1}, once the graph of the original function f is given. (You should try this with the two functions in Example 5.)

For inverse functions, the rule of differentiation is

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

This means that the derivative of the inverse function is the reciprocal of the derivative of the original function; as such, dx/dy must take the same sign as dy/dx, so that if f is strictly increasing (decreasing), then so must be f^{-1}.

As a verification of this rule, we can refer back to Example 5, where dy/dx was found to be 5, and dx/dy equal to $\frac{1}{5}$. These two derivatives are indeed reciprocal to each other and have the same sign.

In that simple example, the inverse function is relatively easy to obtain, so that its derivative dx/dy can be found directly from the inverse function. As Example 6 shows, however, the inverse function is sometimes difficult to express explicitly, and thus direct differentiation may not be practicable. The usefulness of the inverse-function rule then becomes more fully apparent.

Example 6 Given $y = x^5 + x$, find dx/dy. First of all, since

$$\frac{dy}{dx} = 5x^4 + 1 > 0$$

for any value of x, the given function is strictly increasing, and an inverse function exists. To solve the given equation for x may not be such an easy task, but the derivative of the inverse function can nevertheless be found quickly by use of the inverse-function rule:

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{5x^4 + 1}$$

The inverse-function rule is, strictly speaking, applicable only when the function involved is a one-to-one mapping. In fact, however, we do have some leeway. For instance, when dealing with a U-shaped curve (not strictly monotonic), we may consider the downward- and the upward-sloping segments of the curve as representing two *separate* functions, each with a restricted domain, and each being strictly monotonic in the restricted domain. To each of these, the inverse-function rule can then again be applied.

EXERCISE 7.3

1. Given $y = u^3 + 2u$, where $u = 5 - x^2$, find dy/dx by the chain rule.
2. Given $w = ay^2$ and $y = bx^2 + cx$, find dw/dx by the chain rule.
3. Use the chain rule to find dy/dx for the following:
   (*a*) $y = (3x^2 - 13)^3$ (*b*) $y = (7x^3 - 5)^9$ (*c*) $y = (ax + b)^5$
4. Given $y = (16x + 3)^{-2}$, use the chain rule to find dy/dx. Then rewrite the function as $y = 1/(16x + 3)^2$ and find dy/dx by the quotient rule. Are the answers identical?
5. Given $y = 7x + 21$, find its inverse function. Then find dy/dx and dx/dy, and verify the inverse-function rule. Also verify that the graphs of the two functions bear a mirror-image relationship to each other.
6. Are the following functions strictly monotonic?
   (*a*) $y = -x^6 + 5 \quad (x > 0)$
   (*b*) $y = 4x^5 + x^3 + 3x$
   For each strictly monotonic function, find dx/dy by the inverse-function rule.

7.4 Partial Differentiation

Hitherto, we have considered only the derivatives of functions of a single independent variable. In comparative-static analysis, however, we are likely to encounter the situation in which several parameters appear in a model, so that the equilibrium value of each endogenous variable may be a function of more than one parameter. Therefore, as a final preparation for the application of the concept of derivative to comparative statics, we must learn how to find the derivative of a function of more than one variable.

Partial Derivatives

Let us consider a function

$$y = f(x_1, x_2, \ldots, x_n) \tag{7.12}$$

where the variables x_i $(i = 1, 2, \ldots, n)$ are all *independent* of one another, so that each can vary by itself without affecting the others. If the variable x_1 undergoes a change Δx_1 while

$x_2, \ldots, x_n$ all remain fixed, there will be a corresponding change in y, namely, Δy. The difference quotient in this case can be expressed as

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_1} \tag{7.13}$$

If we take the limit of $\Delta y/\Delta x_1$ as $\Delta x_1 \to 0$, that limit will constitute a derivative. We call it the *partial derivative* of y with respect to x_1, to indicate that all the other independent variables in the function are held constant when taking this particular derivative. Similar partial derivatives can be defined for infinitesimal changes in the other independent variables. The process of taking partial derivatives is called *partial differentiation.*

Partial derivatives are assigned distinctive symbols. In lieu of the letter d (as in dy/dx), we employ the symbol ∂, which is a variant of the Greek δ (lowercase delta). Thus we shall now write $\partial y/\partial x_i$, which is read: "the partial derivative of y with respect to x_i." The partial-derivative symbol sometimes is also written as $\frac{\partial}{\partial x_i} y$; in that case, its $\partial/\partial x_i$ part can be regarded as an operator symbol instructing us to take the partial derivative of (some function) with respect to the variable x_i. Since the function involved here is denoted in (7.12) by f, it is also permissible to write $\partial f/\partial x_i$.

