OPMT 5701
Two Variable Optimization

Using Calculus For Maximization Problems

One Variable Case

If we have the following function

$$y = 10x - x^2$$

we have an example of a *dome shaped* function. To find the maximum of the dome, we simply need to find the point where the slope of the dome is zero, or

$$\frac{dy}{dx} = 10 - 2x = 0$$
$$10 = 2x$$
$$x = 5$$
$$and$$
$$y = 25$$

Two Variable Case

Suppose we want to maximize the following function

$$z = f(x, y) = 10x + 10y + xy - x^2 - y^2$$

Note that there are two unknowns that must be solved for: x and y. This function is an example of a *three-dimensional dome.* (i.e. the roof of *BC Place*)

To solve this maximization problem we use **partial derivatives.** We take a partial derivative for each of the unknown choice variables and set them equal to zero

$$\frac{\partial z}{\partial x} = f_x = 10 + y - 2x = 0 \quad \textit{The slope in the "x" direction = 0}$$
$$\frac{\partial z}{\partial y} = f_y = 10 + x - 2y = 0 \quad \textit{The slope in the "y" direction = 0}$$

This gives us a set of equations, one equation for each of the unknown variables. When you have the same number of independent equations as unknowns, you can solve for each of the unknowns.

rewrite each equation as

$$y = 2x - 10$$

$$x = 2y - 10$$

substitute one into the other

$$x = 2(2x - 10) - 10$$

$$x = 4x - 30$$

$$3x = 30$$

$$x = 10$$

similarly,

$$y = 10$$

REMEMBER: To maximize (minimize) a function of many variables you use the technique of partial differentiation. This produces a set of equations, one equation for each of the unknowns. You then solve the set of equations simulaneously to derive solutions for each of the unknowns.

Second order Conditions (second derivative Test)

To test for a maximum or minimum we need to check the second partial derivatives. Since we have two first partial derivative equations (f_x, f_y) and two variable in each equation, we will get four *second partials* $(f_{xx}, f_{yy}, f_{xy}, f_{yx})$

Using our original first order equations and taking the partial derivatives for each of them (a second time) yields:

$$f_x = 10 + y - 2x = 0 \quad f_y = 10 + x - 2y = 0$$

$$f_{xx} = -2 \qquad\qquad f_{yy} = -2$$
$$f_{xy} = 1 \qquad\qquad f_{yx} = 1$$

The two partials, f_{xx}, and f_{yy} are the direct effects of of a small change in x and y on the respective slopes in in the x and y direction. The partials, f_{xy} and f_{yx} are the indirect effects, or the cross effects of one variable on the slope in the other variable's direction. For both *Maximums and Minimums*, the direct effects must outweigh the cross effects

Rules for two variable Maximums and Minimums

1. Maximum

$$f_{xx} \quad < \quad 0$$
$$f_{yy} \quad < \quad 0$$
$$f_{yy}f_{xx} - f_{xy}f_{yx} \quad > \quad 0$$

2. Minimum

$$f_{xx} \quad > \quad 0$$
$$f_{yy} \quad > \quad 0$$
$$f_{yy}f_{xx} - f_{xy}f_{yx} \quad > \quad 0$$

3. Otherwise, we have a *Saddle Point*

From our second order conditions, above,

$$f_{xx} = -2 < 0 \quad f_{yy} = -2 < 0$$
$$f_{xy} = 1 \qquad\quad f_{yx} = 1$$

and

$$f_{yy}f_{xx} - f_{xy}f_{yx} = (-2)(-2) - (1)(1) = 3 > 0$$

therefore we have a maximum.

Hessian Matrix of Second Partials:

Sometimes the Second Order Conditions are checked in matrix form, using a Hession Matrix. The Hessian is written as

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

where the determinant of the Hessian is

$$|H| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{yy}f_{xx} - f_{xy}f_{yx}$$

which is the measure of the direct versus indirect strengths of the second partials. This is a common setup for checking maximums and minimums, but it is not necessary to use the Hessian.

