Envelope Theorem

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1 Maximum Value Functions

A maximum (or minimum) value function is an objective function where the choice variables have been assigned their optimal values. These optimal values of the choice variables are, in turn, functions of the exogenous variables and parameters of the problem. Once the optimal values of the choice variables have been substituted into the original objective function, the function indirectly becomes a function of the parameters (through the parameters' influence on the optmal values of the choice variables). Thus the maximum value function is also referred to as the indirect objective function.

What is the significance of the indirect objective function? Consider that in any optimization problem the direct objective function is maximized (or minimized) for a given set of parameters. The indirect objective function gives all the maximum values of the objective function as these prameters vary. Hence the indirect objective function is an "envelope" of the set of optimized objective functions generated by varying the parameters of the model. For most students of economics the first illustration of this notion of an "envelope" arises in the comparison of short-run and long-run cost curves. Students are typically taught that the long-run average cost curve is an envelope of all the short-run average cost curves (what parameter is varying along the envelope in this case?). A formal derivation of this concept is one of the exercises we will be considering in the following sections.

To illustrate, consider the following maximization problem with two choice variables x and y, and one parameter, α:

Maximize

$$U = f(x, y, \alpha) \tag{1}$$

The first order necessary condition are

$$f_x(x, y, \alpha) = f_y(x, y, \alpha) = 0 \tag{2}$$

if second-order conditions are met, these two equations implicitly define the solutions

$$x = x^*(\alpha) \quad y = x^*(\alpha) \tag{3}$$

If we subtitute these solutions into the objective function, we obtain a new function

$$V(\alpha) = f(x^*(\alpha), y^*(\alpha), \alpha) \tag{4}$$

where this function is the value of f when the values of x and y are those that maximize $f(x, y, \alpha)$. Therefore, $V(\alpha)$ is *the maximum value function* (or indirect objective function). If we differentiate V with respect to α

$$\frac{\partial V}{\partial \alpha} = f_x \frac{\partial x^*}{\partial \alpha} + f_y \frac{\partial y^*}{\partial \alpha} + f_\alpha \tag{5}$$

However, from the first order conditions we know $f_x = f_y = 0$. Therefore, the first two terms disappear and the result becomes

$$\frac{\partial V}{\partial \alpha} = f_\alpha \tag{6}$$

This result says that, at the optimum, as α varies, with x^* and y^* allowed to adjust optimally gives the same result as if x^* and y^* were held constant! Note that α enters maximum value function (equation 4) in three places: one direct and two indirect (through x^* and y^*). Equations 5 and 6 show that, at the optimimum, only the direct effect of α on the objective function matters. This is the essence of the envelope theorem. The envelope theorem says only the direct effects of a change in an exogenous variable need be considered, even though the exogenous variable may enter the maximum value function indirectly as part of the solution to the endogenous choice variables.

1.1 The Profit Function

Let's apply the above approach to an economic application, namely the profit function of a competitive firm. Consider the case where a firm uses two inputs: capital, K, and labour, L. The profit function is

$$\pi = pf(K, L) - wL - rK \tag{7}$$

where p is the output price and w and r are the wage rate and rental rate respectively.

The first order conditions are

$$\begin{aligned} \pi_L &= f_L(K, L) - w = 0 \\ \pi_K &= f_K(K, L) - r = 0 \end{aligned} \tag{8}$$

which respectively define the factor demand equations

$$\begin{aligned} L &= L^*(w, r, p) \\ K &= K^*(w, r, p) \end{aligned} \tag{9}$$

substituting the solutions K^* and L^* into the objective function gives us

$$\pi^*(w, r, p) = pf(K^*, L^*) - wL^* - rK^* \tag{10}$$

$\pi^*(w, r, p)$ is the *profit function* (or indirect objective function). The profit function gives the maximum profit as a function of the exogenous variables w, r, and p.

Now consider the effect of a change in w on the firm's profits. If we differentiate the original profit function (equation 7) with respect to w, holding all other variables constant and we get

$$\frac{\partial \pi}{\partial w} = -L \tag{11}$$

However, this result does not take into account the profit maximizing firms ability to make a substitution of capital for labour and adjust the level of output in accordance with profit maximizing behavior.

Since $\pi^*(w, r, p)$ is the maximum value of profits for any values of w, r, and p, changes in π^* from a change in w takes all captial for labour subsitutions into account. To evaluate a change in the maximum profit function from a change in w, we differentiate $\pi^*(w, r, p)$ with respect to w yielding

$$\frac{\partial \pi^*}{\partial w} = [pf_L - w] \frac{\partial L^*}{\partial w} + [pf_K - r] \frac{\partial K^*}{\partial w} - L^* \tag{12}$$

From the first order conditions, the two bracketed terms are equal to zero. Therefore, the resulting equation becomes

$$\frac{\partial \pi^*}{\partial w} = -L^*(w, r, p) \tag{13}$$

This result says that, at the the profit maximizing position, a change in profits with respect to a change in the wage is the same whether or not the factors are held constant or allowed to vary as the factor price changes. In this case the derivative of the profit function with respect to w is the negative of the factor demand function $L^*(w, r, p)$. Following the above procedure, we can also show the additional comparative statics results

$$\frac{\partial \pi^*(w, r, p)}{\partial r} = -K^*(r, w, p) \tag{14}$$

and

$$\frac{\partial \pi^*(w, r, p)}{\partial p} = f(K^*, L^*) = q^* \tag{15}$$

The simple comparative static results derived from the profit function is known as "Hotelling's Lemma". Hotelling's Lemma is simply an application of the envelope theorem.