Is there also a partial-derivative counterpart for the symbol $f'(x)$ that we used before? The answer is yes. Instead of f', however, we now use f_1, f_2, etc., where the subscript indicates which independent variable (alone) is being allowed to vary. If the function in (7.12) happens to be written in terms of unsubscripted variables, such as $y = f(u, v, w)$, then the partial derivatives may be denoted by f_u, f_v, and f_w rather than f_1, f_2, and f_3.

In line with these notations, and on the basis of (7.12) and (7.13), we can now define

$$f_1 \equiv \frac{\partial y}{\partial x_1} \equiv \lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x_1}$$

as the first in the set of n partial derivatives of the function f.

Techniques of Partial Differentiation

Partial differentiation differs from the previously discussed differentiation primarily in that we must hold $(n - 1)$ independent variables *constant* while allowing *one* variable to vary. Inasmuch as we have learned how to handle *constants* in differentiation, the actual differentiation should pose little problem.

Example 1

Given $y = f(x_1, x_2) = 3x_1^2 + x_1 x_2 + 4x_2^2$, find the partial derivatives. When finding $\partial y/\partial x_1$, (or f_1), we must bear in mind that x_2 is to be treated as a constant during differentiation. As such, x_2 will drop out in the process if it is an *additive* constant (such as the term $4x_2^2$) but will be retained if it is a *multiplicative* constant (such as in the term $x_1 x_2$). Thus we have

$$\frac{\partial y}{\partial x_1} \equiv f_1 = 6x_1 + x_2$$

Similarly, by treating x_1 as a constant, we find that

$$\frac{\partial y}{\partial x_2} \equiv f_2 = x_1 + 8x_2$$

Note that, like the primitive function f, both partial derivatives are themselves functions of the variables x_1 and x_2. That is, we may write them as two derived functions

$$f_1 = f_1(x_1, x_2) \qquad \text{and} \qquad f_2 = f_2(x_1, x_2)$$

For the point $(x_1, x_2) = (1, 3)$ in the domain of the function f, for example, the partial derivatives will take the following specific values:

$$f_1(1, 3) = 6(1) + 3 = 9 \qquad \text{and} \qquad f_2(1, 3) = 1 + 8(3) = 25$$

Example 2 Given $y = f(u, v) = (u + 4)(3u + 2v)$, the partial derivatives can be found by use of the product rule. By holding v constant, we have

$$f_u = (u + 4)(3) + 1(3u + 2v) = 2(3u + v + 6)$$

Similarly, by holding u constant, we find that

$$f_v = (u + 4)(2) + 0(3u + 2v) = 2(u + 4)$$

When $u = 2$ and $v = 1$, these derivatives will take the following values:

$$f_u(2, 1) = 2(13) = 26 \qquad \text{and} \qquad f_v(2, 1) = 2(6) = 12$$

Example 3 Given $y = (3u - 2v)/(u^2 + 3v)$, the partial derivatives can be found by use of the quotient rule:

$$\frac{\partial y}{\partial u} = \frac{3(u^2 + 3v) - 2u(3u - 2v)}{(u^2 + 3v)^2} = \frac{-3u^2 + 4uv + 9v}{(u^2 + 3v)^2}$$

$$\frac{\partial y}{\partial v} = \frac{-2(u^2 + 3v) - 3(3u - 2v)}{(u^2 + 3v)^2} = \frac{-u(2u + 9)}{(u^2 + 3v)^2}$$

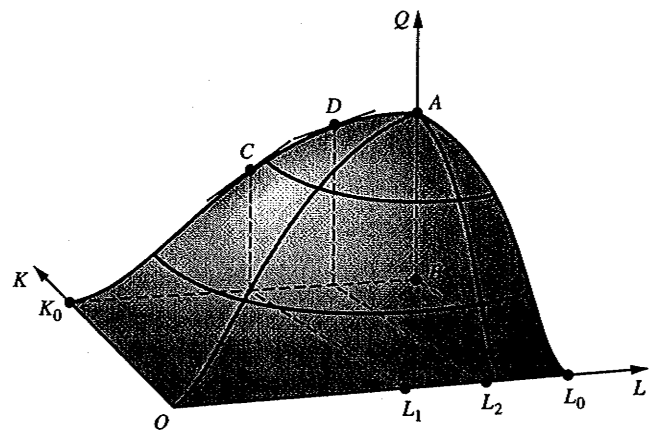
Geometric Interpretation of Partial Derivatives

As a special type of derivative, a partial derivative is a measure of the instantaneous rates of change of some variable, and in that capacity it again has a geometric counterpart in the slope of a particular curve.