Example: Profit Maximization

A monopolist offers two different products, each having the following market demand functions

$$q_1 = 14 - \tfrac{1}{4}p_1$$
$$q_2 = 24 - \tfrac{1}{2}p_2$$

The monopolist's joint cost function is

$$C(q_1, q_2) = q_1^2 + 5q_1q_2 + q_2^2$$

The monopolist's profit function can be written as

$$\pi = p_1q_1 + p_2q_2 - C(q_1, q_2) = p_1q_1 + p_2q_2 - q_1^2 - 5q_1q_2 - q_2^2$$

which is the function of four variables: $p_1, p_2, q_1,$ and q_2. Using the market demand functions, we can eliminate p_1 and p_2 leaving us with a two variable maximization problem. First, rewrite the demand functions to get the inverse functions

$$p_1 = 56 - 4q_1$$
$$p_2 = 48 - 2q_2$$

Substitute the inverse functions into the profit function

$$\pi = (56 - 4q_1)q_1 + (48 - 2q_2)q_2 - q_1^2 - 5q_1q_2 - q_2^2$$

The first order conditions for profit maximization are

$$\tfrac{\partial \pi}{\partial q_1} = 56 - 10q_1 - 5q_2 = 0$$
$$\tfrac{\partial \pi}{\partial q_2} = 48 - 6q_2 - 5q_1 = 0$$

Solve the first order conditions using Cramer's rule. First, rewrite in matrix form

$$\begin{bmatrix} 10 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 56 \\ 48 \end{bmatrix}$$

where $|A| = 35$

$$q_1^* = \frac{\begin{vmatrix} 56 & 5 \\ 48 & 6 \end{vmatrix}}{35} = 2.75$$

$$q_2^* = \frac{\begin{vmatrix} 10 & 56 \\ 5 & 48 \end{vmatrix}}{35} = 5.7$$

Using the inverse demand functions to find the respective prices, we get

$$p_1^* = 56 - 4(2.75) = 45$$
$$p_2^* = 48 - 2(5.7) = 36.6$$

From the profit function, the maximum profit is

$$\pi = 213.94$$

Next, check the second order conditions to verify that the profit is at a maximum. The various second derivatives can be set up in a matrix called a *Hessian* The Hessian for this problem is

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -10 & -5 \\ -5 & -6 \end{bmatrix}$$

The sufficient conditions are

$$|H_1| = \pi_{11} = -10 < 0 \qquad \text{(First Principle Minor of Hessian)}$$
$$|H_2| = \pi_{11}\pi_{22} - \pi_{12}\pi_{21} = (-10)(-6) - (-5)^2 = 35 > 0 \qquad \text{(determinant)}$$

Therefore the function is at a maximum. Further, since the signs of $|H_1|$ and $|H_2|$ are invariant to the values of q_1 and q_2, we know that the profit function is strictly concave.

Example: Profit Max Capital and Labour

Suppose we have the following production function

$$q = f(K, L) = L^{\frac{1}{2}} + K^{\frac{1}{2}} \qquad \begin{aligned} q &= \text{ Output} \\ L &= \text{ Labour} \\ K &= \text{ Capital} \end{aligned}$$

Then the profit function for a competitive firm is

$$\begin{aligned} \pi &= Pq - wL - rK \\ &\text{or} \\ \pi &= PL^{\frac{1}{2}} + PK^{\frac{1}{2}} - wL - rK \end{aligned} \qquad \begin{aligned} P &= \text{ Market Price} \\ w &= \text{ Wage Rate} \\ r &= \text{ Rental Rate} \end{aligned}$$

First order conditions

$$\text{General Form}$$

1. $\frac{\partial \pi}{\partial L} = \frac{P}{2}L^{\frac{-1}{2}} - w = 0 \qquad Pf_L - w = 0$
2. $\frac{\partial \pi}{\partial k} = \frac{P}{2}K^{\frac{-1}{2}} - r = 0 \qquad Pf_K - r = 0$

Solving (1) and (2), we get

$$L^* = \left(\frac{2w}{P}\right)^{-2} \qquad K^* = \left(\frac{2r}{P}\right)^{-2}$$

