ch 11: OPTIMIZATION WITH MORE THAN ONE VARIABLE

I USING DIFFERENTIALS

given: $z = f(x, y)$

then
$$dz = f_x\,dx + f_y\,dy$$

if $dx \neq 0,\ dy \neq 0$

and $dz = 0$ {critical Point}

Then it must be true that

$$f_x = f_y = 0 \qquad \text{or} \left\{ \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0 \right\}$$

$f_x = 0$: means z is Not changing in the x-direction

$f_y = 0$: means z is not changing in the y-direction

This is the FIRST ORDER NECESSARY condition for a max. or min.

SECOND ORDER CONDITIONS

given $z = f(x, y)$

The first derivative (differential) is

$$\Rightarrow \quad dz = f_x\,dx + f_y\,dy$$

TAKE THE TOTAL DIFFERENTIAL A SECOND TIME, TREATING dx and dy as CONSTANTS

$$d^2z = f_{xx}\,dx\,dx + f_{yy}\,dy\,dy + f_{xy}\,dx\,dy + f_{yx}\,dy\,dx$$

$$d^2z = f_{xx}\,dx^2 + f_{yy}\,dy^2 + f_{xy}\,dx\,dy + f_{yx}\,dy\,dx$$

where $f_{xx} \equiv$ 2nd Partial derivative w.r.t. x,

$f_{yy} \equiv$ 2nd Partial derivative w.r.t. y

CROSS PARTIAL DERIV. $\left\{ \begin{array}{l} f_{xy} \equiv \text{CHANGE IN } \left(\frac{\partial z}{\partial x}\right) \text{ from a } \Delta \text{ in } y \\ f_{yx} \equiv \text{CHANGE IN } \left(\frac{\partial z}{\partial y}\right) \text{ from a } \Delta \text{ in } x \end{array} \right.$

YOUNG'S THEOREM

→ FOR CROSS PARTIAL "EFFECTS" the ORDER of DIFFERENTIATION is IMMATERIAL

Therefore:

$$f_{xy} = f_{yx}$$

As long as the cross partials are continuous.

⇒ In the case of GENERAL FUNCTIONS this will always be assumed to be TRUE!

EXAMPLE: $z = x^3 + 5xy - y^2$

$f_x = 3x^2 + 5y$	$f_y = 5x - 2y$
$\Rightarrow f_{xx} = 6X$	$\Rightarrow f_{yy} = -2$
$\Rightarrow f_{xy} = 5$	$\Rightarrow f_{yx} = 5$

$$f_{xy} = f_{yx}$$

QUADRATIC FORM

The function

$$q = ax^2 + 2bxy + cy^2$$

is a quadratic form

A quadratic can be written in matrix form as:

$$\underset{(1\times 1)}{q} = \underset{(1\times 2)}{\begin{bmatrix} x & y \end{bmatrix}} \underset{(2\times 2)}{\begin{bmatrix} a & b \\ b & c \end{bmatrix}} \underset{(2\times 1)}{\begin{bmatrix} x \\ y \end{bmatrix}} \qquad \text{Det} = (ac - b^2)$$

POSITIVE + NEGATIVE DEFINITENESS

q is said to be

$$\left.\begin{array}{l} \text{i, Positive definite} \\ \text{ii, Positive semi-definite} \\ \text{iii, Negative semi-definite} \\ \text{iv, Negative definite} \end{array}\right\} \text{ if } q \text{ is always } \left\{\begin{array}{l} >0 \\ \geq 0 \\ \leq 0 \\ <0 \end{array}\right.$$

{for all x, y}

$$q \text{ is } \left\{\begin{array}{l} \text{POSITIVE DEFINITE} \\ \text{NEGATIVE DEFINITE} \end{array}\right\} \text{ if } \left\{\begin{array}{l} a > 0 \\ a < 0 \end{array}\right\} \text{ and } (ac - b^2) > 0$$