Let us consider a production function $Q = Q(K, L)$, where Q, K, and L denote output, capital input, and labor input, respectively. This function is a particular two-variable version of (7.12), with $n = 2$. We can therefore define two partial derivatives $\partial Q/\partial K$ (or Q_K) and $\partial Q/\partial L$ (or Q_L). The partial derivative Q_K relates to the rates of change of output with respect to infinitesimal changes in capital, while labor input is held constant. Thus Q_K symbolizes the marginal-physical-product-of-capital (MPP_K) function. Similarly, the partial derivative Q_L is the mathematical representation of the MPP_L function.

Geometrically, the production function $Q = Q(K, L)$ can be depicted by a *production surface* in a 3-space, such as is shown in Fig. 7.4. The variable Q is plotted vertically, so that for any point (K, L) in the base plane (KL plane), the height of the surface will indicate the output Q. The domain of the function should consist of the entire nonnegative quadrant of the base plane, but for our purposes it is sufficient to consider a subset of it, the

FIGURE 7.4



rectangle OK_0BL_0. As a consequence, only a small portion of the production surface is shown in the figure.

Let us now hold capital fixed at the level K_0 and consider only variations in the input L. By setting $K = K_0$, all points in our (curtailed) domain become irrelevant except those on the line segment K_0B. By the same token, only the curve K_0CDA (a cross section of the production surface) is germane to the present discussion. This curve represents a total-physical-product-of-labor (TPP_L) curve for a fixed amount of capital $K = K_0$; thus we may read from its slope the rate of change of Q with respect to changes in L while K is held constant. It is clear, therefore, that the slope of a curve such as K_0CDA represents the geometric counterpart of the partial derivative Q_L. Once again, we note that the slope of a total (TPP_L) curve is its corresponding marginal ($\text{MPP}_L \equiv Q_L$) curve.

As mentioned earlier, a partial derivative is a function of all the independent variables of the primitive function. That Q_L is a function of L is immediately obvious from the K_0CDA curve itself. When $L = L_1$, the value of Q_L is equal to the slope of the curve at point C; but when $L = L_2$, the relevant slope is the one at point D. Why is Q_L also a function of K? The answer is that K can be fixed at various levels, and for each fixed level of K, there results a different TPP_L curve (a different cross section of the production surface), with inevitable repercussions on the derivative Q_L. Hence Q_L is also a function of K.

An analogous interpretation can be given to the partial derivative Q_K. If the labor input is held constant instead of K (say, at the level of L_0), the line segment L_0B will be the relevant subset of the domain, and the curve L_0A will indicate the relevant subset of the production surface. The partial derivative Q_K can then be interpreted as the slope of the curve L_0A—bearing in mind that the K axis extends from southeast to northwest in Fig. 7.4. It should be noted that Q_K is again a function of both the variables L and K.

Gradient Vector

All the partial derivatives of a function $y = f(x_1, x_2, \ldots, x_n)$ can be collected under a single mathematical entity called the *gradient vector,* or simply the *gradient,* of function f:

$$\text{grad } f(x_1, x_2, \ldots, x_n) = (f_1, f_2, \ldots, f_n)$$

where $f_i \equiv \partial y/\partial x_i$. Note that we are using parentheses rather than brackets here in writing the vector. Alternatively, the gradient can be denoted by $\nabla f(x_1, x_2, \ldots, x_n)$, where ∇ (read: "del") is the inverted version of the Greek letter Δ.

Since the function f has n arguments, there are altogether n partial derivatives; hence, grad f is an n-vector. When these derivatives are evaluated at a specific point $(x_{10}, x_{20}, \ldots, x_{n0})$ in the domain, we get grad $f(x_{10}, x_{20}, \ldots, x_{n0})$, a vector of specific derivative *values*.

Example 4 The gradient vector of the production function $Q = Q(K, L)$ is

$$\nabla Q = \nabla Q(K, L) = (Q_K, Q_L)$$

EXERCISE 7.4

1. Find $\partial y/\partial x_1$ and $\partial y/\partial x_2$ for each of the following functions:
   (*a*) $y = 2x_1^3 - 11x_1^2x_2 + 3x_2^2$
   (*b*) $y = 7x_1 + 6x_1x_2^2 - 9x_2^3$
   (*c*) $y = (2x_1 + 3)(x_2 - 2)$
   (*d*) $y = (5x_1 + 3)/(x_2 - 2)$
2. Find f_x and f_y from the following:
   (*a*) $f(x, y) = x^2 + 5xy - y^3$
   (*b*) $f(x, y) = (x^2 - 3y)(x - 2)$
   (*c*) $f(x, y) = \dfrac{2x - 3y}{x + y}$
   (*d*) $f(x, y) = \dfrac{x^2 - 1}{xy}$
3. From the answers to Prob. 2, find $f_x(1, 2)$—the value of the partial derivative f_x when $x = 1$ and $y = 2$—for each function.
4. Given the production function $Q = 96K^{0.3}L^{0.7}$, find the MPP_K and MPP_L functions. Is MPP_K a function of K alone, or of both K and L? What about MPP_L?
5. If the utility function of an individual takes the form