Second order conditions (Hessian)

$$\pi_{LL} = Pf_{LL} = \frac{-P}{4}L^{\frac{-3}{2}} < 0$$
$$\pi_{KK} = Pf_{KK} = \frac{-P}{4}K^{\frac{-3}{2}} < 0$$
$$\pi_{LK} = \pi_{KL} = Pf_{LK} = Pf_{KL} = 0$$

or, in matrix form

$$H = \begin{vmatrix} \pi_{LL} & \pi_{LK} \\ \pi_{KL} & \pi_{KK} \end{vmatrix} = \begin{vmatrix} \frac{-P}{4}L^{\frac{-3}{2}} & 0 \\ 0 & \frac{-P}{4}K^{\frac{-3}{2}} \end{vmatrix}$$

$$P\left[f_{LL}f_{KK} - (f_{LK})^2\right] = \left(\frac{-P}{4}L^{\frac{-3}{2}}\right)\left(\frac{-P}{4}K^{\frac{-3}{2}}\right) - 0 > 0$$

Differentiate first order of conditions with respect to capital (K) and labour (L)
$\implies$Therefore profit maximization

Example: If $P = 1000$, $w = 20$, and $r = 10$

1. Find the optimal K, L, and π

2. Check second order conditions

Example: Cobb-Douglas production function and a competitive firm

Consider a competitive firm with the following profit function

$$\pi = TR - TC = PQ - wL - rK$$

where P is price, Q is output, L is labour and K is capital, and w and r are the input prices for L and K respectively. Since the firm operates in a competitive market, the exogenous variables are P,w and r. There are three endogenous variables, K, L and Q. However output, Q, is in turn a function of K and L via the production function

$$Q = f(K, L)$$

which in this case, is the Cobb-Douglas function

$$Q = L^a K^b$$

where a and b are positive parameters. If we further assume decreasing returns to scale, then a + b < 1. For simplicity, let's consider the symmetric case where $a = b = \frac{1}{4}$

$$Q = L^{\frac{1}{4}} K^{\frac{1}{4}}$$

Substituting Equation 3 into Equation 1 gives us

$$\pi(K, L) = PL^{\frac{1}{4}} K^{\frac{1}{4}} - wL - rK$$

The first order conditions are

$$\frac{\partial \pi}{\partial L} = P\left(\frac{1}{4}\right) L^{-\frac{3}{4}} K^{\frac{1}{4}} - w = 0$$
$$\frac{\partial \pi}{\partial K} = P\left(\frac{1}{4}\right) L^{\frac{1}{4}} K^{-\frac{3}{4}} - r = 0$$

This system of equations define the optimal L and K for profit maximization. But first, we need to check the second order conditions to verify that we have a maximum.

The Hessian for this problem is

$$H = \begin{bmatrix} \pi_{LL} & \pi_{LK} \\ \pi_{KL} & \pi_{KK} \end{bmatrix} = \begin{bmatrix} P(-\frac{3}{16})L^{-\frac{7}{4}} K^{\frac{1}{4}} & P\left(\frac{1}{4}\right)^2 L^{-\frac{3}{4}} K^{-\frac{3}{4}} \\ P\left(\frac{1}{4}\right)^2 L^{-\frac{3}{4}} K^{-\frac{3}{4}} & P\left(-\frac{3}{16}\right) L^{\frac{1}{4}} K^{\frac{7}{4}} \end{bmatrix}$$

The sufficient conditions for a maximum are that $|H_1| < 0$ and $|H| > 0$. Therefore, the second order conditions are satisfied.

We can now return to the first order conditions to solve for the optimal K and L. Rewriting the first equation in Equation 5 to isolate K

$$P\left(\frac{1}{4}\right) L^{-\frac{3}{4}} K^{\frac{1}{4}} = w$$
$$K = \left(\frac{4w}{p} L^{\frac{3}{4}}\right)^4$$

Substituting into the second equation of Equation 5

$$\frac{P}{4} L^{\frac{1}{4}} K^{-\frac{3}{4}} = \left(\frac{P}{4}\right) L^{\frac{1}{4}} \left[\left(\frac{4w}{p} L^{\frac{3}{4}}\right)^4\right]^{-\frac{3}{4}} = r$$
$$= P^4 \left(\frac{1}{4}\right)^4 w^{-3} L^{-2} = r$$

Re-arranging to get L by itself gives us

$$L^* = \left(\frac{P}{4} w^{-\frac{3}{4}} r^{-\frac{1}{4}}\right)^2$$

Taking advantage of the symmetry of the model, we can quickly find the optimal K

$$K^* = \left(\frac{P}{4} r^{-\frac{3}{4}} w^{-\frac{1}{4}}\right)^2$$

L^* and K^* are the firm's factor demand equations.