1.2 The Envelope Theorem and Constrained Optimization

Now let us turn our attention to the case of constrained optimization. Again we will have an objective function (U), two choice variables, (x and y) and one prarameter (α) except now we introduce the following constraint:

$$g(x, y; \alpha) = 0$$

The derivation of the envelope theorem for the models with one constraint is as follows:

The problem then becomes

Maximize

$$U = f(x, y; \alpha) \tag{16}$$

subject to

$$g(x, y; \alpha) = 0 \tag{17}$$

The Lagrangian for this problem is

$$Z = f(x, y; \alpha) + \lambda g(x, y; \alpha) \tag{18}$$

The first order conditions are

$$\begin{aligned} Z_x &= f_x + \lambda g_x = 0 \\ Z_y &= f_y + \lambda g_y = 0 \\ Z_\lambda &= g(x, y; \alpha) = 0 \end{aligned} \tag{19}$$

Solving this system of equations gives us

$$x = x^*(\alpha) \quad y = y^*(\alpha) \quad \lambda = \lambda^*(\alpha) \tag{20}$$

Substituting the solutions into the objective function, we get

$$U^* = f(x^*(\alpha), y^*(\alpha), \alpha) = V(\alpha) \tag{21}$$

where $V(\alpha)$ is the indirect objective function, or maximum value function. This is the maximum value of y for any α and x_i's that satisfy the constraint.

How does $V(\alpha)$ change as α changes? First, we differentiate V with respect to α

$$\frac{\partial V}{\partial \alpha} = f_x \frac{\partial x^*}{\partial \alpha} + f_y \frac{\partial y^*}{\partial \alpha} + f_\alpha \tag{22}$$

In this case,equation 22 will not simplify to $\frac{\partial V}{\partial \alpha} = f_\alpha$ since $f_x \neq 0$ and $f_y \neq 0$. However, if we substitute the solutions to x and y into the constraint (producing an identity)

$$g(x^*(\alpha), y^*(\alpha), \alpha) \equiv 0 \tag{23}$$

and differentiating with respect to α yields

$$g_x \frac{\partial x^*}{\partial \alpha} + g_x \frac{\partial x^*}{\partial \alpha} + g_\alpha \equiv 0 \tag{24}$$

If we multiply equation 24 by λ and combine the result with equation 22 and rearranging terms, we get

$$\frac{\partial V}{\partial \alpha} = (f_x + \lambda g_x)\frac{\partial x^*}{\partial \alpha} + (f_y + \lambda g_y)\frac{\partial y^*}{\partial \alpha} + f_\alpha + \lambda g_\alpha = Z_\alpha \tag{25}$$

Where Z_α is the partial deviative of the Lagrangian function with respect to α, holding all other variable constant. In this case, the Langrangian functions serves as the objective function in deriving the indirect objective function.

While the results in equation 25 nicely parallel the unconstrained case, it is important to note that some of the comparative static results depend critically on whether the parameters enter only the objective function or whether they enter only the constraints, or enter both. If a parameter enters only in the objective function then the comparative static results are the same as for unconstrained case. However, if the parameter enters the constraint, the relation

$$V_{\alpha\alpha} \geq f_{\alpha\alpha}$$

will no longer hold.

1.3 Interpretation of the Lagrange Multiplier

In the consumer choice problem in chapter 12 we derived the result that the Lagrange multiplier, λ, represented the change in the value of the Lagrange function when the consumer's budget changed. We loosely interpreted λ as the marginal utility of income. Now let us derive a more general interpretation of the Lagrange multiplier with the assistance of the envelope theorem.Consider the problem

Maximize

$$U = f(x, y) \tag{26}$$

Subject to

$$c - g(x, y) = 0 \tag{27}$$

where c is a constant. The Lagrangian for this problem is

$$Z = f(x, y) + \lambda(c - g(x, y)) \tag{28}$$

The first order equations are

$$\begin{aligned} Z_x &= f_x(x, y) - \lambda g_x(x, y) = 0 \\ Z_y &= f_y(x, y) - \lambda g_y(x, y) = 0 \\ Z_\lambda &= c - g(x, y) = 0 \end{aligned} \tag{29}$$

From the firs twot equations in (29), we get

$$\lambda = \frac{f_x}{g_x} = \frac{f_y}{g_y} \tag{30}$$

which gives us the condition that the slope of the level curve of the objective function must equal the slope of the constraint at the optimum.