   $$U = U(x_1, x_2) = (x_1 + 2)^2(x_2 + 3)^3$$

   where U is total utility, and x_1 and x_2 are the quantities of two commodities consumed:
   (*a*) Find the marginal-utility function of each of the two commodities.
   (*b*) Find the value of the marginal utility of the first commodity when 3 units of each commodity are consumed.
6. The total money supply M has two components: bank deposits D and cash holdings C, which we assume to bear a constant ratio $C/D = c$, $0 < c < 1$. The high-powered money H is defined as the sum of cash holdings held by the public and the reserves held by the banks. Bank reserves are a fraction of bank deposits, determined by the reserve ratio r, $0 < r < 1$.
   (*a*) Express the money supply M as a function of high-powered money H.
   (*b*) Would an increase in the reserve ratio r raise or lower the money supply?
   (*c*) How would an increase in the cash-deposit ratio c affect the money supply?
7. Write the gradients of the following functions:
   (*a*) $f(x, y, z) = x^2 + y^3 + z^4$
   (*b*) $f(x, y, z) = xyz$

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7.5 Applications to Comparative-Static Analysis

Equipped with the knowledge of the various rules of differentiation, we can at last tackle the problem posed in comparative-static analysis: namely, how the equilibrium value of an endogenous variable will change when there is a change in any of the exogenous variables or parameters.

Market Model

First let us consider again the simple one-commodity market model of (3.1). That model can be written in the form of two equations:

$$Q = a - bP \qquad (a, b > 0) \qquad \text{[demand]}$$
$$Q = -c + dP \qquad (c, d > 0) \qquad \text{[supply]}$$

with solutions

$$P^* = \frac{a + c}{b + d} \tag{7.14}$$

$$Q^* = \frac{ad - bc}{b + d} \tag{7.15}$$

These solutions will be referred to as being in the *reduced form:* The two endogenous variables have been reduced to explicit expressions of the four mutually independent parameters a, b, c, and d.

To find how an infinitesimal change in one of the parameters will affect the value of P^*, one has only to differentiate (7.14) partially with respect to each of the parameters. If the *sign* of a partial derivative, say, $\partial P^*/\partial a$, can be determined from the given information about the parameters, we shall know the direction in which P^* will move when the parameter a changes; this constitutes a qualitative conclusion. If the magnitude of $\partial P^*/\partial a$ can be ascertained, it will constitute a quantitative conclusion.

Similarly, we can draw qualitative or quantitative conclusions from the partial derivatives of Q^* with respect to each parameter, such as $\partial Q^*/\partial a$. To avoid misunderstanding, however, a clear distinction should be made between the two derivatives $\partial Q^*/\partial a$ and $\partial Q/\partial a$. The latter derivative is a concept appropriate to the demand function taken alone, and without regard to the supply function. The derivative $\partial Q^*/\partial a$ pertains, on the other hand, to the equilibrium quantity in (7.15) which, being in the nature of a solution of the model, takes into account the interaction of demand and supply together. To emphasize this distinction, we shall refer to the partial derivatives of P^* and Q^* with respect to the parameters as *comparative-static derivatives.* The possibility of confusion between $\partial Q^*/\partial a$ and $\partial Q/\partial a$ is precisely the reason why we have chosen to use the asterisk notation, as in Q^* to denote the equilibrium value.

Concentrating on P^* for the time being, we can get the following four partial derivatives from (7.14):

$$\frac{\partial P^*}{\partial a} = \frac{1}{b + d} \qquad \left[\text{parameter } a \text{ has the coefficient } \frac{1}{b + d}\right]$$

$$\frac{\partial P^*}{\partial b} = \frac{0(b + d) - 1(a + c)}{(b + d)^2} = \frac{-(a + c)}{(b + d)^2} \qquad \text{[quotient rule]}$$

$$\frac{\partial P^*}{\partial c} = \frac{1}{b+d}\left(= \frac{\partial P^*}{\partial a}\right)$$

$$\frac{\partial P^*}{\partial d} = \frac{0(b+d) - 1(a+c)}{(b+d)^2} = \frac{-(a+c)}{(b+d)^2}\left(= \frac{\partial P^*}{\partial b}\right)$$

