331

Chapter 10: exponentials & Logarithms

I Simple Compound interest

$\Rightarrow$ Suppose you have x dollars to invest at an interest rate of r percent per year.

$\Rightarrow$ In one year you will have y dollars, where:

$$y = x + rx = x(1+r)$$

In two years,

$$y = [x(1+r)](1+r) = x(1+r)^2$$

In three years,

$$y = [x(1+r)^2](1+r) = x(1+r)^3$$

The Present Value of y 3 yrs from now is:

$$x = \frac{y}{(1+r)^3} = y(1+r)^{-3}$$

PV: Tells you "how much to invest now" in order to have y dollars in 3 years

II Compounding within a year

(A.) Semi-Annual compounding

at six months

$$y = x\left(1+\frac{r}{2}\right) = x + \frac{xr}{2}$$

at one year

$$y = \left[x\left(1+\frac{r}{2}\right)\right]\left(1+\frac{r}{2}\right) = x\left(1+\frac{r}{2}\right)^2$$

B. Monthly Compounding

$$y = x\left(1+\frac{r}{12}\right)^{12} \quad \text{for one year}$$

For two years

$$y = \left[x\left(1+\frac{r}{12}\right)^{12}\right]\left(1+\frac{r}{12}\right)^{12} = x\left(1+\frac{r}{12}\right)^{24}$$

For n-years

$$y = x\left(1+\frac{r}{12}\right)^{12n}$$

III Converting compound interest into an Annual yield

⇒ Suppose you are offered a choice:

(1) 10% compounded semi-annually

or

(2) 10.2% annually

⇒ Which do you choose?

⇒ We know for Semi-Annual

$$y = x\left(1+\frac{r}{2}\right)^2 = x\left(1+\frac{.10}{2}\right)^2 = (1.05)^2 x$$

or

$$y = 1.1025x$$

⇒ yield = y − Principal = y − x

⇒ yield = 1.1025x − x = .1025x

or you earn 10.25% annually

Since 10.25% > 10.20%

pick option (1).

IV Continuous Compounding

(A.) Daily interest for one year

$$y = x\left(1+\frac{r}{365}\right)^{365}$$

Suppose $x = \$1$ and $r = 100\%$ $\{or\ r=1\}$

$$y = 1\left(1+\frac{1}{365}\right)^{365} = \$2.71456$$

(B.) Compound hourly $(365 \times 24) = 8760$

$$y = 1\left(1+\frac{1}{8760}\right)^{8760} = 2.71812$$

or if

$$y = 1\left(1+\frac{1}{m}\right)^{m}$$

If we let $m \to$ infinity (∞)

$$y = \left(1+\frac{1}{m}\right)^{m} \Rightarrow 2.71828\ldots \equiv e$$

for any r as $m \to \infty$ $\{$and $x = \$1\}$

$$y = \left(1+\frac{r}{m}\right)^{m} \Rightarrow e^{r}$$

V The number "e"

(A.) The number $e = 2.71828...$ is the value of \$1 compounded continuously for one year at an interest rate of 100%
(or one period)

(B.) Continuous Compounding at r percent for t years (Periods) of a principal equal to x