Equations (29) implicitly define the solutions

$$x = x^*(c) \quad y = y(c) \quad \lambda = \lambda^*(c) \tag{31}$$

substituting (31) back into the Lagrangian yields the mamximum value function

$$V(c) = Z^*(c) = f\left(x^*(c), y^*(c)\right) + \lambda^*(c)\left(c - g(x_1^*(c), y^*(c))\right) \tag{32}$$

differentiating with respect to c yields

$$\frac{\partial Z^*}{\partial c} = f_x \frac{\partial x^*}{\partial c} + f_y \frac{\partial y^*}{\partial c} + \left(c - g(x^*(c), y^*(c))\right)\frac{\partial \lambda^*}{\partial c} - \lambda^*(c) g_x \frac{\partial x^*}{\partial c} - \lambda^*(c) g_y \frac{\partial y^*}{\partial c} + \lambda^*(c)\frac{\partial c}{\partial c} \tag{33}$$

by rearranging we get

$$\frac{\partial Z^*}{\partial c} = (f_x - \lambda^* g_x)\frac{\partial x^*}{\partial c} + (f_y - \lambda^* g_y)\frac{\partial y^*}{\partial c} + (c - g(x^*, y^*))\frac{\partial \lambda^*}{\partial c} + \lambda^* \tag{34}$$

Note that the three terms in brackets are nothing more than the first order equations and, at the optimal values of x, y and λ, these terms are all equal to zero. Therefore this expression simplifies to

$$\frac{\partial V(c)}{\partial c} = \frac{\partial Z^*}{\partial c} = \lambda^* \tag{35}$$

Therefore equals the rate of change of the maximum value of the objective function when c changes (λ is sometimes referred to as the "shadow price" of c).Note that, in this case, c enters the problem only through the constraint; it is not an argument of the original objective function.

2 Duality and the Envelope Theorem

A consumer's expenditure function and his indirect utility function are the minimum and maximum value functions for dual problems. An expenditure function specifies the minimum expenditure required to obtain a fixed level of utility given the utility function and the prices of consumption goods. An indirect utility function specifies the maximum utility that can be obtained given prices, income and the utility function.

Let $U(x, y)$ be a utility function in x and y are consumption goods. The consumer has a budget, B, and faces market prices P_x and P_y for goods x and y respectively.

Setting up the Lagrangian:

$$Z = U(x, y) + \lambda(B - P_x x - P_y y) \tag{36}$$

The first order conditions are

$$\begin{aligned} Z_x &= U_x - \lambda P_x = 0 \\ Z_y &= U_y - \lambda P_y = 0 \\ Z_\lambda &= B - P_x X - P_y Y = 0 \end{aligned} \tag{37}$$

This system of equations implicity defines a solution for x^M, y^M and λ^M as a function of the exogenous variables B, P_x, P_y.

$$\begin{aligned} x^M &= x^M(P_x, P_y, B) \\ y^M &= y^M(P_x, P_y, B) \\ \lambda^M &= \lambda^M(P_x, P_y, B, \alpha) \end{aligned} \tag{38}$$

The solutions to x^M and y^M are the consumer's ordinary demand functions, sometimes called the "Marshallian" demand functions.[1]

Substituting the solutions to x^* and y^* into the utility function yields

$$U^* = U^*(x^M(B, P_x, P_y), y^M(B, P_x, P_y)) = V(B, P_x, P_y) \tag{39}$$

Where V is the maximum value function, or indirect utility function.

Now consider the alternative, or dual, problem for the consumer; minimize total expenditure on x and y while maintaining a given level of utility, U^*. The Langranian for this problem is

$$Z = P_x x + P_y y + \lambda(U^* - U(x, y)) \tag{40}$$

The first order conditions are

$$\begin{aligned} Z_x &= P_x - \lambda U_x = 0 \\ Z_y &= P_y - \lambda U_y = 0 \\ Z_\lambda &= U^* - U(x, y; \alpha) = 0 \end{aligned} \tag{41}$$

This system of equations implicitly define the solutions to x^h, y^h and λ^h

$$\begin{aligned} x^h &= x^h(U^*, P_x, P_y) \\ y^h &= y^h(U^*, P_x, P_y) \\ \lambda^h &= \lambda^h(U^*, P_x, P_y) \end{aligned} \tag{42}$$

[1] Named after the famous economist Alfred Marshall, known to most economic students as "another dead guy."

x^h and y^h are the compensated, or "real income" held constant demand functions. They are commonly referred to as "Hicksion" demand functions, hence the h superscript.[2]

If we compare the first two equations from the first order conditions in both utility maximization problem and expenditure minimization problem (Z_x, Z_y), we see that both sets can be combined (eliminating λ) to give us

$$\frac{Px}{Py} = \frac{Ux}{Uy}(= MRS) \tag{43}$$

This is the tangency condition in which the consumer chooses the optimal bundle where the slope of the indifference curve equals the slope of the budget constraint. The tangency condition is identical for both problems. If the target level of utility in the minimization problem is set equal to the value of the utility obtained in the solution to the maximization problem, namely U^*, we obtain the following

$$\begin{aligned} x^M(B, P_x, P_y) = x^h(U^*, P_x, P_y) \\ y^M(B, P_x, P_y) = y^h(U^*, P_x, P_y) \end{aligned} \tag{44}$$

or the solution to both the maximization problelm and the minimization problem produce identical values for x and y. However, the solutions are functions of different exogenous variables so any comparative statics exercises will produce different results.