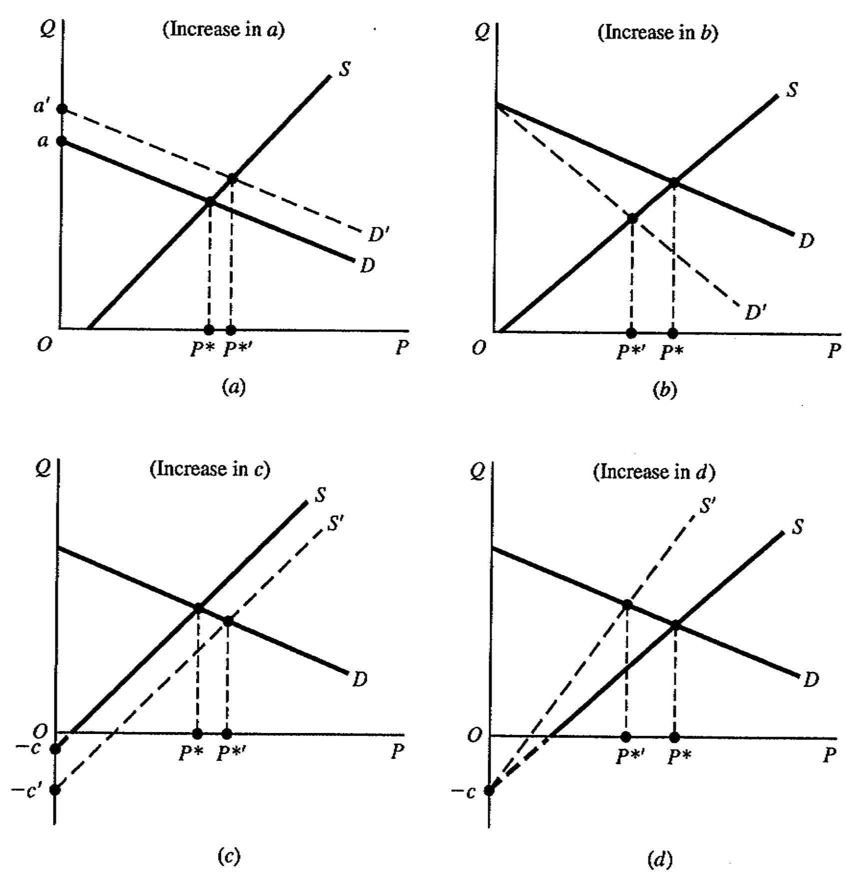
Since all the parameters are restricted to being positive in the present model, we can conclude that

$$\frac{\partial P^*}{\partial a} = \frac{\partial P^*}{\partial c} > 0 \qquad \text{and} \qquad \frac{\partial P^*}{\partial b} = \frac{\partial P^*}{\partial d} < 0 \tag{7.16}$$

For a fuller appreciation of the results in (7.16), let us look at Fig. 7.5, where each diagram shows a change in *one* of the parameters. As before, we are plotting Q (rather than P) on the vertical axis.

Figure 7.5*a* pictures an increase in the parameter a (to a'). This means a higher vertical intercept for the demand curve, and inasmuch as the parameter b (the slope parameter) is unchanged, the increase in a results in a parallel upward shift of the demand curve from D

FIGURE 7.5



to D'. The intersection of D' and the supply curve S determines an equilibrium price $P^{*\prime}$, which is greater than the old equilibrium price P^*. This corroborates the result that $\partial P^*/\partial a > 0$, although for the sake of exposition we have shown in Fig. 7.5*a* a much larger change in the parameter a than what the concept of derivative implies.

The situation in Fig. 7.5*c* has a similar interpretation; but since the increase takes place in the parameter c, the result is a parallel shift of the supply curve instead. Note that this shift is downward because the supply curve has a vertical intercept of $-c$; thus an increase in c would mean a change in the intercept, say, from -2 to -4. The graphical comparative-static result, that $P^{*\prime}$ exceeds P^*, again conforms to what the positive sign of the derivative $\partial P^*/\partial c$ would lead us to expect.

Figures 7.5*b* and 7.5*d* illustrate the effects of changes in the slope parameters b and d of the two functions in the model. An increase in b means that the slope of the demand curve will assume a larger numerical (absolute) value; i.e., it will become steeper. In accordance with the result $\partial P^*/\partial b < 0$, we find a decrease in P^* in this diagram. The increase in d that makes the supply curve steeper also results in a decrease in the equilibrium price. This is, of course, again in line with the negative sign of the comparative-static derivative $\partial P^*/\partial d$.