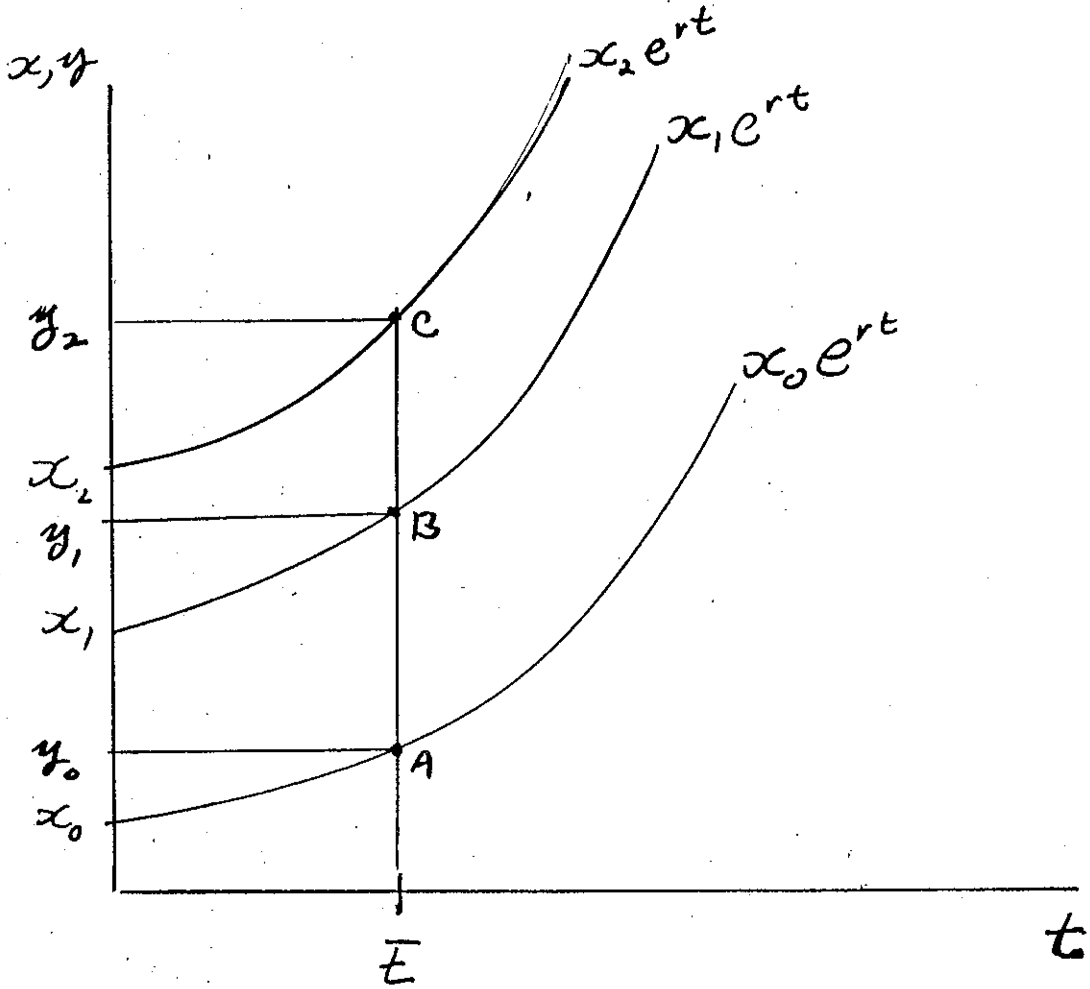
$$y = x e^{rt}$$

$\Rightarrow$ The present value of y is

$$x = \frac{y}{e^{rt}} = y e^{-rt}$$

$\Rightarrow$ which tells you the amount needed to invest today that will be worth y dollars in t years of continuous compounding

Present Value (xe^{-rt}) Graphically



slope $= \frac{dy}{dt} = rxe^{rt}$

Properties of e

1. $y = e^x$ $\qquad \frac{dy}{dx} = e^x$

2. $y = e^{f(x)}$ $\qquad \frac{dy}{dx} = f'(x)\, e^{f(x)}$

3. EXAMPLES:

a) $y = e^{3x}$ $\qquad \frac{dy}{dx} = 3e^{3x}$

b) $y = e^{-rt}$ $\qquad \frac{dy}{dt} = -r e^{-rt}$

c) $y = ae^{(t^2-t)}$ $\qquad \frac{dy}{dt} = a(2t-1)\, e^{(t^2-t)}$

4. $e^{-\infty} = \frac{1}{e^{\infty}} \approx 0$ $\qquad e^0 = 1$

GROWTH Rates

given $y = xe^{rt}$

The change in y is,

$$\frac{dy}{dt} = rxe^{rt} = ry$$

$\Rightarrow$ However, the percentage change in y, or the "growth rate" is

$$\text{Growth rate} = \frac{\Delta \text{ in } Y}{Y} \approx \frac{dy}{y}$$

Therefore

$$\text{Growth Rate} = \frac{dy/dt}{y} = \frac{rxe^{rt}}{xe^{rt}} = r$$

$\Rightarrow$ where r is the continuous rate of growth of y over time

Note: the growth rate is constant however the slope of $y=xe^{rt}$ is not constant