The 2nd order total differential

$$d^2z = (f_{xx})dx^2 + (2f_{xy})dx\,dy + (f_{yy})dy^2 \quad \{f_{xy} = f_{yx}\}$$

Young's Theorm

is a quadratic form

Therefore

$$d^2z = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

The matrix of 2nd partial derivatives is called the HESSIAN

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \quad \text{where } |H| = f_{xx}f_{yy} - f_{xy}^2$$

SECOND ORDER CONDITIONS

$$d^2 z \text{ is } \begin{Bmatrix} \text{POSITIVE DEFINITE} \\ \text{NEGATIVE DEFINITE} \end{Bmatrix}$$

$$\text{iff } \begin{Bmatrix} f_{xx} > 0 \\ f_{xx} < 0 \end{Bmatrix} \text{ and } \left\{ f_{xx} f_{yy} - f_{xy}^2 > 0 \right\}$$

Note: $f_{xx} f_{yy} - f_{xy}^2 > 0$

implies that f_{yy} must have the same sign as f_{xx}

<u>Therefore if</u> (1): $f_x = f_y = 0$ (F.O.C.)

<u>and if</u> (2): (S.O.C.)

$$d^2 z \text{ is } \begin{Bmatrix} \text{Negative definite} \\ \text{Positive definite} \end{Bmatrix} \text{ then } z \text{ is } \begin{Bmatrix} \text{a Maximum} \\ \text{a Minimum} \end{Bmatrix}$$

n - VARIABLE CASE

given $z = f(x_1, x_2, \cdots, x_n)$

For an Extremum (Max or Min):

$$f_1 = f_2 = f_3 = \cdots = f_n = 0 \quad \text{(First Order Conditions)}$$

Then d^2z in Matrix form is

$$\underset{(1 \times n)}{[dx_1 \; dx_2 \cdots dx_n]} \underset{(n \times n)}{\begin{bmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & \cdots & f_{2n} \\ f_{31} & \cdots & f_{33} & \cdots & \vdots \\ \vdots & & & \ddots & \vdots \\ f_{n1} & \cdots & \cdots & \cdots & f_{nn} \end{bmatrix}} \underset{(n \times 1)}{\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \\ \vdots \\ dx_n \end{bmatrix}}$$

⇒ <u>For a Max</u>: (Principal minors alternate signs)

$$|H_1| = f_{11} < 0, \; |H_2| = f_{11}f_{22} - f_{12}^2 > 0, \; |H_3| < 0, \cdots (-1)^n |H_n| > 0$$

⇒ <u>For a Min</u>: (Principal minors have the same sign)

$$|H_1| > 0 \quad |H_2| > 0, \cdots, |H_n| > 0$$

EXAMPLE: Output Maximization

Let

$$Q = 10L + 10K + LK - L^2 - K^2$$

F. O. C.

$$\Rightarrow \frac{\partial Q}{\partial L} = 10 + K - 2L = 0$$

$$\Rightarrow \frac{\partial Q}{\partial K} = 10 + L - 2K = 0$$

or

$$\left\{ \begin{array}{l} 2L - K = 10 \\ -L + 2K = 10 \end{array} \right\}$$

2 Equations with 2 unknowns from F. O. C.

Matrix form:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} L \\ K \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \quad \{ Det = 3 \}$$

CRAMER's Rule

$$L = \frac{\begin{vmatrix} 10 & -1 \\ 10 & 2 \end{vmatrix}}{3} = \frac{20+10}{3} = \underline{\underline{10}}$$

$$K = \frac{\begin{vmatrix} 2 & 10 \\ -1 & 10 \end{vmatrix}}{3} = \frac{20+10}{3} = \underline{\underline{10}}$$

EXAMPLE (cont...)