Thus far, all the results in (7.16) seem to have been obtainable graphically. If so, why should we bother to use differentiation at all? The answer is that the differentiation approach has at least two major advantages. First, the graphical technique is subject to a dimensional restriction, but differentiation is not. Even when the number of endogenous variables and parameters is such that the equilibrium state cannot be shown graphically, we can nevertheless apply the differentiation techniques to the problem. Second, the differentiation method can yield results that are on a higher level of generality. The results in (7.16) will remain valid, regardless of the specific values that the parameters a, b, c, and d take, as long as they satisfy the sign restrictions. So the comparative-static conclusions of this model are, in effect, applicable to an infinite number of combinations of (linear) demand and supply functions. In contrast, the graphical approach deals only with some specific members of the family of demand and supply curves, and the analytical result derived therefrom is applicable, strictly speaking, only to the specific functions depicted.

This discussion serves to illustrate the application of partial differentiation to comparative-static analysis of the simple market model, but only half of the task has actually been accomplished, for we can also find the comparative-static derivatives pertaining to Q^*. This we shall leave to you as an exercise.

National-Income Model

In place of the simple national-income model discussed in Chap. 3, let us now work with a slightly enlarged model with three endogenous variables, Y (national income), C (consumption), and T (taxes):

$$\begin{aligned} Y &= C + I_0 + G_0 \\ C &= \alpha + \beta(Y - T) \qquad (\alpha > 0; \quad 0 < \beta < 1) \\ T &= \gamma + \delta Y \qquad (\gamma > 0; \quad 0 < \delta < 1) \end{aligned} \tag{7.17}$$

The first equation in this system gives the equilibrium condition for national income, while the second and third equations show, respectively, how C and T are determined in the model.

The restrictions on the values of the parameters α, β, γ, and δ can be explained thus: α is positive because consumption is positive even if disposable income $(Y - T)$ is zero; β is a positive fraction because it represents the marginal propensity to consume; γ is positive because even if Y is zero the government will still have a positive tax revenue (from tax bases other than income); and finally, δ is a positive fraction because it represents an income tax rate, and as such it cannot exceed 100 percent. The exogenous variables I_0 (investment) and G_0 (government expenditure) are, of course, nonnegative. All the parameters and exogenous variables are assumed to be independent of one another, so that any one of them can be assigned a new value without affecting the others.

This model can be solved for Y^* by substituting the third equation of (7.17) into the second and then substituting the resulting equation into the first. The equilibrium income (in reduced form) is

$$Y^* = \frac{\alpha - \beta\gamma + I_0 + G_0}{1 - \beta + \beta\delta} \tag{7.18}$$

Similar equilibrium values can also be found for the endogenous variables C and T, but we shall concentrate on the equilibrium income.

From (7.18), there can be obtained six comparative-static derivatives. Among these, the following three have special policy significance:

$$\frac{\partial Y^*}{\partial G_0} = \frac{1}{1 - \beta + \beta\delta} > 0 \tag{7.19}$$

$$\frac{\partial Y^*}{\partial \gamma} = \frac{-\beta}{1 - \beta + \beta\delta} < 0 \tag{7.20}$$

$$\frac{\partial Y^*}{\partial \delta} = \frac{-\beta(\alpha - \beta\gamma + I_0 + G_0)}{(1 - \beta + \beta\delta)^2} = \frac{-\beta Y^*}{1 - \beta + \beta\delta} < 0 \qquad \text{[by (7.18)]} \tag{7.21}$$

The partial derivative in (7.19) gives us the *government-expenditure multiplier.* It has a positive sign here because β is less than 1, and $\beta\delta$ is greater than zero. If numerical values are given for the parameters β and δ, we can also find the numerical value of this multiplier from (7.19). The derivative in (7.20) may be called the *nonincome-tax multiplier,* because it shows how a change in γ, the government revenue from nonincome-tax sources, will affect the equilibrium income. This multiplier is negative in the present model because the denominator in (7.20) is positive and the numerator is negative. Lastly, the partial derivative in (7.21)—which is not in the nature of a multiplier, since it does not relate a dollar change to another dollar change as the derivatives in (7.19) and (7.20) do—tells us the extent to which an increase in the income tax rate δ will lower the equilibrium income.

Again, note the difference between the two derivatives $\partial Y^*/\partial G_0$ and $\partial Y/\partial G_0$. The former is derived from (7.18), the expression for the equilibrium income. The latter, obtainable from the first equation in (7.17), is $\partial Y/\partial G_0 = 1$, which is altogether different in magnitude and in concept.