LOGARITHMS

(I) Common log {or log base 10}

Given $10^2 = 100$

The Exponent 2 is defined as the LOGARITHM of 100 to the Base 10

E.g.

$\log 1000 = 3$ because $\{10^3 = 1000\}$

$\log 10 = 1$ " $\{10^1 = 10\}$

$\log 1 = 0$ " $\{10^0 = 1\}$

$\log 0.1 = -1$ " $\{10^{-1} = .1\}$

$\log 0.01 = -2$ " $\{10^{-2} = .001\}$

(II) NATURAL LOGARITHM

of $y = e^x$ $\ln y = \ln e^x = x$

where ln is the logarithm to base e

RULES OF logarithms

RULE I: $\ln(AB) = \ln A + \ln B$

RULE II: $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$

RULE III: $\ln(A^b) = b \ln A$

EXAMPLE: $\ln(x^3 y^2) = 3\ln x + 2\ln y$

OTHER PROPERTIES

if $x = y$ then $\ln x = \ln y$

if $x > y$ then $\ln x > \ln y$

⇒* $\ln(-3)$ does not exist!
you cannot take a logarithms
of a negative number

⇒* $\ln(A+B) \neq \ln A + \ln B$!!!

Derivatives of the Natural logarithm

I. $y = \ln x$ $\quad \dfrac{dy}{dx} = \dfrac{1}{x}$ or $dy = \dfrac{dx}{x}$

II. $y = \ln ax$ $\quad \dfrac{dy}{dx} = \dfrac{a}{ax} = \dfrac{1}{x}$

or

$$y = \ln ax = \ln x + \ln a$$

$$\frac{dy}{dx} = \frac{1}{x} \quad \left\{ \text{since } \frac{d(\ln a)}{dx} = 0 \right\}$$

III. $y = \ln(x^2+2x)$

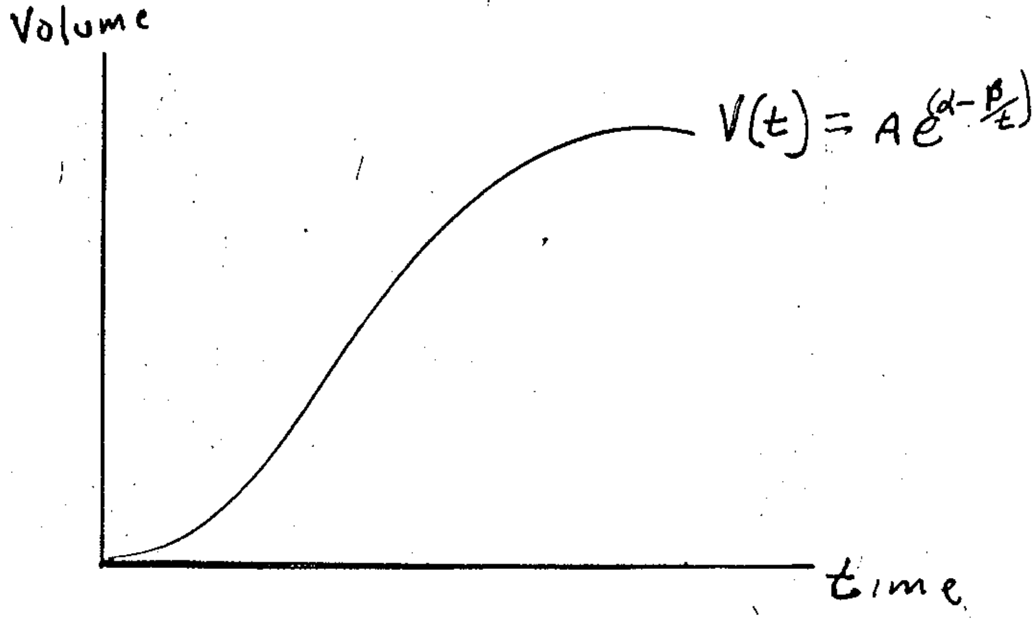
$$\frac{dy}{dx} = \frac{1}{(x^2+2x)}(2x+2) = \frac{2x+2}{x^2+2x} = \frac{1}{x+2} + \frac{1}{x}$$