NOW CHECK 2nd ORDER CONDITIONS

from F.O.C.

$$10 + K - 2L \equiv 0$$
$$10 + L - 2K \equiv 0$$

when $K, L = 10$ F.O.C's are identities

S.O.C.

$$dK - 2dL \equiv 0$$
$$dL - 2dK \equiv 0$$

Find the HESSIAN

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} dL \\ dK \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$H = \begin{bmatrix} Q_{LL} & Q_{LK} \\ Q_{KL} & Q_{KK} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$Q_{LL} = -2 < 0 \qquad Q_{LL}Q_{KK} - Q_{KL}^2 = (-2)(-2) - (1)^2 > 0$$

THEREFORE Q is Max @ $K=10, L=10$

ECONOMIC INTERPRETATION OF the 2ND ORDER CONDITIONS

Given the production function

$$Q = Q(K, L)$$

$Q_L > 0$, $Q_{LL} < 0$ "implies the "Law of diminishing Returns"

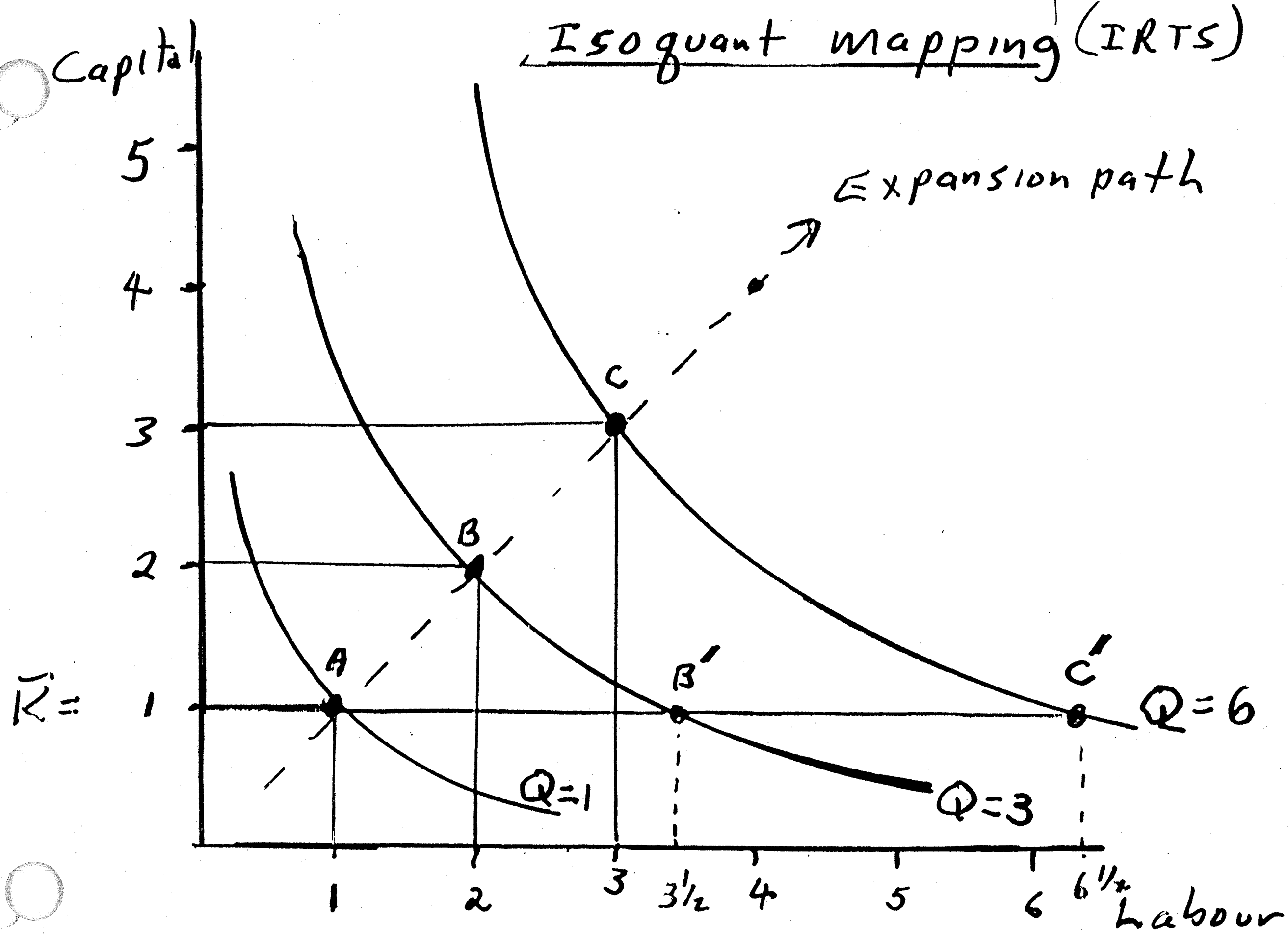
The condition

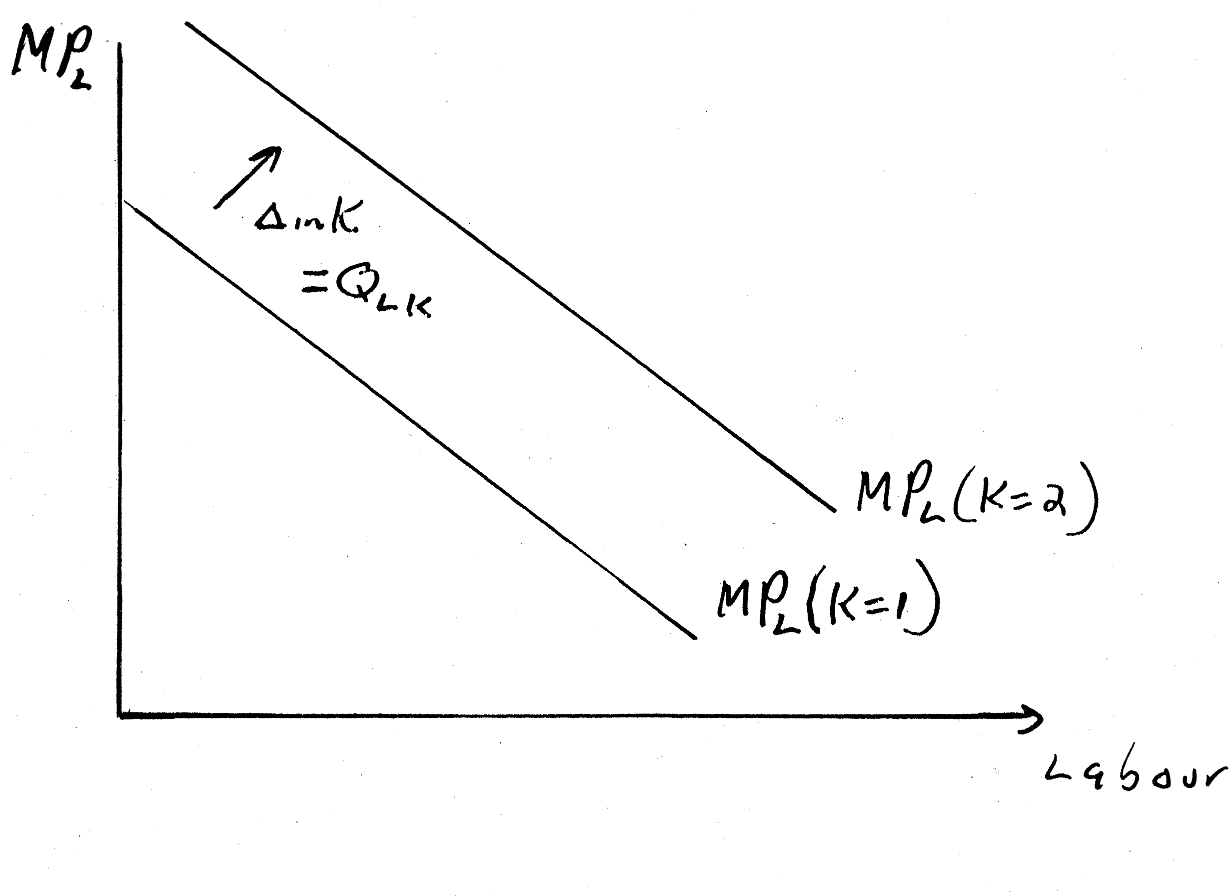
$$Q_{LL} Q_{KK} - Q_{KL}^2 > 0$$

⇒ says that for a Maximum the direct effects (Q_{LL}, Q_{KK}) must outweigh the indirect effects (Q_{KL}, Q_{LK})

⇒ A production function can have the properties of "the Law of diminishing Returns" and "increasing returns to Scale" at the same time