Cournot Doupoly (Game Theory)

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Assumption: Each firm takes the other firms output as exogenous and chooses output to maximize its own profits.

Market Demand:
$$P = a - bq$$
or
$$P = a - b(q_1 + q_2) \quad (q_1 + q_2 = q)$$

Where q_i is firm i's output
$$i = 1, 2$$

Each firm faces the same cost function
$$TC = K + cq_i$$
$$(i = 1, 2)$$

Each firm's profit function is
$$\pi_i = Pq_i - cq_i - K$$

Firm 1
$$\pi_1 = Pq_1 - cq_1 - K$$
$$\pi_1 = (a - bq_1 - bq_2)q_1 - cq_1 - K$$

Max π_1, treating q_2 as constant

$\frac{\partial \pi}{\partial q_1} = a - bq_2 - 2bq_1 - c = 0$
$2bq_1 = a - c - bq_2$
$q_1 = \frac{a-c}{2b} - \frac{q_2}{2}$ $\qquad \Rightarrow$ "Best Response Function"

Best Response Function tells Firm 1 the profit maximizing q_1 for any level of q_2.

For Firm 2

$\pi_2 = (a - bq_1 - bq_2)q_2 - cq_2 - K$
Max π_2 $\qquad\qquad$ Treating q_1 as constant
$q_2 = \frac{a-c}{2b} - \frac{q_1}{2}$ $\qquad\qquad$ Firm 2's "Best Response Function"

The two "Best Response Functions"

$$\begin{array}{ll} \text{Firm 1} & q_1 = \frac{a-c}{2b} - \frac{q_2}{2} \\ \text{Firm 2} & q_2 = \frac{a-c}{2b} - \frac{q_1}{2} \end{array}$$

gives us two equations and two unknowns.

The solution to this system of equations is the equilibrium to the "Cournot Duopoly Game."

Using Cramers Rule

$$\begin{array}{lll} 1. & q_1^* = \frac{a-c}{3b} \\ 2. & q_2^* = \frac{a-c}{3b} \\ \text{Market Output} & q_1^* + q_2^* = \frac{2(a-c)}{3b} \end{array}$$

Review of Some Derivative Rules

1. Partial Derivative Rules:

$$U = xy \qquad\qquad \partial U/\partial x = y \qquad\qquad \partial U/\partial y = x$$
$$U = x^a y^b \qquad\qquad \partial U/\partial x = ax^{a-1}y^b \qquad \partial U/\partial y = bx^a y^{b-1}$$
$$U = x^a y^{-b} = \frac{x^a}{y^b} \qquad \partial U/\partial x = ax^{a-1}y^{-b} \qquad \partial U/\partial y = -bx^a y^{-b-1}$$
$$U = ax + by \qquad\qquad \partial U/\partial x = a \qquad\qquad \partial U/\partial y = b$$
$$U = ax^{1/2} + by^{1/2} \quad \partial U/\partial x = a\left(\tfrac{1}{2}\right)x^{-1/2} \quad \partial U/\partial y = b\left(\tfrac{1}{2}\right)y^{-1/2}$$

2. Logarithm (Natural log) $\ln x$

   (a) Rules of natural log

$$\begin{array}{ll} If & Then \\ y = AB & \ln y = \ln(AB) = \ln A + \ln B \\ y = A/B & \ln y = \ln A - \ln B \\ y = A^b & \ln y = \ln(A^b) = b\ln A \end{array}$$

$$\text{NOTE: } \ln(A + B) \neq \ln A + \ln B$$

   (b) derivatives

$$\begin{array}{ll} IF & THEN \\ y = \ln x & \frac{dy}{dx} = \frac{1}{x} \\ y = \ln(f(x)) & \frac{dy}{dx} = \frac{1}{f(x)} \cdot f'(x) \end{array}$$

   (c) Examples

$$\begin{array}{ll} If & Then \\ y = \ln(x^2 - 2x) & dy/dx = \frac{1}{(x^2 - 2x)}(2x - 2) \\ y = \ln(x^{1/2}) = \frac{1}{2}\ln x & dy/dx = \left(\frac{1}{2}\right)\left(\frac{1}{x}\right) = \frac{1}{2x} \end{array}$$