Input-Output Model

The solution of an open input-output model appears as a matrix equation $x^* = (I - A)^{-1}d$. If we denote the inverse matrix $(I - A)^{-1}$ by $V = [v_{ij}]$, then, for instance, the solution for a three-industry economy can be written as $\bar{x} = Vd$, or

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \tag{7.22}$$

What will be the rates of change of the solution values x_j^* with respect to the exogenous final demands d_1, d_2, and d_3? The general answer is that

$$\frac{\partial x_j^*}{\partial d_k} = v_{jk} \qquad (j, k = 1, 2, 3) \tag{7.23}$$

To see this, let us multiply out Vd in (7.22) and express the solution as

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} v_{11}d_1 + v_{12}d_2 + v_{13}d_3 \\ v_{21}d_1 + v_{22}d_2 + v_{23}d_3 \\ v_{31}d_1 + v_{32}d_2 + v_{33}d_3 \end{bmatrix}$$

In this system of three equations, each one gives a particular solution value as a function of the exogenous final demands. Partial differentiation of these produces a total of nine comparative-static derivatives:

$$\begin{array}{lll} \dfrac{\partial x_1^*}{\partial d_1} = v_{11} & \dfrac{\partial x_1^*}{\partial d_2} = v_{12} & \dfrac{\partial x_1^*}{\partial d_3} = v_{13} \\ \dfrac{\partial x_2^*}{\partial d_1} = v_{21} & \dfrac{\partial x_2^*}{\partial d_2} = v_{22} & \dfrac{\partial x_2^*}{\partial d_3} = v_{23} \\ \dfrac{\partial x_3^*}{\partial d_1} = v_{31} & \dfrac{\partial x_3^*}{\partial d_2} = v_{32} & \dfrac{\partial x_3^*}{\partial d_3} = v_{33} \end{array} \tag{7.23'}$$

This is simply the expanded version of (7.23).

Reading (7.23′) as three distinct columns, we may combine the three derivatives in each column into a matrix (vector) derivative:

$$\frac{\partial x^*}{\partial d_1} \equiv \frac{\partial}{\partial d_1}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix} \qquad \frac{\partial x^*}{\partial d_2} = \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix} \qquad \frac{\partial x^*}{\partial d_3} = \begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix} \tag{7.23''}$$

Since the three column vectors in (7.23″) are merely the columns of the matrix V, by further consolidation we can summarize the nine derivatives in a single matrix derivative $\partial x^*/\partial d$. Given $x^* = Vd$, we can simply write

$$\frac{\partial x^*}{\partial d} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} = V \equiv (I - A)^{-1}$$

Thus, $(I - A)^{-1}$, the inverse of the Leontief matrix, gives us an ordered display of all the comparative-static derivatives of our open input-output model. Obviously, this matrix derivative can easily be extended from the present three-industry model to the general n-industry case.

Comparative-static derivatives of the input-output model are useful as tools of economic planning, for they provide the answer to the question: If the planning targets, as reflected in

$(d_1, d_2, \ldots, d_n)$, are revised, and if we wish to take care of all direct and indirect requirements in the economy so as to be completely free of bottlenecks, how must we change the output goals of the n industries?

EXERCISE 7.5

1. Examine the comparative-static properties of the equilibrium quantity in (7.15), and check your results by graphic analysis.
2. On the basis of (7.18), find the partial derivatives $\partial Y^*/\partial I_0$, $\partial Y^*/\partial \alpha$, and $\partial Y^*/\partial \beta$. Interpret their meanings and determine their signs.
3. The numerical input-output model (5.21) was solved in Sec. 5.7.
   (*a*) How many comparative-static derivatives can be derived?
   (*b*) Write out these derivatives in the form of (7.23') and (7.23'').

7.6 Note on Jacobian Determinants

Our study of partial derivatives was motivated solely by comparative-static considerations. But partial derivatives also provide a means of testing whether there exists functional (linear *or* nonlinear) dependence among a set of n functions in n variables. This is related to the notion of Jacobian determinants (named after Jacobi).

Consider the two functions

$$\begin{aligned} y_1 &= 2x_1 + 3x_2 \\ y_2 &= 4x_1^2 + 12x_1x_2 + 9x_2^2 \end{aligned} \qquad (7.24)$$

If we get all the four partial derivatives

$$\frac{\partial y_1}{\partial x_1} = 2 \qquad \frac{\partial y_1}{\partial x_2} = 3 \qquad \frac{\partial y_2}{\partial x_1} = 8x_1 + 12x_2 \qquad \frac{\partial y_2}{\partial x_2} = 12x_1 + 18x_2$$

and arrange them into a square matrix in a prescribed order, called a Jacobian matrix and denoted by J, and then take its determinant, the result will be what is known as a *Jacobian determinant* (or a *Jacobian,* for short), denoted by $|J|$:

$$|J| \equiv \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ (8x_1 + 12x_2) & (12x_1 + 18x_2) \end{vmatrix} \qquad (7.25)$$