⇒ Therefore $Q(K, L)$ has no unconstrained maximum


Isoquant mapping (IRTS)
Capital
5
4
3
2
$\bar{K}$ = 1
Expansion path
C
B
A
B'
C'
Q=1
Q=3
Q=6
1
2
3
3½
4
5
6
6¼
Labour


MP_L
Δ in K
$= Q_{LK}$
$MP_L(K=2)$
$MP_L(K=1)$
Labour

PROFIT MAXIMIZATION & COMPARATIVE STATICS

Let $q = f(x_1, x_2)$ be the production function

The Profit function is

$$\pi = Pf(x_1, x_2) - w_1 x_1 - w_2 x_2$$

where $P \equiv$ output Price $w_1, w_2 \equiv$ factor prices of x_1 and x_2 respectively

The F.O.C's for profit Max.

$$\left.\begin{aligned} \frac{\partial \pi}{\partial x_1} &= Pf_1 - w_1 = 0 \\ \frac{\partial \pi_2}{\partial x_2} &= Pf_2 - w_2 = 0 \end{aligned}\right\}$$

This gives us two equations and two unknowns, x_1^*, x_2^*

Solving the F.O.C's we get:

$$X_1^* = X_1^*(w_1, w_2, P) \qquad X_2^* = X_2^*(w_1, w_2, P)$$

{ X_1^*, X_2^* as functions of Exogenous Variables }

2nd order Conditions

from the F.O.C's

$$\left.\begin{array}{l} p f_1 - w_1 \equiv 0 \\ p f_2 - w_2 \equiv 0 \end{array}\right\} \text{ at } x_1 = x_1^*,\ x_2 = x_2^*$$

differentiate again for the Hessian