or $y = \ln(x^2+2x) = \ln[(x+2)x]$

$$= \ln(x+2) + \ln x$$

$$\frac{dy}{dx} = \frac{1}{x+2} + \frac{1}{x}$$

Optimal Timing Problems



The Forest Harvesting Problem

⇒ Assume a stand of trees grows according to the following function:

$$\underline{\text{Volume} = V(t) = A e^{\alpha - \frac{\beta}{t}}} \quad \left\{ \text{Measured in } (ft)^3 \right\}$$

<u>Question</u>: When is the best time to harvest the stand of trees?

⇒ For Simplicity assume that the Price per ft^3 for lumber is \$1 and it remains constant over time

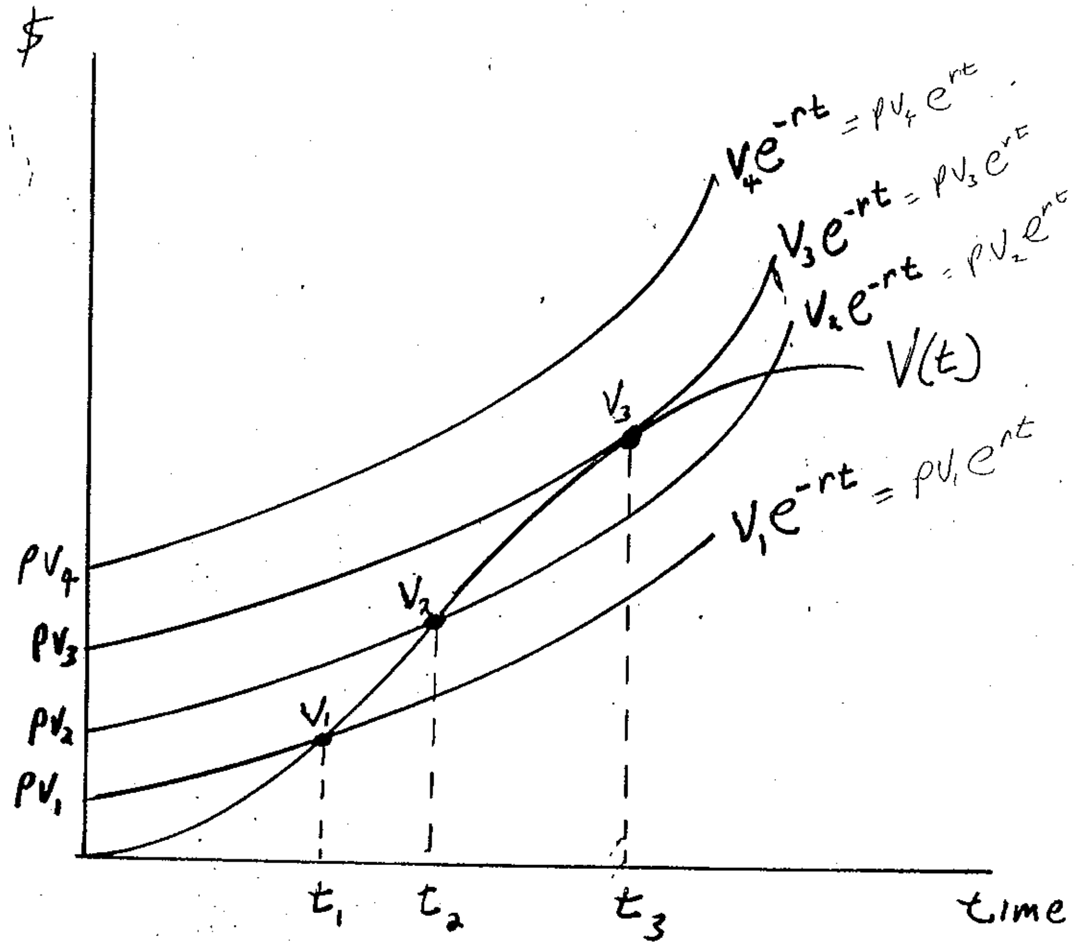
⇒ if the market rate of interest is r then the problem is to choose a time to harvest the Trees that maximizes the Present Value of the asset