For economy of space, this Jacobian is sometimes also expressed as

$$|J| \equiv \left| \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \right|$$

More generally, if we have n differentiable functions in n variables, not necessarily linear,

$$\begin{aligned} y_1 &= f^1(x_1, x_2, \ldots, x_n) \\ y_2 &= f^2(x_1, x_2, \ldots, x_n) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n &= f^n(x_1, x_2, \ldots, x_n) \end{aligned} \qquad (7.26)$$

where the symbol f^n denotes the nth function (and *not* the function raised to the nth power), we can derive a total of n^2 partial derivatives. Adopting the notation $f_j^i \equiv \partial y^i/\partial x_j$, we can write the Jacobian

$$|J| \equiv \left|\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)}\right| \equiv \begin{vmatrix} \partial y_1/\partial x_1 & \cdots & \partial y_1/\partial x_n \\ \vdots & & \vdots \\ \partial y_n/\partial x_1 & \cdots & \partial y_n/\partial x_n \end{vmatrix} \equiv \begin{vmatrix} f_1^1 & \cdots & f_n^1 \\ \vdots & & \vdots \\ f_1^n & \cdots & f_n^n \end{vmatrix} \tag{7.27}$$

A Jacobian test for the existence of functional dependence among a set of n functions is provided by the following theorem: The Jacobian $|J|$ defined in (7.27) will be identically zero for all values of $x_1, \ldots, x_n$ if and only if the n functions $f^1, \ldots, f^n$ in (7.26) are functionally (linearly or nonlinearly) dependent.

As an example, for the two functions in (7.24) the Jacobian as given in (7.25) has the value

$$|J| = (24x_1 + 36x_2) - (24x_1 + 36x_2) = 0$$

That is, the Jacobian vanishes for all values of x_1 and x_2. Therefore, according to the theorem, the two functions in (7.24) must be dependent. You can verify that y_2 is simply y_1 squared; thus they are indeed functionally dependent—here *non*linearly dependent.

Let us now consider the special case of *linear* functions. We have earlier shown that the rows of the coefficient matrix A of a linear-equation system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= d_n \end{aligned} \tag{7.28}$$

are linearly dependent if and only if the determinant $|A| = 0$. This result can now be interpreted as a special application of the Jacobian criterion of functional dependence.

Take the left side of each equation in (7.28) as a separate function of the n variables $x_1, \ldots, x_n$, and denote these functions by $y_1, \ldots, y_n$. The partial derivatives of these functions will turn out to be $\partial y_1/\partial x_1 = a_{11}$, $\partial y_1/\partial x_2 = a_{12}$, etc., so that we may write, in general, $\partial y_i/\partial x_j = a_{ij}$. In view of this, the elements of the Jacobian of these n functions will be precisely the elements of the coefficient matrix A, already arranged in the correct order. That is, we have $|J| = |A|$, and thus the Jacobian criterion of functional dependence among $y_1, \ldots, y_n$—or, what amounts to the same thing, linear dependence among the rows of the coefficient matrix A—is equivalent to the criterion $|A| = 0$ in the present linear case.

We have discussed the Jacobian in the context of a system of n functions in n variables. It should be pointed out, however, that the Jacobian in (7.27) is defined even if each function in (7.26) contains more than n variables, say, $n + 2$ variables:

$$y_i = f^i(x_1, \ldots, x_n, x_{n+1}, x_{n+2}) \qquad (i = 1, 2, \ldots, n)$$

In such a case, if we hold any two of the variables (say, x_{n+1} and x_{n+2}) constant, or treat them as parameters, we will again have n functions in exactly n variables and can form a

Jacobian. Moreover, by holding a different pair of the x variables constant, we can form a different Jacobian. Such a situation will indeed be encountered in Chap. 8 in connection with the discussion of the implicit-function theorem.

EXERCISE 7.6

1. Use Jacobian determinants to test the existence of functional dependence between the paired functions.
   (*a*) $y_1 = 3x_1^2 + x_2$
   $y_2 = 9x_1^4 + 6x_1^2(x_2 + 4) + x_2(x_2 + 8) + 12$
   (*b*) $y_1 = 3x_1^2 + 2x_2^2$
   $y_2 = 5x_1 + 1$
2. Consider (7.22) as a set of three functions $x_i^* = f^i(d_1, d_2, d_3)$ (with $i = 1, 2, 3$).
   (*a*) Write out the 3×3 Jacobian. Does it have some relation to (7.23′)? Can we write $|J| = |V|$?
   (*b*) Since $V \equiv (I - A)^{-1}$, can we conclude that $|V| \neq 0$? What can we infer from this about the three equations in (7.22)?