$$\begin{bmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{bmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$H = \begin{bmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{bmatrix} = p \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

FOR MAXIMUM

$$|H_1| = f_{11} < 0 \qquad |H_2| = f_{11} f_{22} - f_{12}^2 > 0$$

Comparative Statics

by substituting

$$X_1^* = X_1^*(\omega_1, \omega_2, P), \quad X_2^* = X_2^*(\omega_1, \omega_2, P)$$

back into the F.O.C's

$$\Rightarrow \quad P f_1\left(X_1^*(\omega_1, \omega_2, P), X_2^*(\omega_1, \omega_2, P)\right) - \omega_1 \equiv 0$$

$$\Rightarrow \quad P f_2\left(X_1^*(\omega_1, \omega_2, P), X_2^*(\omega_1, \omega_2, P)\right) - \omega_2 \equiv 0$$

$\Rightarrow$ The FOC's become identities that implicitly define X_1, X_2 as functions of ω_1, ω_2, and P

$\Rightarrow$ Therefore to find $\underline{\frac{\partial X_1^*}{\partial \omega_1}}$, $\underline{\frac{\partial X_2^*}{\partial \omega_1}}$ etc

we can use the <u>implicit function</u> theorem by finding the <u>Jacobian</u> of the F.O.Cs

Find: $\frac{\partial x_1^*}{\partial w_1}$, $\frac{\partial x_2^*}{\partial w_1}$