Substituting x^h and y^h into the objective function of the minimization problem yields

$$P_x x^h(P_x, P_y, U^*) + P_y y^h(P_x, P_y, U^*) = E(P_x, P_y, U^*) \tag{45}$$

where E is the minimum value function or expenditure function. The duality relationship in this case is

$$E(P_x, P_y, U^*, \alpha) = B \tag{46}$$

where B is the exogenous budget from the maximization problem.

Finally, it can be shown from the first order conditions of the two problems that

$$\lambda^M = \frac{1}{\lambda^h} \tag{47}$$

2.1 Roy's Identity

One application of the envelope theorem is the derivation of Roy's identity. Roy's identity states that the individual consumer's marshallian demand function is equal to the ratio of partial derviatives of the maximum value function. Substituting the optimal values of x^M, y^M and λ^M into the Lagrangian gives us

$$V(B, P_x, P_y) = U(x^M, y^M) + \lambda^M(B - P_x x^M - P_y y^M) \tag{48}$$

[2]Yet another famous, but dead economist, Sir John Hicks.

First differentiate with respect to P$_x$

$$\frac{\partial V}{\partial P_x} = (U_x - \lambda^M P_x)\frac{\partial x^M}{\partial P_x} + (U_y - \lambda^M P_y)\frac{\partial y^M}{\partial P_x} + (B - P_x x^M - P_y y^M)\frac{\partial \lambda^M}{\partial P_x} - \lambda^M x^M \quad (49)$$

$$\frac{\partial V}{\partial P_x} = (0)\frac{\partial x^M}{\partial P_x} + (0)\frac{\partial y^M}{\partial P_x} + (0)\frac{\partial \lambda^M}{\partial P_x} - \lambda^M x^M = -\lambda^M x^M \quad (50)$$

Next, differentiate the value function with respect to B

$$\frac{\partial V}{\partial B} = (U_x - \lambda^M P_x)\frac{\partial x^M}{\partial B} + (U_y - \lambda^M P_y)\frac{\partial y^M}{\partial B} + B - P_x x^M - P_y y^M)\frac{\partial \lambda^M}{\partial B} + \lambda^M \quad (51)$$

$$\frac{\partial V}{\partial B} = (0)\frac{\partial x^M}{\partial B} + (0)\frac{\partial y^M}{\partial B} + (0)\frac{\partial \lambda^M}{\partial B} + \lambda^M = \lambda^M \quad (52)$$

Finally, taking the ratio of the two partial derivatives

$$\frac{\frac{\partial V}{\partial P_x}}{\frac{\partial V}{\partial B}} = \frac{-\lambda^M x^M}{\lambda^M} = x^M \quad (53)$$

which is Roy's identity.

2.2 Shephard's Lemma

Earlier in the chapter an application of the envelope theorem was the derivation of Hotelling's Lemma, which states that the partial derivatives of the maximum value of the profit function yields the firm's factory demand functions and the supply functions. A similar approach applied to the expenditure function yields Shepard's Lemma.

Consider the consumer's minimization problem. The Lagrangian is

$$Z = P_x x + P_y y + \lambda(U^* - U(x, y)) \quad (54)$$

From the first order conditions, the solutions are implicitly defined

$$\begin{aligned} x^h &= x^h(P_x, P_y, U^*) \\ y^h &= y^h(P_x, P_y, U^*) \\ \lambda^h &= \lambda^h(P_x, P_y, U^*) \end{aligned} \quad (55)$$

Substituting these solutions into the Lagrangian yields the minimum value function

$$V(P_x, P_y, U^*) = P_x x^h + P_y y^h + \lambda^h(U^* - U(x^h, y^h)) \quad (56)$$

The partial derivatives of the value function with respect to P_x and P_y are the consumer's conditional, or Hicksian, demands:

$$\begin{aligned} \tfrac{\partial V}{\partial P_x} &= (P_x - \lambda^h U_x)\tfrac{\partial x^h}{\partial P_x} + (P_y - \lambda^h U_y)\tfrac{\partial y^h}{\partial P_x} + (U^* - U(x^h, y^h))\tfrac{\partial \lambda^h}{\partial P_x} + x^h \\ \tfrac{\partial V}{\partial P_x} &= (0)\tfrac{\partial x^h}{\partial P_x} + (0)\tfrac{\partial y^h}{\partial P_x} + (0)\tfrac{\partial \lambda^h}{\partial P_x} + x^h = x^h \end{aligned} \tag{57}$$

and

$$\begin{aligned} \tfrac{\partial V}{\partial P_y} &= (P_x - \lambda^h U_x)\tfrac{\partial x^h}{\partial P_y} + (P_y - \lambda^h U_y)\tfrac{\partial y^h}{\partial P_y} + (U^* - U(x^h, y^h))\tfrac{\partial \lambda^h}{\partial P_y} + y^h \\ \tfrac{\partial V}{\partial P_y} &= (0)\tfrac{\partial x^h}{\partial P_y} + (0)\tfrac{\partial y^h}{\partial P_y} + (0)\tfrac{\partial \lambda^h}{\partial P_y} + y^h = y^h \end{aligned} \tag{58}$$