3. The Number e

$$\text{if } y = e^x \text{ then } \frac{dy}{dx} = e^x$$

$$\text{if } y = e^{f(x)} \text{ then } \frac{dy}{dx} = e^{f(x)} \cdot f'(x)$$

   (a) Examples

$$\begin{array}{ll} y = e^{3x} & \frac{dy}{dx} = e^{3x}(3) \\ y = e^{7x^3} & \frac{dy}{dx} = e^{7x^3}(21x^2) \\ y = e^{rt} & \frac{dy}{dt} = re^{rt} \end{array}$$

Finding the MRS from Utility functions

EXAMPLE: Find the total differential for the following utility functions

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1. $U(x_1, x_2) = ax_1 + bx_2$ where $(a, b > 0)$

2. $U(x_1, x_2) = x_1^2 + x_2^3 + x_1 x_2$

3. $U(x_1, x_2) = x_1^a x_2^b$ where $(a, b > 0)$

4. $U(x_1, x_2) = \alpha \ln c_1 + \beta \ln c_2$ where $(\alpha, \beta > 0)$

Answers:

1. $\frac{\partial U}{\partial x_1} = U_1 = a$ $\quad$ $\frac{\partial U}{\partial x_2} = U_2 = b$

and

$$dU = U_1 dx_1 + U_2 dx_2 = a dx_1 + b dx_2 = 0$$

If we rearrange to get dx_2/dx_1

$$\frac{dx_2}{dx_1} = -\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = -\frac{U_1}{U_2} = -\frac{a}{b}$$

The MRS is the Absolute value of $\frac{dx_2}{dx_1}$:

$$MRS = \frac{a}{b}$$

2. $\frac{\partial U}{\partial x_1} = U_1 = 2x_1 + x_2$ $\quad$ $\frac{\partial U}{\partial x_2} = U_2 = 3x_2^2 + x_1$

and

$$dU = U_1 dx_1 + U_2 dx_2 = (2x_1 + x_2) dx_1 + (3x_2^2 + x_1) dx_2 = 0$$

Find dx_2/dx_1

$$\frac{dx_2}{dx_1} = -\frac{U_1}{U_2} = -\frac{(2x_1 + x_2)}{(3x_2^2 + x_1)}$$

The MRS is the Absolute value of $\frac{dx_2}{dx_1}$:

$$MRS = \frac{(2x_1 + x_2)}{(3x_2^2 + x_1)}$$

iii) $\frac{\partial U}{\partial x_1} = U_1 = ax_1^{a-1} x_2^b$ $\qquad$ $\frac{\partial U}{\partial x_2} = U_2 = bx_1^a x_2^{b-1}$

and

$$dU = \left(ax_1^{a-1} x_2^b\right) dx_1 + \left(bx_1^a x_2^{b-1}\right) dx_2 = 0$$

Rearrange to get

$$\frac{dx_2}{dx_1} = -\frac{U_1}{U_2} = -\frac{ax_1^{a-1} x_2^b}{bx_1^a x_2^{b-1}} = -\frac{ax_2}{bx_1}$$

The MRS is the Absolute value of $\frac{dx_2}{dx_1}$:

$$MRS = \frac{ax_2}{bx_1}$$

iv) $\frac{\partial U}{\partial c_1} = U_1 = \alpha \left(\frac{1}{c_1}\right) dc_1 = \left(\frac{\alpha}{c_1}\right) dc_1 \qquad \frac{\partial U}{\partial x_2} = U_2 = \beta \left(\frac{1}{c_2}\right) dc_2 = \left(\frac{\beta}{c_2}\right) dc_2$

and

$$dU = \left(\frac{\alpha}{c_1}\right) dc_1 + \left(\frac{\beta}{c_2}\right) dc_2 = 0$$

Rearrange to get

$$\frac{dc_2}{dc_1} = -\frac{U_1}{U_2} = \frac{\left(\frac{\alpha}{c_1}\right)}{\left(\frac{\beta}{c_2}\right)} = -\frac{\alpha c_2}{\beta c_1}$$

The MRS is the Absolute value of $\frac{dc_2}{dc_1}$:

$$MRS = \frac{\alpha c_2}{\beta c_1}$$