Totally differentiate w.r.t w_1

$$pf_{11}\frac{\partial x_1^*}{\partial w_1} + pf_{12}\frac{\partial x_2^*}{\partial w_1} - \frac{dw_1}{dw_1} \equiv 0 \qquad \left\{ \frac{dw_1}{\partial w_1} = 1 \right\}$$

$$pf_{21}\frac{\partial x_1^*}{\partial w_1} + pf_{22}\frac{\partial x_2^*}{\partial w_1} \equiv 0$$

MATRIX FORM:

$$\begin{bmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1^*}{\partial w_1} \\ \frac{\partial x_2^*}{\partial w_1} \end{bmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The Jacobian determinant

$$|J| = p\left(f_{11}f_{22} - f_{12}^2\right) > 0$$

The Jacobian of the F.O.C's is also the Hessian of the S.O.C's

Solving by Cramer's Rule

$$\Rightarrow \frac{\partial x_1^*}{\partial w_1} = \frac{\begin{vmatrix} 1 & Pf_{12} \\ 0 & Pf_{22} \end{vmatrix}}{|H|} = \frac{Pf_{22}}{P(f_{11}f_{22}-f_{12}^2)} = \frac{f_{22}}{f_{11}f_{22}-f_{12}^2} \quad \underline{\underline{<0}}$$

$$\Rightarrow \frac{\partial x_2^*}{\partial w_1} = \frac{\begin{vmatrix} Pf_{11} & 1 \\ Pf_{21} & 0 \end{vmatrix}}{|H|} = \frac{-f_{21}}{f_{11}f_{22}-f_{12}^2} \gtreqless 0 \ ?$$

$\Rightarrow \frac{\partial x_1^*}{\partial w_1} < 0$ implies downward sloping factor demand curve

$\Rightarrow \frac{\partial x_2^*}{\partial w_1}$ this sign depends on the relationship in production between x_1 and x_2

EXAMPLE: PROFIT MAXIMIZATION

⇒ Suppose we have the following Production function:

⇒ $$q = f(K,L) = L^{1/2} + K^{1/2}$$

$$\left\{ \begin{array}{l} q \equiv \text{output} \\ L \equiv \text{Labour} \\ K \equiv \text{Capital} \end{array} \right\}$$

⇒ Then the PROFIT FUNCTION for a Competitive firm is

$$\pi = Pq - wL - rK$$

{where $P \equiv$ Market Price, $w \equiv$ Wage rate, $r \equiv$ rental rate}

or

$$\pi = PL^{1/2} + PK^{1/2} - wL - rK$$

⇒ First order conditions

general form

(1) $\frac{\partial \pi}{\partial L} = \frac{P}{2} L^{-1/2} - w = 0$ $\qquad \{Pf_L - w = 0\}$

(2) $\frac{\partial \pi}{\partial K} = \frac{P}{2} K^{-1/2} - r = 0$ $\qquad \{Pf_K - r = 0\}$

Solving (1) and (2) we get

$$L^* = \left(\frac{2w}{P}\right)^{-2} \qquad K^* = \left(\frac{2r}{P}\right)^{-2}$$

Profit Max Example (Cont...)

Second order Conditions (HESSIAN)

diff. F.O.C.'s w.r.t. K, L

General

$$Pf_{LL}\,dL + Pf_{LK}\,dK = 0$$

$$Pf_{KL}\,dL + Pf_{KK}\,dK = 0$$

Hessian

$$\begin{pmatrix} Pf_{LL} & Pf_{LK} \\ Pf_{KL} & Pf_{KK} \end{pmatrix}\begin{pmatrix} dL \\ dK \end{pmatrix}$$

$$|H_1| = Pf_{LL} < 0$$

$$|H_2|\; P[f_{LL}f_{KK} - (f_{LK})^2] > 0$$

Specific

$$-\frac{P}{4}L^{-3/2}\,dL + (0)\,dK = 0$$

$$-P/4\,K^{-3/2}\,dK + (0)\,dL = 0$$

Hessian

$$\begin{pmatrix} -\frac{P}{4}L^{-3/2} & 0 \\ 0 & -\frac{P}{4}K^{-3/2} \end{pmatrix}\begin{pmatrix} dL \\ dK \end{pmatrix}$$

$$H_1 = -P/4\,L^{-3/2}$$

$$|H_2| = \left(-\frac{P}{4}L^{-3/2}\right)\left(-\frac{P}{4}K^{-3/2}\right) - 0 > 0$$