Differentiating V with respect to the constraint U* yields λ^h, the marginal cost of the constraint

$$\begin{aligned} \frac{\partial V}{\partial U^*} &= (P_x - \lambda^h U_x)\frac{\partial x^h}{\partial U^*} + (P_y - \lambda^h U_y)\frac{\partial y^h}{\partial P_y} + (U^* - U(x^h, y^h))\frac{\partial \lambda^h}{\partial U^*} + \lambda^h \\ \frac{\partial V}{\partial U^*} &= (0)\frac{\partial x^h}{\partial U^*} + (0)\frac{\partial y^h}{\partial U^*} + (0)\frac{\partial \lambda^h}{\partial U^*} + y^h = \lambda^h \end{aligned}$$

Together, these three partial derivatives are Shepard's Lemma.

2.3 Example of duality for the consumer choice problem

2.3.1 Utility Maximization

Consider a consumer with the utility function $U = xy$, who faces a budget constraint of $B = P_x x P_y y$, where all variables are defined as before.

The choice problem is

Maximize

$$U = xy \tag{59}$$

Subject to

$$B = P_x x P_y y \tag{60}$$

The Lagrangian for this problem is

$$Z = xy + \lambda(B - P_x x P_y y) \tag{61}$$

The first order conditions are

$$\begin{aligned} Z_x &= y - \lambda P_x = 0 \\ Z_y &= x - \lambda P_y = 0 \\ Z_\lambda &= B - P_x x - P_y y = 0 \end{aligned} \tag{62}$$

Solving the first order conditions yield the following solutions

$$x^M = \tfrac{B}{2P_x} \quad y^M = \tfrac{B}{2P_y} \quad \lambda = \tfrac{B}{2P_x P_y} \tag{63}$$

where x^M and y^M are the consumer's Marshallian demand functions. Checking second order conditions, the bordered Hessian is

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & 1 & -P_x \\ 1 & 0 & -P_y \\ -P_x & -P_y & 0 \end{vmatrix} = 2P_xP_y > 0 \tag{64}$$

Therefore the solution does represent a maximum . Substituting x^M and y^M into the utility function yields the indirect utility function

$$V(P_x, P_y, B) = \left(\frac{B}{2P_x}\right)\left(\frac{B}{2P_y}\right) = \frac{B^2}{4P_xP_y} \tag{65}$$

If we denote the maximum utility by U_0 and re-arrange the indirect utility function to isolate B

$$\frac{B^2}{4P_xP_y} = U_0 \tag{66}$$

$$B = (4P_xP_yU_0)^{\frac{1}{2}} = 2P_x^{\frac{1}{2}}P_y^{\frac{1}{2}}U_0^{\frac{1}{2}} = E(P_x, P_y, U_0) \tag{67}$$

We have the expenditure function

Roy's Identity Let's verify Roy's identity which states

$$x^M = -\frac{\frac{\partial V}{\partial P_x}}{\frac{\partial V}{\partial B}} \tag{68}$$

Taking the partial derivative of V

$$\frac{\partial V}{\partial P_x} = -\frac{B^2}{4P_x^2P_y} \tag{69}$$

and

$$\frac{\partial V}{\partial B} = -\frac{B}{P_xP_y} \tag{70}$$

Taking the negative of the ratio of these two partials

$$-\frac{\frac{\partial V}{\partial P_x}}{\frac{\partial V}{\partial B}} = -\frac{\left(\frac{B^2}{4P_x^2P_y}\right)}{\left(\frac{B}{P_xP_y}\right)} = \frac{B}{2P_x} = x^M \tag{71}$$

Thus we find that Roy's Identity does hold.

2.3.2 The dual and Shepard's Lemma

Now consider the dual problem of cost minimization given a fixed level of utility. Letting U_0 denote the target level of utility, the problem is

Minimize

$$P_x x + P_y y \tag{72}$$

Subject to

$$U_0 = xy \tag{73}$$

The Lagrangian for the problem is

$$Z = P_x x + P_y y + \lambda(U_0 - xy) \tag{74}$$

The first order conditions are

$$\begin{array}{c} Z_x = P_x - \lambda y = 0 \\ Z_y = P_y - \lambda x = 0 \\ Z_\lambda = U_0 - xy = 0 \end{array} \tag{75}$$

Solving the system of equations for x, y and λ

$$\begin{array}{c} x^h = \left(\frac{P_y U_0}{P_x}\right)^{\frac{1}{2}} \\ y^h = \left(\frac{P_x U_0}{P_y}\right)^{\frac{1}{2}} \\ \lambda^h = \left(\frac{P_x P_y}{U_0}\right)^{\frac{1}{2}} \end{array} \tag{76}$$

where x^h and y^h are the consumer's compensated (Hicksian) demand functions. Checking the second order conditions for a minimum

$$\left|\overline{H}\right| = \left| \begin{array}{ccc} 0 & -\lambda & -y \\ -\lambda & 0 & -x \\ -y & -x & 0 \end{array} \right| = -2xy\lambda < 0 \tag{77}$$

Thus the sufficient conditions for a minimum are satisfied.