⇒ at any time, t_0 the Present Value is:

$$PV = V(t_0)\, e^{-rt_0}$$

$$= \left[A e^{\alpha - \frac{\beta}{t_0}}\right] e^{-rt}$$

$$PV = A e^{\alpha - \beta/t - rt}$$



$\Rightarrow$ optimal harvest time: $\underline{t_3}$

$\Rightarrow$ Maximum Present Value: $\underline{PV_3}$ $\left\{ \frac{V_3'}{V_3} = r \right\}$

$\Rightarrow$ at V_1 the growth rate of tree's excedes the growth rate of a financial asset since $\frac{V_1(t)'}{V_1} > r$

I Present Value is

$$PV(t) = V(t)\, e^{-rt} \qquad \left\{ V(t) = A e^{\alpha - \beta/t} \right\}$$

$$PV(t) = A e^{\alpha - \beta/t - rt}$$

Max PV with respect to t $\left\{ \frac{dPV}{dt} = 0 \right\}$

$$\frac{dPV}{dt} = A e^{\alpha - \beta/t - rt}\left(\beta/t^2 - r \right) = 0$$

$$\frac{dPV}{dt} = 0 \quad \text{iff} \quad \beta/t^2 - r = 0$$

Therefore $\frac{\beta}{t^2} = r$ or $t = \sqrt{\beta/r}$

II Logarithmic Approach

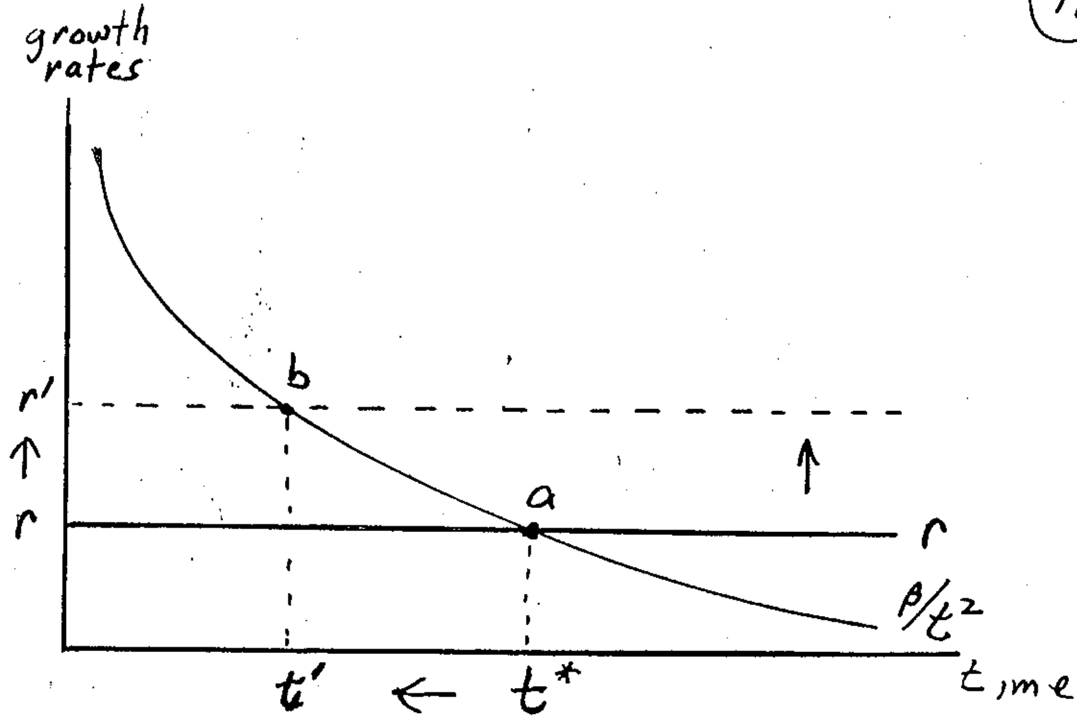
$$\ln PV = \ln\left(e^{\alpha - \frac{\beta}{t} - rt} \right) = \alpha - \beta/t - rt$$

