OPMT 5701 Lecture Notes

1 The Chain Rule

Of all the basic rules of derivatives, the most challenging one is the chain rule. However, like the other rules, if you break it down to simple steps, it too is quite manageble. There are a couple of approaches to learning the chain rule. Both are equally good, it just comes down to preference.

The good news is that once you have mastered the chain rule – combined with the first five – you are ready to tackle about 90% of the calculus problems found in business courses (including graduate programs like the MBA!)

1.1 The Rule:

Suppose y is a "nested" function of x, where nested mean "a function inside a function".

$$y = f\left[g(x)\right]$$

where $g(x)$ is nested inside f. then the derivative is

$$\frac{dy}{dx} = y' = f'\left[g(x)\right] \times g'(x) \tag{1}$$

The process is to start with the outside function, taking the derivative of f but leaving $g(x)$ inside unchanged. Then find $g'$and multiply it by f'.

For example, if

$$y = \left(x^2 + 1\right)^3$$

then $f = (\quad)^3$ and $g = x^2 + 1$. From the power function rule, we know that the derivative of x^3 is $3x^2$. This is true for anything cubed! So

$$f' = 3\left(\quad\right)^2$$

(and $g' = 2x$)Using equation 1, we get

$$y' = f'\left[g(x)\right] \times g'(x) = 3(x^2+1)^2 \cdot (2x)$$

1.2 Some Examples

1. <u>**Example**</u>: if

$$y = (x^3 + x)^{1/2}$$

where $f = (\quad)^{1/2}$, then

$$y' = f'\left[g(x)\right] \times g'(x) = \frac{1}{2}\left(x^3 + x\right)^{-1/2}\left(3x^2 + 1\right)$$

2. **Example**: suppose

$$y = \frac{1}{\sqrt{3x+7}}$$

For this one, we need to re-write this to look more like a "power-function" problem. (Remember the basics: $\sqrt{x} = x^{1/2}$ and $\frac{1}{x} = x^{-1}$) therefore

$$y = (3x+7)^{-1/2}$$

Here $f = (\quad)^{-1/2}$. apply equation 1

$$y' = f'\left[g(x)\right] \times g'(x) = -\frac{1}{2}\left(3x+7\right)^{-3/2}(3)$$

1.3 Combining the Chain Rule with Product and Quotient Rule

The chain rule, or "function in a function" rule: if $y = f(g(x))$ then $y' = f'(g(x)) \times g'(x)$

It is usually in the form like this

$$y = \overbrace{[g(x)]^n}^{f(x)}$$

where the derivative is

$$y' = \frac{dy}{dx} = \overbrace{n\left[g(x)\right]^{n-1}}^{f'(x)} \times g'(x)$$

Sometimes the inside function can be a bit more complex. For example

$$y = \left[h(x)j(x)\right]^n$$

by breaking it down, we see that

$$y = \overbrace{\left[\underbrace{h(x) \cdot j(x)}_{g(x)}\right]^n}^{f(x)}$$

In this case the inside function, $g(x)$ is two functions, $h(x)$ and $j(x)$. The steps are the same as before except, in this case, we need to use the product rule on $g(x)$ $(g' = h'j + hj')$

1. **Example**: let y be

$$y = \left[(2x+1)(x^{1/2}-4)\right]^3$$

First, Identify the individual parts:

$$y = \overbrace{\left[\underbrace{\overset{h}{(2x+1)}\overset{j}{(x^{1/2}-4)}}_{g}\right]^3}^{f}$$

The derivative of f (power-function rule) is

$$f' = 3\left[(2x+1)(x^{1/2}-4)\right]^2$$

and the derivative of g (product rule) is

$$\begin{aligned} g' &= h'j + hj' = (2)(x^{1/2}-4) + (2x+1)\left(\tfrac{1}{2}x^{-1/2}\right) \\ \text{(simplify)} &= 2x^{1/2} - 8 + x^{1/2} + \tfrac{1}{2}x^{-1/2} \\ &= x^{1/2} + \tfrac{1}{2}x^{-1/2} - 8 \end{aligned}$$

Now put it all together

$$y' = f' \cdot g' = 3\left[(2x+1)(x^{1/2}-4)\right]^2 \left(x^{1/2} + \tfrac{1}{2}x^{-1/2} - 8\right)$$

2. **Example**: suppose

$$y = \left(\frac{6x+1}{x^3}\right)^2$$

Here this one is a little more complicated. The outside function (f) is

$$f = (\quad)^2$$

but the nested function (g) is

$$g(x) = \frac{6x+1}{x^3}$$

.which will require the quotient rule rule. This one should be done in parts seperately then subsitute into equation 1. First

$$\begin{aligned} g' &= \frac{(6)(x^3) - (2x+1)(3x^2)}{(x^3)^2} \\ &= \frac{6x^3 - 6x^3 - 3x^2}{x^6} \\ &= \frac{-3x^2}{x^6} = -3x^{-4} \end{aligned}$$

remember that $f = (\quad)^2$ and $f' = 2(\quad)$, we can use equation 1

$$y' = f'\left[g(x)\right] \times g'(x) = 2\overbrace{\left(\frac{6x+1}{x^3}\right)}^{f'} \cdot \overbrace{\left(-3x^{-4}\right)}^{g'}$$