Substituting x^h and y^h into the orginal objective function gives us the minimum value function, or expenditure function

$$\begin{array}{l} P_x x^h + P_y y^h = P_x \left(\frac{P_y U_0}{P_x}\right)^{\frac{1}{2}} + P_y \left(\frac{P_x U_0}{P_y}\right)^{\frac{1}{2}} \\ = (P_x P_y U_0)^{\frac{1}{2}} + (P_x P_y U_0)^{\frac{1}{2}} \\ = 2P_x^{\frac{1}{2}} P_y^{\frac{1}{2}} U_0^{\frac{1}{2}} \end{array} \tag{78}$$

Note that the expenditure function derived here is identical to the expenditure function obtained by re-arranging the indirect utility function from the maximization problem.

Shepard's Lemma We can now test Shepard's Lemma by differentiating the expenditure function directly.

First, we derive the conditional demand functions

$$\frac{\partial E(P_x, P_y, U_0)}{\partial P_x} = \frac{\partial}{\partial P_x}\left(2P_x^{\frac{1}{2}}P_y^{\frac{1}{2}}U_0^{\frac{1}{2}}\right) = \frac{P_y^{\frac{1}{2}}U_0^{\frac{1}{2}}}{P_x^{\frac{1}{2}}} = x^h \tag{79}$$

and

$$\frac{\partial E(P_x, P_y, U_0)}{\partial P_y} = \frac{\partial}{\partial P_y}\left(2P_x^{\frac{1}{2}}P_y^{\frac{1}{2}}U_0^{\frac{1}{2}}\right) = \frac{P_y^{\frac{1}{2}}U_0^{\frac{1}{2}}}{P_y^{\frac{1}{2}}} = y^h \tag{80}$$

Next, we can find the marginal cost of utility (the Lagrange multiplier)

$$\frac{\partial E(P_x, P_y, U_0)}{\partial U^0} = \frac{\partial}{\partial U^0}\left(2P_x^{\frac{1}{2}}P_y^{\frac{1}{2}}U_0^{\frac{1}{2}}\right) = \frac{P_x^{\frac{1}{2}}P_y^{\frac{1}{2}}}{U_0^{\frac{1}{2}}} = \lambda^h \tag{81}$$

Thus, Shepard's Lemma holds in this example.

3 Income and Substitution Effects: The Slutsky Equation

3.1 The Traditional Approach

Consider a representative consumer who chooses only two goods: x and y. The price of both goods are determined in the market and are therefore exogenous. As well, the consumer's budget is also exogenously determined. The consumer choice problem then is

Maximize

$$U(x, y) \tag{82}$$

Subject to

$$B = P_x X + P_y Y \tag{83}$$

The Langrangian function for this optimization problem is

$$Z = U(x, y) + \lambda(B - P_x x + P_y y) \tag{84}$$

The first order conditions yield the following set of simultaneous equations:

$$\begin{aligned} Z_\lambda &= B - P_x x - P_y y = 0 \\ Z_x &= U_x - \lambda P_x = 0 \\ Z_y &= U_y - \lambda P_y = 0 \end{aligned} \tag{85}$$

Solving this system will allow us to express the optimal values of the endogenous variables as implicit functions of the exogenous variables:

$$\begin{aligned} \lambda^* &= \lambda^*(P_x, P_y, B) \\ x^* &= x^*(P_x, P_y, B) \\ y^* &= y^*(P_x, P_y, B) \end{aligned}$$

If the bordered Hessian in the present problem is positive

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & -P_x & -P_y \\ -P_x & U_{xx} & U_{xy} \\ -P_y & U_{yx} & U_{yy} \end{vmatrix} = 2P_xP_yU_{xy} - P_y^2U_{xx} - P_x^2U_{yy} > 0 \tag{86}$$

then the value of U will be a maximum.

.