$$\frac{d(\ln PV)}{dt} = \frac{dPV}{PV} = \beta/t^2 - r = 0$$

$$\Rightarrow \beta/t^2 = r \qquad \left\{ t = \sqrt{\beta/r} \right\}$$

$\Rightarrow \beta/t^2$ = Growth rate of the value of uncut trees

$\Rightarrow r$ = Growth rate of optimally invested Money



β/t^2 = the growth in your wealth from leaving trees uncut

r = Growth in your wealth if you cut down the trees, sell them, and put the money into a savings account paying e^{rt}

COMPARATIVE STATICS:

If interest rate rises: $r \to r'$

then the optimal cutting time falls: $t^* \to t'$

JJI

REVIEW: WHEN TO USE THE IMPLICIT FUNCTION THEOREM (JACOBIAN) ??

GENERAL FORM

$$\max\ U(x,y) + \lambda(B - P_x X + P_y Y)$$

F.O.C.

$\mathcal{L}_x: U_x - \lambda P_x = 0$ (Eq 1)

$\mathcal{L}_y: U_y - \lambda P_y = 0$ (Eq 2)

$\mathcal{L}_\lambda: B - P_x X - P_y Y = 0$ (Eq 3)

Equations 1, 2, and 3 IMPLICITLY Define

$x^* = x^*(B, P_x, P_y)$

$y^* = y^*(B, P_x, P_y)$

$\lambda^* = \lambda^*(B, P_x, P_y)$

S.O.C.

$$|\bar{H}| = \begin{vmatrix} 0 & -P_x & -P_y \\ -P_x & U_{xx} & U_{xy} \\ -P_y & U_{yx} & U_{yy} \end{vmatrix} > 0 \text{ by assumption}$$

Find $\frac{\partial X^*}{\partial P_x}$: Use implicit Function Theorem

SPECIFIC FORM

$$\max\ xy + \lambda(B - P_x X - P_y y)$$

F.O.C.

$\mathcal{L}_x: y - \lambda P_x = 0$ (Eq 1)

$\mathcal{L}_y: x - \lambda P_y = 0$ (Eq 2)

$\mathcal{L}_\lambda: B - P_x X - P_y Y = 0$ (Eq 3)

Equations 1, 2, and 3 EXPLICITLY Define

$x^* = \frac{B}{2P_x}$ $\quad y^* = \frac{B}{2P_y}$

$\lambda^* = \frac{B}{2P_x P_y}$

S.O.C.

$$|\bar{H}| = \begin{vmatrix} 0 & -P_x & -P_y \\ -P_x & 0 & 1 \\ -P_y & 1 & 0 \end{vmatrix} = 2P_x P_y > 0$$

Find: $\frac{\partial X^*}{\partial P_x}$: Differentiate X^* directly

$$\frac{\partial X^*}{\partial P_x} = -\frac{B}{2P_x^2} < 0$$