$\Downarrow$

Therefore Profit Max

Profit Max Example (cont...)

from the F.O.C.s we know:

$$L^* = \left(\frac{2w}{p}\right)^{-2} \qquad K^* = \left(\frac{2r}{p}\right)^{-2}$$

by subbing K^* and L^* into the profit function we get:

$$\Rightarrow \pi^* = pL^{1/2} + pK^{1/2} - wL - rK$$

$$\Rightarrow \pi^* = p\left[\left(\frac{2w}{p}\right)^{-2}\right]^{1/2} + p\left[\left(\frac{2r}{p}\right)^{-2}\right]^{1/2} - w\left(\frac{2w}{p}\right)^{-2} - r\left(\frac{2r}{p}\right)^{-2}$$

$$\Rightarrow \pi^* = \frac{p^2}{2w} + \frac{p^2}{2r} - \frac{p^2}{4w} - \frac{p^2}{4r}$$

Finally:

$$\Rightarrow \pi^* = \pi^*(w, r, p) = \frac{p^2}{4w} + \frac{p^2}{4r}$$

where $\pi^*(w, r, p)$ is "Maximum profits as a function of w, r, and p"

Profit Max Example (cont...)

Hotelling's Lemma

Hotelling's Lemma states the following conditions about the profit function:

① $$\frac{\partial \pi^*(w,r,p)}{\partial p} = q^*$$

②a $$-\frac{\partial \pi^*(w,r,p)}{\partial w} = L^*$$ ②b $$-\frac{\partial \pi^*(w,r,p)}{\partial r} = K^*$$

Using our profit function: $\pi^*(w,r,p) = \frac{p^2}{4w} + \frac{p^2}{4r}$

condition 1:

$$\frac{\partial \pi^*}{\partial p} = \frac{2p}{4w} + \frac{2p}{4r} = \frac{p}{2w} + \frac{p}{2r}$$

check

$$q = L^{1/2} + K^{1/2} = \left[\left(\frac{2w}{p}\right)^{-2}\right]^{1/2} + \left[\left(\frac{2r}{p}\right)^{-2}\right]^{1/2}$$

$$= \left(\frac{2w}{p}\right)^{-1} + \left(\frac{2r}{p}\right)^{-1} = \frac{p}{2w} + \frac{p}{2r}$$

<u>Condition 2a</u>

$$-\frac{\partial \pi^*(w,r,p)}{\partial w} = -\frac{\partial}{\partial w}\left[\frac{p^2}{4w} + \frac{p^2}{4r}\right] = -\left(-\frac{p^2}{4w^2}\right) = \left(\frac{2w}{p}\right)^{-2}$$

<u>Therefore</u> $-\frac{\partial \pi^*}{\partial w} = L^*$

<u>Condition 2b</u>

$$-\frac{\partial \pi^*(w,r,p)}{\partial r} = -\left[-\frac{p^2}{4r^2}\right] = \left(\frac{2r}{p}\right)^{-2} = K^*$$

<u>Factor Demand Curves</u>

L^* and K^* are the firms demand curves for labour and capital

$$\Rightarrow L^* = \frac{p^2}{4w^2} \Rightarrow \frac{\partial L^*}{\partial w} = -\frac{p^2}{2\cancel{8}w^3} < 0$$

$$\Rightarrow K^* = \frac{p^2}{4r^2} \Rightarrow \frac{\partial K^*}{\partial r} = -\frac{p^2}{2\cancel{8}r^3} < 0$$