By substituting the optimal values x^*, y^* and λ^* into the first order equations, we convert these equations into equilibrium identities:

$$\begin{aligned} B - P_yx^* - P_yy^* &\equiv 0 \\ U_x\left(x^*, y^*\right) - \lambda^*P_x &\equiv 0 \\ U_y\left(x^*, y^*\right) - \lambda^*P_y &\equiv 0 \end{aligned}$$

By taking the total differential of each identity in turn, and noting that $U_{xy} = U_{yx}$ (Young's Theorem), we then arrive at the linear system

$$\begin{aligned} -Pxdx^* - Pydy &= x^*dPx + y^*dPy - dB \\ -Pxd\lambda^* + U_{xx}dx^* + U_{xy}dy^* &= \lambda^*dPx \\ -Pyd\lambda^* + U_{yx}dx^* + U_{yy}dy^* &= \lambda^*dPy \end{aligned} \tag{87}$$

Writing these equations in matrix form

$$\begin{bmatrix} 0 & -P_x & -P_y \\ -P_x & U_{xx} & U_{xy} \\ -P_y & U_{yx} & U_{yy} \end{bmatrix} \begin{pmatrix} d\lambda^* \\ dx^* \\ dy^* \end{pmatrix} = \begin{bmatrix} x^*dP_x + y^*dP_y - dB \\ \lambda^*dP_x \\ \lambda^*dP_y \end{bmatrix}$$

To study the effect of a change in the budget, let the other exogenous differentials equal zero $(dP_x = dP_y = 0, dB \neq 0)$. Then dividing through by dB, and applying the implicit function theorem, we have

$$\begin{bmatrix} 0 & -P_x & -P_y \\ -P_x & U_{xx} & U_{xy} \\ -P_y & U_{yx} & U_{yy} \end{bmatrix} \begin{pmatrix} d\lambda^*/\partial B \\ dx^*/\partial B \\ dy^*/\partial B \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \tag{88}$$

The coefficient matrix of this system is the Jacobian matrix, which has the same value as the bordered Hessian $\left|\overline{H}\right|$ which is positive if the second order conditions are met. By using Cramer's rule we can solve for the following comparative static

$$\frac{\partial x^*}{\partial B} = \frac{1}{\left|\overline{H}\right|} \begin{vmatrix} 0 & -1 & -P_y \\ -P_x & 0 & U_{xy} \\ -P_y & 0 & U_{yy} \end{vmatrix} = \frac{1}{\left|\overline{H}\right|} \begin{vmatrix} -P_x & U_{xy} \\ -P_y & U_{yy} \end{vmatrix} = \frac{\left|\bar{H}_{12}\right|}{\left|\overline{H}\right|} \lesseqgtr 0 \tag{89}$$

As before, in the absence of additional information about the relative magnitudes of P_x, P_y and the cross partials, U_{ij}, we are unable to ascertain the sign of this comparative-static derivative. This means that the optimal x^* may increase in the budget, B, depending on whether it is a *normal* or *inferior* good (ambiguous income effect)

Next, we may analyze the effect of a change in P_x. Letting $dP_y = dB = 0$ but keeping $dP_x \neq 0$ and dividing Equation 87 by dP_x we obtain

$$\begin{bmatrix} 0 & -P_x & -P_y \\ -P_x & U_{xx} & U_{xy} \\ -P_y & U_{yx} & U_{yy} \end{bmatrix} \begin{pmatrix} \partial\lambda^*/\partial P_x \\ \partial x^*/\partial P_x \\ \partial y^*/\partial P_x \end{pmatrix} = \begin{bmatrix} x^* \\ \lambda^* \\ 0 \end{bmatrix} \tag{90}$$

From this, the following comparative static emerges:

$$\begin{aligned} \tfrac{\partial x^*}{\partial P_x} &= \tfrac{1}{|\overline{H}|}\begin{vmatrix} 0 & x^* & -P_y \\ -P_x & \lambda^* & U_{xy} \\ -P_y & 0 & U_{yy} \end{vmatrix} \\ &= \tfrac{-x^*}{|\overline{H}|}\begin{vmatrix} -P_x & U_{xy} \\ -P_y & U_{yy} \end{vmatrix} + \tfrac{\lambda^*}{|\overline{H}|}\begin{vmatrix} 0 & -P_y \\ -P_y & U_{yy} \end{vmatrix} \\ &= (-x^*)\tfrac{|\bar{H}_{12}|}{|\overline{H}|} + \lambda^*\tfrac{|\bar{H}_{22}|}{|\overline{H}|} \end{aligned} \tag{91}$$

Note that there are two componants in $(\frac{\partial x^*}{\partial P_x})$. By comparing the first term to our previous comparative static $(\frac{\partial x^*}{\partial B})$, we see that

$$(-x^*)\frac{|\bar{H}_{12}|}{|\overline{H}|} = (-x^*)\left(\frac{\partial x^*}{\partial B}\right) \lesseqgtr 0 \tag{92}$$

which can be interpreted as the *income effect of a price change.* The second term is the income compensated version of $\partial x^*/\partial P_x$, or the substitution effect of a price change, which is unambiguously negative:

$$\left(\frac{\partial x^*}{\partial P_x}\right)_{compensated} = \frac{\lambda^*}{|\overline{H}|}\begin{vmatrix} 0 & -P_y \\ -P_y & U_{yy} \end{vmatrix} = \lambda^*\frac{|\bar{H}_{22}|}{|\overline{H}|} = \frac{\lambda^*}{|\overline{H}|}(P_y^2) < 0 \tag{93}$$

Hence, we can express Equation 91 in the form

$$\frac{\partial x^*}{\partial P^*} = \underbrace{-\left(\frac{\partial x^*}{\partial B}\right)x^*}_{\text{Income Effect}} + \underbrace{\left(\frac{\partial x^*}{\partial P_x}\right)_{compensated}}_{\text{Substitution Effect}} \tag{94}$$

This result, which decomposes the comparative static derivative $(\partial x^*/\partial P_x)$ into two componants, an income effect and a substitution effect, is the two-good version of the "Slutsky Equation."

3.2 Duality and the Alternative Slutsky

From the envelope theorem, we can derive the Slutsky decomposition in a more succinct manner. Consider first that from the utility maximum problem we derived solutions for x and y

$$\begin{aligned} x^M &= x^M(P_x, P_y, B) \\ y^M &= y^M(P_x, P_y, B) \end{aligned} \tag{95}$$

which were the marshallian demand functions. Substituting these solutions into the utility function yielded the indirect utility function (or maximum value function)

$$U^* = U(x^M(P_x, P_y, B), y^M(P_x, P_y, B)) = U^*(P_x, P_y, B) \tag{96}$$

which could be rewritten to isolate B and giving us the expenditure function

$$B^* = B(P_x, P_{y,}U^*) \tag{97}$$

Second, from the budget minimization problem we derived the Hicksian, or compensated, demand function

$$x^* = x^h(P_x, P_{y,}U^*) \tag{98}$$

which, by Shephards lemma, is equivalent to the partial derivative of the expenditure function with respect to P_x:

$$\frac{\partial B(P_x, P_{y,}U^*)}{\partial P_x} = x^c(P_x, P_{y,}U^*) \tag{99}$$

Thus we know that if the maximum value of utility obtained from

$$Max \quad U(x, y) + \lambda(B - P_x x - P_y y)$$

is the same value as the exogenous level of utility found in the constrained minimization problem

$$Min \quad P_x x + P_y y + \lambda(U_0 - U(x, y)) \tag{100}$$

the values of x and y that satisfy the first order conditions of both problems will be identical, or

$$x^c(P_x, P_{y,}U_0) = x^m(P_x, P_{y,}B) \tag{101}$$

at the optimum.If we subsitiute the expenditure function into x^M in place of the budget, B, we get

$$x^c(P_x, P_{y,}U_0) = x^M(P_x, P_{y,}B^*(P_x, P_{y,}U_0)) \tag{102}$$

Differentiate both sides of equation 102 with respect to P_x

$$\begin{aligned} \tfrac{\partial x^c(P_x,P_y,U_0)}{\partial P_x} = & \tfrac{\partial x^M(P_x,P_y,B^*(P_x,P_y,U_0))}{\partial P_x} \\ & + \tfrac{\partial x^M(P_x,P_y,B^*(P_x,P_y,U_0))}{\partial B} \tfrac{\partial B(P_x,P_y,U_0)}{\partial P_x} \end{aligned} \tag{103}$$

But we know from Shephard's lemma that

$$\frac{\partial B(P_x, P_y, U_0)}{\partial P_x} = x_c \tag{104}$$

substituting equation 104 in to equation 103 we get

$$\frac{\partial x^c}{\partial P_x} = \frac{\partial x^M}{\partial P_x} + x^c \frac{\partial x^M}{\partial B} \tag{105}$$

Subtract $(x^c \frac{\partial x^M}{\partial B})$ from both sides gives us

$$\frac{\partial x^M}{\partial P_x} = \underbrace{-x^c \frac{\partial x^M}{\partial B}}_{\text{Income effect}} + \underbrace{\frac{\partial x^c}{\partial P_x}}_{\text{Substitution effect}} \tag{106}$$

If we compare equation (106) to equation (94) we see that we have arrived at the identical result. The method of deriving the slutsky decomposition through the application of duality and the envelope theorem is sometimes referred to as the "instant slutsky".

3.2.1 Problems:

1. A consumer has the following utility function: $U(x, y) = x(y+1)$, where x and y are quantities of two consumption goods whose prices are p_x and p_y respectively. The consumer also has a budget of B. Therefore the consumer's maximization problem is

$$x(y + 1) + \lambda(B - p_x x - p_y y)$$

   (a) From the first order conditions find expressions for the demand functions. What kind of good is y? In particular what happens when $p_y > B/2$?

   (b) Verify that this is a maximum by checking the second order conditions. By substituting x^* and y^* into the utility function find an expressions for the indirect utility function

$$U^* = U(p_x, p_y, B)$$

   and derive an expression for the expenditure function

$$B^* = B(p_x, p_y, U^*)$$

   (c) This problem could be recast as the following dual problem

$$Minimize \quad p_x x + p_y y \text{ subject to } U^* = x(y + 1)$$

   Find the values of x and y that solve this minimization problem and show that the values of x and y are equal to the partial derivatives of the expenditure function, $\partial B/\partial p_x$ and $\partial B/\partial p_y$ respectively.