<u>Therefore:</u> Downward sloping factor demand curves

Iso-Profit Curves (Level curves)

Take Total differential of $\pi^*(w,r,p)$; let $d\pi^* = 0$

$$d\pi^* = -\frac{p^2}{4w^2}\,dw + -\frac{p^2}{4r^2}\,dr = 0$$

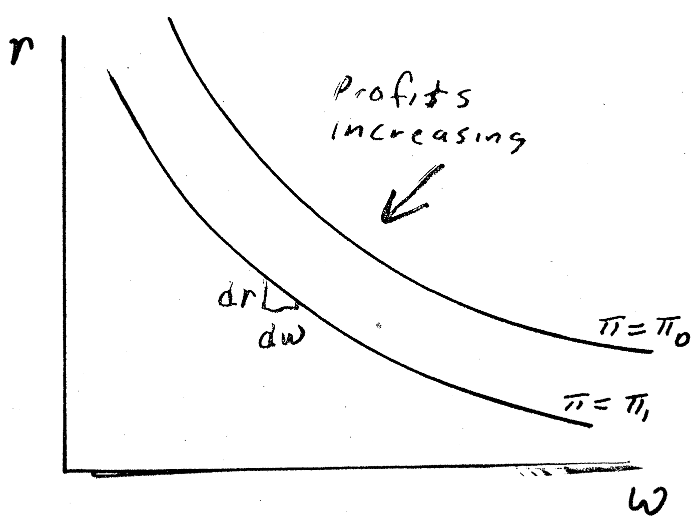
$$\frac{dr}{dw} = -\frac{p^2/4w^2}{p^2/4r^2} = -\frac{r^2}{w^2} < 0 \text{ (slope of Iso Profit Curve)}$$

Concave or Convex ?

$$\frac{d}{dw}\left(\frac{dr}{dw}\right) = -\left(-2\frac{r^2}{w^3}\right) = 2\frac{r^2}{w^3} > 0$$

Therefore the slope of the iso-profit curve is negative (dr/dw) but the slope is becomming less negative: $\left(\frac{d^2r}{dw^2}\right) > 0$

Therefore: Convex



PROFIT MAXIMIZATION

Developing the profit function

$$\pi = TR - TC$$

Where

$$\pi = PQ - C(Q)$$

Therefore π Max is

$$\frac{\partial \pi}{\partial Q} = \frac{\partial TR}{\partial Q} - \frac{\partial C}{\partial Q} = MR - MC = 0$$

Q is the choice variable

Now suppose $Q = f(K, L)$

then

$$\pi = P \cdot f(K, L) - (wL + rK)$$

where $TC = wL + rK$

Now profit Max is

① $\frac{\partial \pi}{\partial L} = p f_L - w = 0$

② $\frac{\partial \pi}{\partial K} = p f_K - r = 0$

Now K, L are the choice variables

The solution

$$\left\{ \begin{array}{l} K^* = K^*(w, r, p) \\ L^* = L^*(w, r, p) \end{array} \right\}$$

are demand curves

Now suppose the firm is a monopolist

Then he faces a downward sloping demand curve

$$P = D(Q)$$

Profit function is

$$\pi = D(Q)Q - wL - rK$$

where $Q = f(K,L)$

differentiate, using the chain role

F.O.C.

$$\pi_L : [D(Q) + QD'(Q)]f_L - w = 0$$

$$\pi_K : [D(Q) + QD'(Q)]f_K - r = 0$$

or

$$MR \cdot MP_L - w = 0$$

$$MR \cdot MP_K - r = 0$$

or finally

$$[D(f(K,L)) + f(K,L)D'(f(K,L))]f_L - w = 0$$

$$[D(f(K,L)) + f(K,L)D'(f(K,L))]f_K - r = 0$$

giving $K^* = K^*(w,r)$ $\quad L^* = L^*(w,r)$